季節變動調整方法에 關한 參考文獻集

第1卷 季節調整方法

1986. 7.

調査統計局 統計分析課

일 러 두 기

當局에서는 景氣綜合指數, 產業生產指數 等의 各種 月別 經濟指標의 季節變動調整系列을 X-II-ARIMA 方法으로 作成하고 있으며, 個別 指標別 ARIMA 模型選定, 休日의 事前調整 等 보다 나은 季節變動調整을 위한 研究作業을 推進하고 있는바, 當課에서 그동안 參考資料로 活用하여 온 미국, 일본, 캐나다 등의 各種 研究論文 資料, Technical Paper 等을 보다 많은 利用·活用을 위하여 拔萃하여 發刊하는 것임.

分量關係로 2卷으로 나누었으며 收錄內容은 다음과 같음.

第1卷. 季節變動調整方法에 관한 全般的인 內容

第2卷. 事前調整, ARIMA 模型, 特異項 等에 관한 內容

統計分析課長

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O.D. Anderson & M.R. Perryman, eds., TIME SERIES ANALYSIS

ESTIMATION OF SEASONAL FACTORS USING BOTH TRADITIONAL METHODS AND BOX-JENKINS TECHNIQUES

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We present here a situation in which an economic investigator believes that the time series under study has an annual seasonal pattern. However, the traditional methods do not detect this pattern satisfactorily. Using Box-Jenkins techniques we are able to find that the seasonality does exist, although it shows up more clearly in the monthly rate of change of the series. Therefore, the Box-Jenkins methodology served, in this instance, to shed some light on the underlying characteristics of the data. Once the appropriate variable is determined, a procedure is developed to produce forecasts under restrictions.

I. INTRODUCTION

The purpose of this work is to determine if the data on Total Financing Granted by the Mexican Bank System (see table 1), depends -as is believed- on monthly seasonal factors. If this hypothesis is true we should (i) isolate the seasonal effect in order to study the trend and (ii) distribute the financing throughout the year. The latter in accordance to exogenous policies or criteria that affect historial trend and/or seasonality.

Use is made here of two elements which constitute Time Series Analysis, namely :

a) The traditional, which uses empirical methods, with a very simple but weak basis, and whose practical utility lies mainly on estimating seasonal factors; and
b) the Box-Jenkins technique, which is an efficient forecasting method but, unfortunately, does not yield seasonal indices in an obvious fashion.

There are many reasons why one may want to estimate seasonal indices. In the present case we want to estimate them to deseasonalize the observed series.

TABLE 1

TOTAL FINANCING GRANTED BY THE MEXICAN BANK SYSTEM
(Billions of pesos)

YEAR	JAN	FEB	MAR	APR	MAY	JUN	JUL	AUG	SEP	OCT	NOV	DEC
1975	376.9	386.6	393.5	402.6	410.1	415.1	421.9	431.4	438.4	448.7	458.5	477.5
1976	481.9	488.8	498.0	507.8	516.1	526.4	541.4	546.8	563.6	565.6	[illegible]	596.1
1977	597.6	604.7	615.9	624.5	631.9	643.8	654.1	671.7	681.8	697.5	[illegible]	[illegible]
1978	747.5	761.2	775.1	777.8	789.2	805.8	827.5	850.1	857.5	882.0	893.4	928.3
1979	927.3	957.1	984.8	1004.6								

II. USE OF TRADITIONAL METHODS.

The computation of seasonal factors from periodical time series is a routine matter due to the availability of "canned" procedures. In this particular study, the Census Method II, and the classical Ratio to Moving Average Method, were made available through the Banco de México's Computer Center. These methods assume implicitly the existence of a multiplicative model such as

Observation = Trend x Seasonality x Irregularity

which some authors (e.g. Hannan, 1963) have translated into statistical terms such as

$$Z_\tau = P_\tau \cdot S_\tau \cdot \exp(\varepsilon_\tau) \qquad (2.1)$$

where Z_τ is the observed datum, P_τ is the trend component, S_τ denotes the seasonal component and ε_τ is a second-order stationary random error with mean zero.

If $Z_\tau > 0$ for all τ and if it is reasonable to assume a stable monthly seasonal effect, model (2.1) can be written as

$$\log(Z_{12\tau+q}) = \log(P_{12\tau+q}) + \log(S_q) + \varepsilon_{12\tau+q}$$

$$\tau = 0,1,\ldots,m-1;\; q = 1,\ldots,12 \qquad (2.2)$$

with $\sum_{q=1}^{12} \log(S_q) = 0$, $E(\varepsilon_{12\tau+q}) = 0$ and $E(\varepsilon_{12\tau+q}, \varepsilon_{12\tau+q+r}) = \gamma_r$.

The reasoning behind the ratio to moving average method should lead us to estimate S_q by means of

$$\hat{S}_q = \left(\prod_{\tau=0}^{m-1} \frac{Z_{12\tau+q}}{GMA(Z_\tau,q)} \right)^{\frac{1}{m}} \qquad (2.3)$$

where $GMA(Z_\tau,q) = (Z_{12\tau+q-6} \cdot Z_{12\tau+q+6} \cdot \prod_{i=-5}^{5} Z^2_{12\tau+q+i})^{\frac{1}{24}}$

is a geometric moving average of Z_τ centered at q.

The estimator (2.3) is more logically sustained than (2.4), which is the classical ratio to moving average estimator, i.e.,

$$\tilde{S}_q = \frac{1}{m}\left(\sum_{\tau=0}^{m-1} \frac{Z_{12\tau+q}}{AMA(Z_\tau,q)} \right), \qquad (2.4)$$

where $AMA(Z_\tau,q) = \frac{1}{24} \; (Z_{12\tau+q-6} + Z_{12\tau+q+6} + 2 \sum_{i=-5}^{5} Z_{12\tau+q+i})$

is an arithmetic moving average of Z_τ centered at q.

Formula (2.3) is defficient in that it does not consider replacement of extreme values, nor does it completely eliminate the trend component (unless the trend is linear). For these reasons, it would be preferable to use Hannan's method or to perform an iterative procedure. However, to implement either possibility was beyond the objectives of this study. Thus, both Census Method II (CM-II) and (2.3) were used in order to make comparisons. The seasonal factors (obtained with both methods under the assumption of stable seasonality) are shown in table 2.

Both methods indicate that May is the month lowest with respect to the trend, and December is its counterpart. In general, however, the factors are discordant, January for instance, lies 0.64% above the trend according to (2.3), but CM-II indicates that it is below the trend by 0.41%. This discrepancy is essentially due to the fact that the trend given by (2.3) is different to that obtained by CM-II. Another reason is that the seasonal effect is not clear enough. Aside from this difference, it can be noticed that all factors are very close to unity, which raises the question of whether or not the seasonal effect is relevant for the series (even though, due to the amount of financing, a seasonal increment greater than 2% in December could be important).

TABLE 2

MONTHLY SEASONAL FACTORS

METHOD 2/	JAN	FEB	MAR	APR	MAY	JUN	JUL	AUG	SEP	OCT	NOV	DEC
(2.3)	1.006	1.002	1.002	.996	.991	.993	.996	.999	.997	.998	.998	1.022
CM-II	0.996	.996	.997	.996	.995	.995	.999	1.001	1.000	1.000	1.001	1.026

The previous results led us to test the significance of the seasonality in the data. For lack of a more appropriate test, we used the Adjacent Month Test as described in Makridakis and Wheelwright (1978, p. 125). The average of the variation ratios used in this test all lie inside [0.9881, 1.0195] which means it is doubtful that seasonality plays an important role in explaining the behavior of the series.

III. USE OF THE BOX-JENKINS TECHNIQUE

The identification phase of the Box-Jenkins Technique was used to determine if a model existed which both takes seasonality into consideration and can adequately represent the data.

A graph of the observed series appears in figure 1. Here we noticed the slight jump in December, which was already detected by traditional methods. First, we looked for a reexpression of the data to obtain homogeneous variance for the whole series, because variance heterogeneity can hide the real seasonal effect. It is well known that the original data are not always expressed in the appropriate scale for their analysis, so we employed Bartlett's (1947) procedure to find the power transformation that best stabilizes variance. Such a procedure suggests carrying out the analysis in the scale where (3.1) is satisfied,

$$\frac{S_Z}{\bar{Z}^\delta} = \frac{\text{Sample Standard Deviation}}{(\text{Sample Mean})^\delta} = \text{constant} \qquad (3.1)$$

An adequate value of δ can be estimated by trial and error until (3.1) is approximately fulfilled, thus the appropriate power transformation becomes

$$T(Z) = \begin{cases} Z^{1-\delta} & \text{if } \delta \neq 1 \\ \log(Z) & \text{if } \delta = 1 \end{cases} \qquad (3.2)$$

In practice, we carried out the procedure by dividing the series into annual subseries, leaving out the four months of 1979, and then constructing table 3. The criterion we employed to decide if (3.1) was approximately true was to minimize the coefficient of variation of $S_Z/\bar{Z}^\delta$. Thus, the transformation suggested became the logarithmic. Afterwards, we performed the procedure again with the series divided into four-month periods, in order to include all observations; the conclusion reached was the same as before.

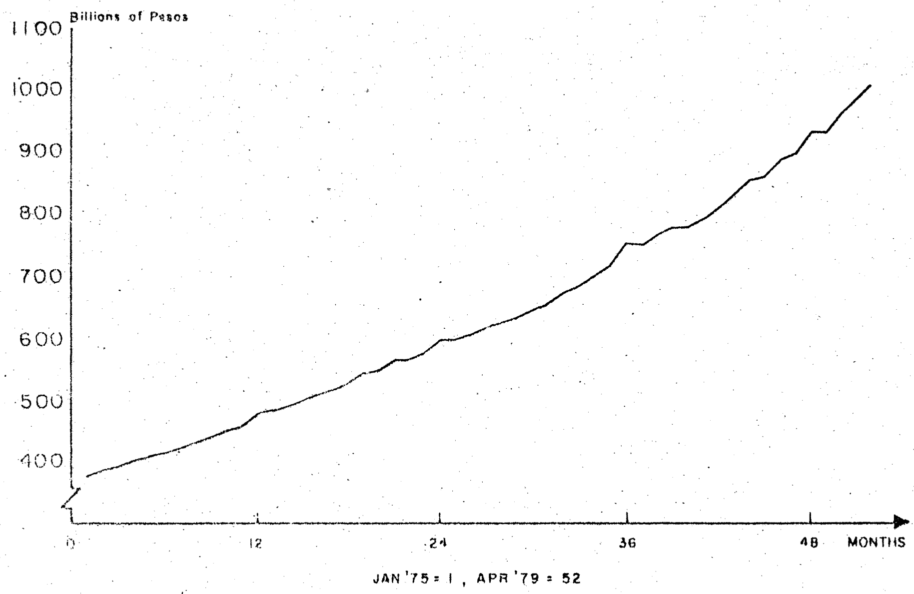


FIGURE 1
TOTAL FINANCING

TABLE 3
COEFFICIENTS OF VARIATION FOR DIFFERENT VALUES OF δ

δ	0	1/2	1	3/2	2
YEAR	S_Z	$S_Z/\bar{Z}^{1/2}$	$S_Z/\bar{Z}$	$S_Z/\bar{Z}^{3/2}$	$S_Z/\bar{Z}^2$
1975	30369.09	46.7624	.0720	1.1087396 -04	1.7072250 -07
1976	36648.72	50.1531	.0686	9.3923871 -05	1.2853312 -07
1977	46535.21	57.4084	.0708	8.7369968 -05	1.0778434 -07
1978	57783.20	63.6322	.0701	7.7166280 -05	8.4977292 -08
Coefficient of variation	.2798	.1384	.0202	.1533	.2963

Taking logarithms from the data has been recommended by several authors in many situations. With regard to this suggestion, Box and Jenkins (1970, p. 303) say: "Logarithms are often taken before analyzing sales data and other series of this kind, because it is percentage fluctuation which might be expected to be comparable at different sales volumes".

The identification process applied to the logged series shows that its first difference is roughly stationary, as seen in figure 2. The strong correlation at lag 12 thus indicates that a periodic component does exist in this new series.

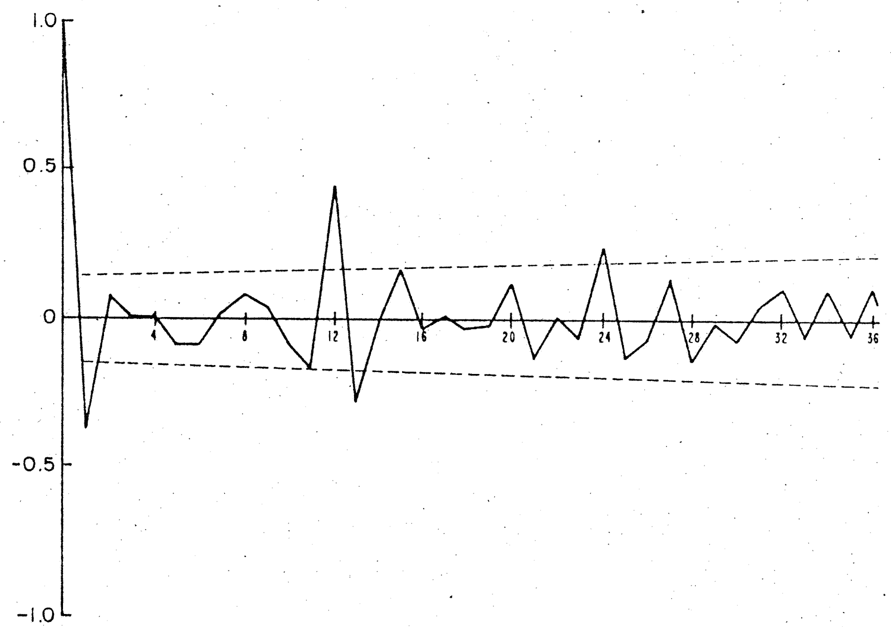


FIGURE 2

AUTOCORRELATIONS OF TRANSFORMED SERIES

Therefore, the steps we followed to visualize the seasonal component were (i) to stabilize variance and (ii) to cancel homogeneous nonstationarity through differencing. By recalling the assumptions of model (2.1), we observe that these steps are also sensible from the traditional viewpoint.

IV. SEASONAL ANALYSIS OF THE PERCENTAGE RATE OF CHANGE SERIES

Given that the seasonal effect showed up in the first difference of the logarithms, we conclude that the seasonal series must be the percentage rate of change because

$$\log (Z_{12\tau+q}) - \log (Z_{12\tau+q-1}) = \log\left(\frac{Z_{12\tau+q}}{Z_{12\tau+q-1}}\right) = \log (1+\gamma_{12\tau+q}) \simeq \gamma_{12\tau+q} \qquad (4.1)$$

where $\gamma_{12\tau+q}$ stands for the rate of change of $Z_{12\tau+q}$ relative to $Z_{12\tau+q-1}$. To corroborate this result we again applied the Adjacent Month Test to this series. Table 4 shows the average of the variation ratios , some of which are clearly far from unity and therefore indicate that seasonality is present.

Once we confirmed the seasonal effect in the percentage rate of change $(100\gamma_{12\tau+q})$, we estimated the seasonal factors by means of CM-II (formula (2.3) could not be used because some rates of change are negative). The corresponding seasonal factors I_q, satisfying $\sum_{q=1}^{12} I_q = 12$, are presented in table 5.

TABLE 4

AVERAGES OF VARIATION RATIOS OF ONE MONTH WITH RESPECT TO THE TWO ADJACENT ONES

JAN	FEB	MAR	APR	MAY	JUN	JUL	AUG	SEP	OCT	NOV	DEC
0.087	1.682	1.214	0.874	0.948	0.991	1.163	1.246	1.543	1.328	0.620	4.584

TABLE 5

SEASONAL FACTOR FOR THE SERIES OF PERCENTAGE RATES

q	1	2	3	4	5	6	7	8	9	10	11	12
I_q	0.126	0.891	0.962	0.781	0.798	0.949	1.164	1.120	0.989	0.980	0.997	2.244

As a result of dividing the percentage rates of change by the seasonal factors, we obtained the deseasonalized version of the series, which is shown together with the original percentage rates in figure 3.

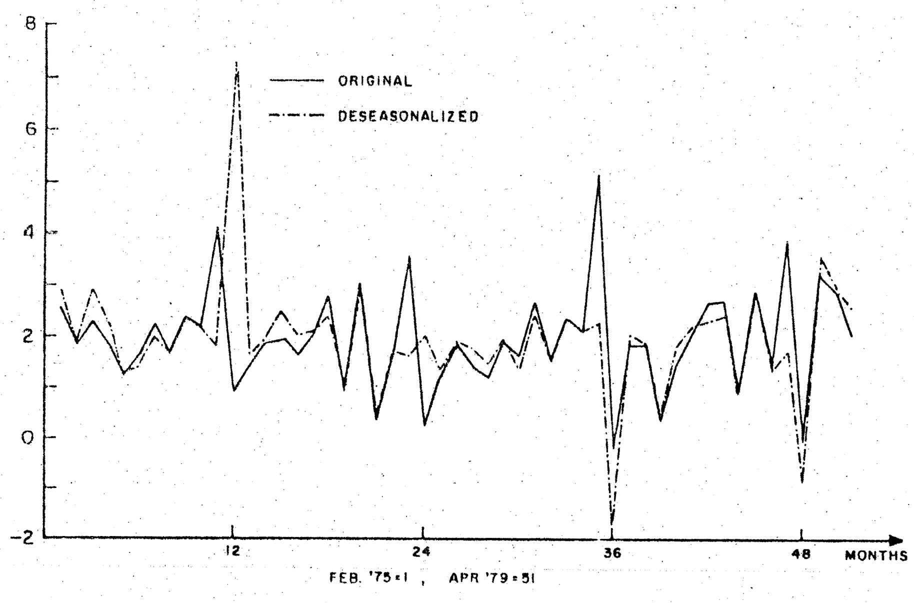


FIGURE 3

PERCENTAGE RATE OF CHANGE

In order to be able to distribute the total financing throughout the year, which is one of the main objectives of this study, we need to forecast the future values of the percentage rates of change, giving the decision maker the freedom to change either the seasonality or the trend components. To do this, we assumed that the deseasonalized percentage rates could be modelled by an ARMA process. Therefore, we have the following model for a monthly seasonal series:

$$(1-B)^d \, T(Z_{12\tau+q}) = I_q X_{12\tau+q} \tag{4.2}$$

with $T(\cdot)$ as given in (3.2), I_q the seasonal index of the stationary time series and $\{X_{12\tau+q}\}$ an ARMA process.

In the present situation, the deseasonalized percentage rate series $(X_{12\tau+q})$ turned out to be nonzero mean white noise, so that to estimate future values was a simple task. In another application of the procedure here described, we found the same result, which suggests that perhaps the steps we followed constitute a filter to extract white noise from a seasonal series.

Now, with the information we have up to this point, it is possible to develop a procedure for carrying out the desired distribution which is to be done in the next section. Only for the sake of completeness, we show here how to obtain a deseasonalized version of the original series. This step is achieved by transforming back the deseasonalized percentage rates series into the original scale by means of

$$Z_i^* = Z_1 \prod_{j=0}^{i-1} (\gamma_j^* + 1) \qquad i = 1,2,\ldots \tag{4.3}$$

where $\gamma_0^* = 0$ and the superscript * denotes deseasonalized.

It should be noticed from (4.3) that $\{Z_i^*\}$ is generated from Z_1 (which is not deseasonalized), therefore a correction must be applied. We propose the estimation of seasonal factors in the original scale by dividing the original series $\{Z_i\}$ by $\{Z_i^*\}$; then, assuming seasonal stability in time, average monthly seasonal factors should be corrected to get I_q' such that $\sum_{q=1}^{12} I_q' = 12$. At last, a deseasonalized series is obtained as $\{Z_{12\tau+q}/I_q'\}$. This procedure was carried out yielding the values in table 6 and the series in figure 4.

TABLE 6

SEASONAL FACTORS FOR ORIGINAL SERIES

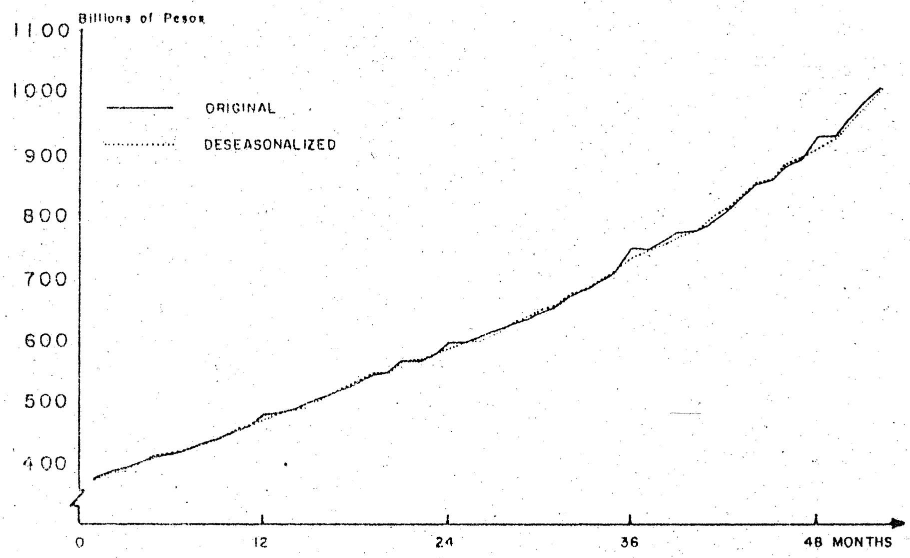
q	1	2	3	4	5	6	7	8	9	10	11	12
I_q'	1.004	1.008	1.010	1.002	0.992	0.991	0.994	0.996	0.996	0.996	0.996	1.018

As was expected, all the seasonal factors are very close to unity, and I_{12}', being the largest factor, corresponds to December.

V. PROCEDURE FOR DISTRIBUTING THE FINANCING

To distribute the financing we use model (4.2) with the added assumption that $\{X_{12\tau+q}\}$ is white noise with nonzero mean, therefore the estimated model becomes

$$(1-B)^d \, \hat{T}(Z_{12\tau+q}) = I_q \bar{X}, \quad \tau=0,1,\ldots,m-1; \; q=1,\ldots,12 \tag{5.1}$$



JAN '75 = 1, APR '79 = 52

FIGURE 4

TOTAL FINANCING

where $\bar{X}$ is the arithmetic mean of $\{X_{12\tau+q}\}$.

Let us suppose that the financing has already been granted up to month c-1 ($1 \leq c < 12$) of year m, i.e. the values $Z_{12m+1},\ldots,Z_{12m+c-1}$ are known and it is required to distribute the rest of the financing assigned to year m with the criterion of preserving the historical seasonality. Thus, it is required that we estimate $Z_{12m+c},\ldots,Z_{12m+12}$ under the restriction that

$$K = Z_{12m+12} - Z_{12m+c-1} \tag{5.2}$$

with K, the constant which determines Z_{12m+12}, given exogenously. Relation (5.2) arises because the series under investigation is of the stock type; if it were of the flow type, the restriction would be $K = \sum_{q=c}^{12} \hat{Z}_{12m+q}$ and the following would not apply.

Two exemplifying cases are now worked out, these are for d = 0 and d = 1, since other cases can be treated in similar fashion.

Case 1. d = 0 : Here we have, from (5.1)

$$\hat{Z}_{12\tau+q} = T^{-1}(I_q\bar{X}),\ \tau=0,1,\ldots,m-1;\ q=1,\ldots,12 \tag{5.3}$$

where $T^{-1}(\cdot)$ is the inverse transformation of (3.2), then

$$\hat{Z}_{12m+q} = T^{-1}\{T(Z_{12m+c-1})(I_q/I_{c-1})\}\quad q=c,c+1,\ldots,12. \tag{5.4}$$

To smooth the effect that fixing Z_{12m+12} exogenously has on the distribution of the financing, we introduce a discordance factor ε, defined as the quotient of Z_{12m+12} -determined by (5.2)- and its estimated value given by (5.4), that is,

$$\varepsilon = Z_{12m+12}/\hat{Z}_{12m+12} \qquad (5.5)$$

which we suggest to dilute among the estimates $\hat{Z}_{12m+c},\ldots,\hat{Z}_{12m+11}$ by modifying (5.4) to

$$\hat{Z}_{12m+q} = \varepsilon^{\frac{q-c+1}{12-c+1}}\ T^{-1}\ \{T(Z_{12m+c-1})(I_q/I_{c-1})\}\quad q=c,c+1,\ldots,11 \qquad (5.6)$$

Formulas (5.2), (5.5) and (5.6) are all we need to distribute the financing as required.

Case 2. $d=1$: This case is summarized in modifications to equations (5.3), (5.4) and (5.6), which become

$$\hat{Z}_{12\tau+q} = T^{-1}\ \{\hat{T}(Z_{12\tau+q-1}) + I_q\bar{X}\},\ \tau=0,1,\ldots,m-1;\ q=1,\ldots,12 \qquad (5.3')$$

$$\hat{Z}_{12m+q} = T^{-1}\ \{T(Z_{12m+c-1}) + \bar{X}\sum_{i=c}^{q} I_i\},\ q=c,c+1,\ldots,12 \qquad (5.4')$$

$$\hat{Z}_{12m+q} = \varepsilon^{\frac{q-c+1}{12-c+1}}\ T^{-1}\ \{T(Z_{12m+c-1}) + \bar{X}\sum_{i=c}^{q} I_i\},\quad q=c,c+1,\ldots,11 \qquad (5.6')$$

Next, as an illustration, let us distribute the financing throughout the year 1979, given that we know the values up to April of 1979. Let us recall relation (4.1) which indicates to work with the percentage rates of change rather than with the first difference of the logarithms. So, instead of using formulas (5.3´), (5.4´) and (5.6´) with $T(\cdot) = \log(\cdot)$, we shall use the following

$$\hat{Z}_{12\tau+q} = \hat{Z}_{12\tau+q-1}(1+I_q\bar{R}),\quad \tau=0,1,\ldots,m-1;\ q=1,\ldots,12 \qquad (5.3'')$$

$$\hat{Z}_{12m+q} = Z_{12m+c-1}\prod_{i=c}^{q}(1+I_i\bar{R}),\ q=c,c+1,\ldots,12 \qquad (5.4'')$$

$$\hat{Z}_{12m+q} = \varepsilon^{\frac{q-c+1}{12-c+1}}\ Z_{12m+c-1}\prod_{i=c}^{q}(1+I_i\bar{R}),\ q=c,c+1,\ldots,11 \qquad (5.6'')$$

where $\bar{R}$ is the arithmetic mean of the deseasonalized rate of change.

Here we have $\bar{R}=0.019584$, $m=4$, $c=5$ and $Z_{52}=1004626.18$. Then, assuming that $K=207309.62$ (in such a way that $Z_{60}=1211935.80$) we get, from (5.4"), $\hat{Z}_{60}=1201157.181$, so that

$$\varepsilon = Z_{60}/\hat{Z}_{60} = 1.008974$$

and the estimates obtained with (5.6") are shown in table 7. In the same table we present the observed values and the estimated/observed ratios to appreciate the goodness of fit.

The rightmost column in table 7 shows that the greatest discrepancy is less than 1.3%, which indicates a reasonable fit.

VI. CONCLUSIONS

Even though the present study has focussed on distributing the financing, this

TABLE 7

ESTIMATED AND OBSERVED VALUES OF TOTAL FINANCING IN 1979

MONTH	TOTAL FINANCING		
	Estimated (1)	Observed (2)	Ratios (1)/(2)
May	1 021 464.49	1 019 151.5	1.00227
June	1 041 602.45	1 044 256.8	0.99746
July	1 066 534.82	1 057 613.5	1.00843
August	1 091 141.17	1 097 565.6	0.99415
September	1 113 506.89	1 111 239.3	1.00204
October	1 136 145.75	1 137 240.1	0.99904
November	1 159 625.60	1 174 882.0	0.98702
December	1 211 935.80	1 211 935.8	—

should not pose a restriction on the general applicability of the proposed method. We think the procedure can be used for seasonally adjusting any series as long as it contains a seasonal effect (maybe hidden by spurious correlation).

On the other hand, to follow our recommendation for distributing the financing may seem complicated at first sight if all we need is to maintain the observed seasonality in the future estimates. We thought of simpler suggestions in that situation, but our aim was to give the decision maker the opportunity to "play" with the seasonality as he pleased.

A possible generalization of our procedure is to consider a dynamic (stochastic) model for the seasonal effect instead of assuming seasonal stability. Even with this latter assumption, the results we have obtained so far are very reasonable for practical purposes.

FOOTNOTES:

1. The ideas contained in this paper are the full responsibility of the authors and do not reflect the standing of Banco de México, S.A.

2. For method (2.3) we required the geometric average of the seasonal factors to be unity, while for CM-II it was the arithmetic average which we required to be unity.

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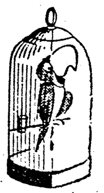


季節変動調整におけるEPA法とセンサス局法の比較

情　報　処　理　課

はじめに

労働省では，労働経済の月次指標のうち「毎勤」及び労働市場関係の主要な指標について季節変動要素を除去した指標を作成し，公表している。この季節変動要素の調整のため，現在はＥＰＡ法（Economic Planning Agency Method）が採用されている。この手法は昭和38年に経済企画庁によって開発されたものであり，労働経済指標のほか各省庁所管の経済指標の多くにこのＥＰＡ法が適用されている。しかし，石油ショックによる昭和48年から51年にかけての激しい経済変動の結果，ＥＰＡ法による季節調整結果にかなりの歪みがでてきたことが問題となり，かねてから統計審議会経済指標部会で専門的な検討が行われてきた。その検討の結果54年９月，同部会から季節調整法の適用にあたっては，現在，センサス局法Ⅱ・Ｘ—11（以下センサス局法と略称する。）以外の方法を適用している指標は，「なるべく早い時期を選んで，センサス局法に切り換えることを十分考慮する」，また「新しく季節調整法を適用する場合は，センサス局法を使用する」といった内容の報告がなされた。

当情報処理課ではこのような状況に対処するため，センサス局法を統計情報部の電算機でも処理できるようにプログラムの開発を進めてきたが，今般，その開発作業が完了した。本稿では，センサス局法に関する今後の検討に必要な材料を提供するため，ＥＰＡ法とセンサス局法による季節調整結果への影響，センサス局法における計算手順を変えることによる影響などについて試算を行い，その結果をもとに若干の問題点の指摘を行うこととする。

なお，本稿作成にあたっては，黒川恒雄氏（日本銀行統計局）の「経済時系列の分析とその季節変動の調整」（日本統計協会「統計」1979年１月～12月号）から多くの示唆を得ていることを付言したい。

1．ＥＰＡ法とセンサス局法の手法上の比較

一般に，経済時系列は長期の傾向をあらわすすう勢変動(Ｔ)，景気循環に伴って変動する循環変動(Ｃ)，１年を周期とする季節変動(Ｓ)，それに，これらだけでは説明できない不規則変動（Ｉ）の４つの要素から成り立っている。したがって，長期的なすう勢や景気動向を分析するためには，少くとも規則的に変動している季節変動の要素を除去して判断することが必要である。

季節変動を除去するいわゆる季節調整は，過去のデータをもとにＴＣＳＩ→ＴＣ→ＳＩ→Ｓという過程で季節要素の予測値を推定しておき，それをもとにＴＣＩ（季節調整済み系列）＝ＴＣＳＩ／Ｓという計算で最新時点のデータの季節要素が除去される。季節調整はこの季節要素の予測値を作る過程が中心である。しかも，一般に季節性自体，時間経過とともに変化していくものであるから，このような変化を適切に季節要素の推定の中に反映させ，しかも，その予測季節要素が翌年になって実勢とくい違いのないように推計することが要請されるわけである。このためには，時系列変動の攪乱要因となる不規則要素の変動が他の規則的な変動に吸収され，その結果，季節要素が本来の季節性を十分追跡できないという状態を避けるための処理が重要である。

現在，この季節調整の方法として，わが国ではセンサス局法，ＥＰＡ法，ＭＩＴＩ法の３つが利用されている。センサス局法は，アメリカの商務省センサス局が1954年に開発したもので，当初，センサス局法Ⅰといわれたものから順次改良されて，現在はセンサス局法Ⅱ・Ｘ—11となっている。ＥＰＡ法は経済企画庁がセンサス局法Ⅱ・Ｘ—３を基礎に，中小型コンピュータでも処理できるように簡易化して

開発したものであり，ＭＩＴＩ法は通産省が自省統計の季節調整用に独自に開発したものである。このうち，センサス局法は日本銀行で全面的に採用されているほか，政府統計でも労働力調査，家計調査，ＳＮＡ統計（国民経済計算）に適用されている。生産，出荷，在庫等の鉱工業指数を除くその他の政府統計にはＥＰＡ法が適用されている。

ところで，これら３つの季節調整の方法は基本的には類似したものであるが，時系列における特異データの処理や，原系列の変化の特性に合わせて移動平均の項数をはじめ各種処理オプションがシステマティックに選択できる点などからセンサス局法が優れているといわれている。では，具体的に，現在使われているＥＰＡ法に比べ，センサス局法の計算手順がどのように違うかをみてみよう。

ＥＰＡ法の計算は次のようにして行われる。

①原系列（ＴＣＳＩ）を中心化12ヵ月移動平均（12ヵ月移動平均の２ヵ月移動平均）し，すう勢・循環要素の近似値（TC_1）を求め，これで原系列を除して予備的季節・不規則要素（SI_1）を算出する。

②予備的季節・不規則要素（SI_1）から２項移動平均等を実施して不規則要素を除去し，予備的季節要素（S_1）を算出し，予備的季節調整済み系列（TCI_1）を得る。

③②で得た予備的季節調整済み系列（TCI_1）に５項反復移動平均を行って最終すう勢・循環要素（ＴＣ）を算出し，これで原系列を除すことによって最終季節・不規則要素（ＳＩ）を算出する。

④この最終季節・不規則要素（ＳＩ）を月別に反復移動平均することによって，季節要素（Ｓ）の最終値が得られる。

これに対し，センサス局法では特異項修正過程が精緻化され，次の４つのパートに分けて計算が行われる。

パートＡ：事前調整計算パート
パートＢ：暫定特異項修正パート
パートＣ：最終特異項修正パート
パートＤ：最終季節要素抽出パート

パートＡでは事前の情報をもとに，月の長さの調整，月間の曜日構成の違いによる調整等を行う。パートＢの計算過程のうち，ＴＣ，Ｓ，Ｉの抽出はＥＰＡ法と類似の計算過程で行われるが，この中にＥＰＡ法にはない特異項修正，回帰による曜日調整等（後述）の過程が組み込まれている。パートＣ，パートＤでもパートＢとほぼ同様の計算が繰り返される。ＥＰＡ法の計算過程は，センサス局法のパートＤの計算部分にほぼ相当する。

センサス局法は，このように事前調整，特異項処理，曜日調整の機能をもち，特異項修正に重点が置かれている一方，すう勢・循環要素，季節要素の抽出過程で使われる移動平均のパターンも拡張されるなど多くの機能を持っていること，しかも，これらの機能をどう使うかは利用者の事前の情報や経験にもまかせられるよう，これらの機能を選択可能なオプション化していることが特徴といえよう（付表参照）。

2．両手法による季節調整結果の比較

センサス局法はＥＰＡ法に比べ多くの計算手順を経ることになっており，その結果，時系列変動に含まれる特異変動の要素を適切に処理しながら安定的な季節変動要素を抽出することが期待されているわけであるが，労働経済の系列について，それがどの程度確保されるかを確認してみる必要がある。またセンサス局法で計算した季節調整結果が，ＥＰＡ法によって現在得られている結果とどの程度の違いになっているかを知ることは，今後，季節調整方式としてセンサス局法を採用するとした場合，過去の時系列変動についての判断の斉合性を考える上で重要な問題である。

1）季節調整済み系列の比較

まず，労働経済の主要系列について，ＥＰＡ法とセンサス局法（標準計算ルーチンによる。）によって計算した季節調整結果を比較してみよう。

第１表は所定外労働時間指数の季節調整済み系列と，すう勢・循環要素の対前月増減率を昭和50年から各月別にみたものである。この表から次のような特徴を指摘することができる。

①季節調整済み系列（ＴＣＩ），すう勢・循環要素（ＴＣ）とも変化方向は両手法で殆んど差がない。

②季節調整済み系列の変動率はセンサス局法の方がＥＰＡ法より大きいが，すう勢・循環要素の変動率の大きさは両手法ともほぼ同程度である。そのため，手法が異なることになる変動率の差は季節調整済み系列で大きく，すう勢・循環要素で相対的に小さくなっている。

③季節調整済み系列の変動率の両手法による差の大きい月は，特定月に集中してあらわれている。しかし，すう勢・循環要素の場合には，そのような特定月に集中する傾向はみられない。つまり，季節調整済み系列については変動率の大きさについての差

第1表　所定外労働時間指数の季節調整結果　(%)

月	対前月増減率								同左の手法間での差（絶対値表示）			
	昭和50年		51年		52年		53年		50年	51年	52年	53年
	EPA法	センサス局法	EPA法	センサス局法	EPA法	センサス局法	EPA法	センサス局法				
季節調整済み系列（TCI）												
1月	−1.9	−3.5	2.1	0.7	1.3	0.5	0.3	−0.7	1.6	1.4	0.8	1.0
2	−8.3	−7.9	5.7	6.4	0.7	1.3	0.5	0.9	0.4	0.7	0.6	0.4
3	−2.7	−2.4	3.4	3.7	0.0	0.1	1.4	1.5	0.3	0.3	0.1	0.1
4	1.6	2.2	0.5	1.3	−0.2	1.1	1.5	3.2	0.6	0.8	1.3	1.7
5	0.0	−0.8	3.7	3.1	−0.7	−1.4	−1.1	−1.9	0.8	0.6	0.7	0.8
6	4.5	5.3	0.8	1.1	−0.5	−0.6	−0.6	−1.3	0.8	0.3	0.1	0.7
7	4.8	5.2	1.4	1.5	−1.7	−1.8	1.6	1.1	0.4	0.1	0.1	0.5
8	3.7	3.1	0.9	0.5	−0.6	−0.8	0.5	0.5	0.6	0.4	0.2	0.0
9	2.2	2.3	0.5	0.6	−0.1	−0.1	1.2	1.3	0.1	0.1	0.0	0.1
10	1.5	2.1	1.2	1.3	1.1	0.7	−0.0	−0.7	0.6	0.1	0.4	0.7
11	1.9	1.7	0.8	1.2	0.3	0.9	1.7	2.8	0.2	0.4	0.6	1.1
12	3.5	3.1	−0.4	−0.8	1.9	2.1	0.3	1.4	0.4	0.4	0.2	0.6
すう勢循環要素（TC）												
1月	−4.5	−4.73	3.1	3.36	0.6	0.45	1.0	0.62	0.2	0.3	0.1	0.2
2	−4.1	−4.12	3.3	3.42	0.4	0.58	0.8	0.79	0.0	0.1	0.2	0.0
3	−2.5	−2.81	3.1	3.08	0.1	0.58	0.7	0.75	0.3	0.0	0.5	0.1
4	−0.4	−0.85	2.5	2.56	−0.3	0.15	0.6	0.59	0.5	0.1	0.5	0.0
5	1.3	1.44	2.2	2.05	−0.6	−0.57	0.3	0.32	0.1	0.1	0.0	0.0
6	2.7	3.29	1.7	1.55	−0.7	−1.12	0.3	0.16	0.6	0.1	0.4	0.1
7	3.5	3.94	1.2	1.12	−0.8	−1.24	0.6	0.29	0.4	0.1	0.4	0.3
8	3.4	3.55	1.0	0.94	−0.5	−0.92	0.7	0.57	0.2	0.1	0.4	0.1
9	2.9	2.71	0.9	0.85	−0.1	−0.11	0.7	0.92	0.2	0.0	0.0	0.2
10	2.7	2.15	0.8	0.75	0.3	0.63	0.7	1.09	0.5	0.0	0.3	0.4
11	2.7	2.28	0.7	0.56	0.6	0.99	0.8	1.14	0.4	0.1	0.4	0.4
12	2.8	2.92	0.6	0.41	1.0	0.97	0.6	1.18	0.1	0.2	0.0	0.6

（注）季節調整の計算期間は昭和30年～53年の24年間である。

第2表　時系列変動要素別対前月変化程度（昭和50～53年）

系列	EPA法						センサス局法					
	TCI	TC	S	I	S/TC	I/TC	TCI	TC	S	I	S/TC	I/TC
常用雇用指数（製造業）	0.20	0.19	0.33	0.06	1.74	0.29	0.22	0.20	0.38	0.12	1.94	0.58
総実労働時間指数（〃）	0.75	0.20	6.37	0.69	31.85	3.46	0.90	0.14	6.33	0.88	44.25	6.18
所定外労働時間指数（〃）	1.62	1.42	6.53	0.89	4.61	0.58	1.89	1.51	6.34	0.98	4.20	0.65
現金給与総額指数（〃）	0.85	0.68	33.46	0.59	49.49	0.87	1.39	0.77	33.12	1.25	43.12	1.63
出勤日数（〃）	0.80	0.15	6.32	0.76	43.08	5.18	1.00	0.08	6.18	1.00	77.87	12.61
有効求人数	1.77	1.39	10.19	0.94	7.36	0.68	1.94	1.45	10.00	1.03	6.89	0.71
有効求職件数	1.21	0.84	5.07	0.88	6.01	1.05	1.69	0.84	5.54	1.42	6.63	1.69

（注1）対前月変化程度は昭和50～53年間における対前月増減率の絶対値の平均である。
（注2）季節調整の計算期間は昭和30年～53年の24年間である。ただし，有効求人数，有効求職件数については昭和38年～53年の16年間である。以下各表とも同じ。

は両手法間で無視できない大きさになっているが，すう勢・循環要素については，その差はかなり縮小していることがわかる。

これまでは，季節調整法が異なることによる影響を所定外労働時間指数を例にみてきたが，その他の系列についても整理してみよう。第2表は主要系列の変動構成要素別の月別変化程度を指標化したものである。この表中，季節調整済み系列（TCI），すう勢・循環要素（TC），季節要素（S），不規則要素（I）の数値は，季節調整の結果得られる各変動構成要素の昭和50年から53年迄の対前月増減率の絶対値の平均値として求めてある。この表から所定外労働時間を月別にみたときに指摘した特徴点のいくつかを読みとることができる。

その1つは，各系列とも季節調整済み系列と不規則要素についてはセンサス局法での変動が大きいということである。そのことからセンサス局法での季節調整済み系列の変動が大きいのは，その場合の不規則要素の変動が大きいことと関連していることが明らかである。つまり，センサス局法では特異項の処理など，不規則要素が季節性にもぐり込むことを防ぐ措置がとられているため，それだけEPA法に比べ不規則要素の変動が大きくなる傾向があるといえよう。

第２は，両手法によるすう勢・循環要素の変動率の差は各系列ともあまり大きくないことである。センサス局法への切り換えに伴い，過去に分析してきた時系列の変化方向に対する判断を変更しなければならないという危険を避けるためには，季節調整済み系列のほかに，すう勢・循環要素も同時に作成して時系列変動の判断材料に加えることが必要であろう。

2) 抽出された季節要素，不規則要素の妥当性

季節調整の結果，季節調整済み系列についてはその計算方法の違いにより多少の差があらわれているが，時系列分析上重要なすう勢・循環要素に関してはあまり差のないことがわかった。しかし，その過程で抜き出された季節要素と不規則要素の関係に多少の差があり，そこに両手法の性格の違いがあると考えられる。そこで，季節要素と不規則要素の抽出度の違いをもとに，季節調整法としての両手法の優劣を検討してみよう。その判断基準として次の２つを考える。

１つは，季節要素の予測値の安定性ということである。季節調整計算の場合，当年迄のデータをもとに１年先の季節要素を予測し，その予測季節要素を使って翌年の季節調整が行われるが，この季節要素の予測値が，翌年，最新データを加えて改めて計算したときもあまり変化しないことが望ましいわけである。かって，石油ショック後の51年時点で，ＥＰＡ法によって計算したときの51年の季節要素の予測値が，その後判明した実績値に比べかなりの差になったときがあった。そこで，この51年時点についてセンサス局法によって同様の試算をし，ＥＰＡ法の場合と比較してみることにする。第３表は季節要素の51年１月～12月についての予測値と実績データにもとずく値の差の絶対値を積み上げたものであるが一部の系列を除きセンサス局法での乖離が小さくなっていることがわかる。つまり，センサス局法が予測季節要素の安定性という面でＥＰＡ法より優れていることを示すものといえよう。

判断基準の第２は，不規則要素の変化がランダム系列になっているかどうかということである。不規則要素は原系列から季節要素，すう勢・循環要素の規則的に変動する要素を除去した残差であるから，規則的な要素が除去されていれば，つまり，そのことは季節調整が良好に行われていればということにもなるが，そうであれば，不規則要素の変動は全くランダムな動きを示すはずである。不規則要素の変動がランダムな変化を示すかどうかの判定は，不規則要素の変化系列に「連(Run)」の考え方を適用して，この連の平均月数をもとに行うことができる。

第３表　季節要素，不規則要素の特性値

系　　列	予測季節要素の安定性（51年季節要素の予測値と実績値の乖離度）		不規則要素変動のランダム性（不規則要素変動の連(Run)の平均月数(ＡＤＲ)(45年～53年)）	
	ＥＰＡ法	センサス局法	ＥＰＡ法	センサス局法
常用雇用指数(製造業)	0.95	1.36	1.89	1.77
総実労働時間(〃)	4.46	3.75	1.50	1.50
所定外労働時間(〃)	14.79	6.98	1.69	1.50
現金給与総額(〃)	19.94	10.76	1.57	1.40
出勤日数(〃)	4.38	5.04	1.42	1.50
有効求人	27.66	16.39	1.96	1.66
有効求職	18.39	9.23	1.71	1.59

（注） 51年季節指数の予測値（$\hat{S}_{51,i}$）と実績値（$S_{51,i}$）の

$$\text{乖離度}=\sum_{i=1}^{12}\left|\hat{S}_{51,i}-S_{51,i}\right|$$

ここで連とは，ある系列について変化方向がプラスの場合を(＋)，マイナスの場合を（－）として，系列の変化の順序に従って，例えば，＋＋－＋－－－＋＋…………と符号を並べたとき，同じ符号の続く間を連といい，この同じ符号の続く数が連の長さである。上の例では連の長さは２，１，１，３，２，……となる。したがって，連の平均月数とは同一方向（変化の方向が＋か－）に平均何カ月連続して変化したかを示すものである。そして，ランダム系列の変化方向による連の長さの平均（ＡＤＲ＝Average Duration of Run）は1.5で，95％の確率で1.36～1.75の範囲にあるといわれている（"X―11. Information for The User" p.73, U.S. Department of Commerce, 1969）。第３表の右欄は不規則要素変動の連の平均月数を計算したものであるが，この表から，計算した系列についてはその多くが不規則要素のランダム性が確保されていること，しかもその場合でもセンサス局法で計算した場合の方がその数が多くなることを読みとることができる。

（注） ランダム系列の連の長さの平均が1.5になることは次のようにして示される。

時期を $t=\cdots\cdots-1,\ 0,\ 1,\ 2,\ \cdots\cdots$ ランダム系列を $x(t)$，その前期差を $y(t)=x(t)-x(t-1)$ とする。連は $y(t)$ の同一符号の連続として表わされる。

ランダム系列では任意のｎ個の時点 $\left\{t_i\right\}_{i=1}^{n}$ の系列の値のうち，ある時点における系列の値 $x(t_i)$ の順位が m（$1\leqq m\leqq n$）である確率は一定で，$1/n$ である。

さて，$t=1$ より連が始まる（すなわち $y(0)$ と $y(1)$

の符号が異る，$y(0)\cdot y(1)<0$）としよう。$t=n-1$までがこの連に属する〔条件P〕とき，$x(-1)>x(0)<x(1)<\cdots<x(n-1)$であるが，さらに$t=n$がこの連に属する〔条件Q〕ためには，(A)，$x(-1)<x(n-1)$かつ$x(n)$はSの中で最大であるか，または，(B)，$x(-1)>x(n-1)$かつ$x(n)$はSの中で2番目以上である，ことが条件である（ただしSは時点$-1, 0, 1, \cdots, n$におけるxの値の集合である）。したがって条件Pのもとで条件Qの起きる確率 $P_{n|n-1}$は，

$$
\begin{aligned}
P_{n|n-1} &= \mathrm{prob}\{(A)\}+\mathrm{prob}\{(B)\} \\
&= \frac{n-1}{n}\cdot\frac{1}{n+2}+\frac{1}{n}\cdot\frac{2}{n+2} \\
&= \frac{n+1}{n(n+2)} \qquad (n\geqq 2)
\end{aligned}
$$

したがって$t=1$より連が始まるとき，$t=n$がこの連に属する確率 P_nは，

$$
P_n=\begin{cases} \prod_{i=2}^{n} P_{i|i-1}=\dfrac{3(n+1)}{(n+2)!} & (n\leqq 2) \\ 1 & (n=1) \end{cases}
$$

となり，$t=1$より始まる連の長さの期待値$E(\ell)$は，$t=1, 2, 3, \cdots\cdots, n, \cdots\cdots$についてこの連に対する寄与 P_nを合計することにより得られる。すなわちランダム系列の変化に関する連の長さの期待値は，

$$
E(\ell)=\sum_{n=1}^{\infty} P_n=\frac{3}{2}
$$

3. センサス局法における選択的計算手順（オプション）とそれによる効果比較

前節のセンサス局法による計算は「標準オプション」といわれている機能を使った(付表参照)。センサス局法は，時系列変動の特性に合わせて，より適切な季節調整を行うために，いくつかの計算手順を選択できる点に特徴があるが，労働経済の系列の場合，どのようなオプションを選択するのが良いかを次に検討してみよう。

第1節でも触れたように，センサス局法で利用できるオプションは多いが，その中で特徴的とされている①特異項の処理，②移動平均項数の選択及び③曜日変動の調整の3つの計算機能について，その機能，効果，問題点などについて検討してみる。

1) 特異項の処理

異常な社会現象や自然現象などによって経済時系列が著るしく乱され，例外的な不規則変動があらわれるような場合には，それが季節変動要素の算出に影響を及ぼし，季節要素をゆがめてしまう可能性がある。そこで，季節要素，すう勢・循環要素を的確に推計するためには，これらの例外的な変動を特異項としてとり出し，それらをあらかじめ除去するかもしくは修正しておくことが必要となる。

センサス局法ではこの特異項を処理するため，次のような方法をとることになっている。

①不規則要素をもとに5カ年間の移動標準偏差（σ）を算出し，それをもとに特異項に対する管理限界を設定する。

②1年経過してσが変化し，特異項に指定されるデータが変更になったとしても，それによる影響が大きくならないようにするため，一定範囲にはいる特異項は0～1の間のウェートによって修正される。例えば管理限界の下限が1.5σ，上限が2.5σの場合，2.5σを超えるデータは純然たる特異項としてゼロのウェートを，また1.5σ以内にはいるデータは1.0のウェートを与えてそのまま生かし，1.5σ～2.5σの間にあるデータには1.0（1.5σに対応）からゼロ（2.5σに対応）にいたるまでの直線的に低下するウェートで特異項を修正する。

③このような手順で特異項修正した不規則要素をもとの不規則要素と置換えることにより，原系列の値を修正し，この修正原系列をもとに季節要素，すう勢・循環要素が再推計されるので，季節要素，すう勢・循環要素に対する特異項の影響は補正される。

この上限・下限の値は原系列の特性に応じ0.1σ～9.9σの中から任意に選択することができる。一般的には経験的に決められた"下限1.5σ～上限2.5σ"が標準のオプションとして使われている。しかし，特定の系列によってはこの標準オプションでは必ずしも十分な季節調整が確保されない場合もでてくる。これが問題になるケースを考えてみよう。

まず第1に，特異項が毎年，特定の月に集中的にあらわれているような系列の場合である。この場合もし，特異項管理限界が低いと，特定月の変動が季節要素を含んだまま傾向的に特異項として除外されてしまうため，特異項補正後の推定季節要素が季節性の変化を追跡しえず，一方，不規則要素の中に季節性が含まれて不規則要素のランダム性も損われる可能性がでることにもなる。このような場合には特異項管理限界を引上げる必要があろう。ただ，特異項が特定の月に集中する場合であっても，もともと特定の月に不規則変動が大きいという性質をもつ時系列の場合には，その変動はランダムな形に分布することになるので，季節要素の推定には影響を及ぼさないはずであり，このような性質をもつ時系列の場合は標準オプションの処理で十分である。

第４表　不規則要素の時系列とその中での特異項の出現の可能性

1)　総実労働時間指数（F）

年	1月	2	3	4	5	6	7	8	9	10	11	12
45年	**98.0**	100.3	99.7	99.8	100.7	100.4	99.8	100.2	98.8	100.4	100.1	99.6
46	100.9	100.1	100.3	100.4	98.9	98.9	100.7	100.7	99.6	100.8	99.9	99.0
47	100.4	98.9	101.1	100.8	99.7	99.6	100.1	99.5	100.7	99.2	99.2	100.5
48	100.2	100.7	100.0	99.9	99.7	100.9	99.5	99.2	99.9	100.3	100.4	100.4
49	**96.9**	100.6	99.6	99.2	100.0	100.5	100.3	99.4	100.7	99.8	**102.5**	101.0
50	**98.0**	99.9	**98.5**	98.8	100.4	100.6	100.1	100.4	100.2	100.2	100.2	99.2
51	**98.2**	100.4	**101.9**	100.7	99.8	99.1	100.6	100.5	99.1	100.2	100.5	99.4
52	**101.7**	99.1	100.4	101.0	100.1	99.7	99.8	100.2	99.5	99.7	99.4	100.5
53	101.1	99.5	100.4	100.0	100.0	100.4	99.5	99.1	100.7	100.0	99.9	100.7

2)　現金給与総額指数（F）

年	1月	2	3	4	5	6	7	8	9	10	11	12
45年	98.2	100.4	99.5	100.0	99.9	99.7	100.8	**103.6**	99.2	99.7	99.6	101.6
46	99.5	100.2	100.0	99.6	**96.4**	100.5	99.8	**103.2**	100.1	100.0	99.8	98.7
47	100.9	100.2	100.0	100.2	99.4	101.0	98.7	100.4	100.5	99.9	99.9	99.8
48	99.9	100.7	99.7	98.9	101.1	102.5	99.3	99.5	100.4	99.9	100.2	**109.2**
49	**95.5**	97.5	**95.4**	**96.9**	**104.3**	102.0	101.5	99.8	99.0	99.1	100.6	**105.8**
50	99.8	99.7	100.6	100.1	99.2	**96.5**	**105.7**	100.7	100.2	100.2	99.0	100.2
51	99.8	100.7	99.9	100.2	99.5	**95.3**	100.9	99.9	99.6	100.4	99.6	100.4
52	99.8	100.0	99.9	100.3	100.5	**98.2**	100.3	100.4	100.0	100.1	99.9	99.6
53	100.2	100.1	100.1	100.5	99.9	99.3	100.3	**99.5**	100.6	99.9	100.7	**99.1**

3)　有効求職件数

年	1月	2	3	4	5	6	7	8	9	10	11	12
45月	100.0	99.8	100.0	**101.2**	99.6	99.9	100.1	99.6	100.4	100.3	99.7	99.4
46	100.2	99.7	100.7	100.7	99.2	100.0	100.1	99.7	99.7	99.7	101.1	100.9
47	99.5	99.5	99.5	99.2	101.0	100.7	99.3	100.6	100.5	99.8	98.9	**97.5**
48	100.1	100.6	100.5	100.1	99.9	99.4	99.1	101.4	100.3	99.9	**95.8**	**96.8**
49	100.4	100.9	99.6	**101.8**	100.1	**97.8**	100.0	99.8	99.5	100.1	99.5	101.8
50	99.3	100.6	**101.8**	99.1	98.7	99.4	101.4	100.4	100.9	100.0	99.2	101.0
51	99.1	99.3	98.8	101.4	100.4	101.0	99.8	99.0	99.3	98.9	101.8	100.3
52	101.2	99.4	**96.9**	98.5	100.6	100.8	99.4	99.8	99.9	101.0	**103.4**	99.2
53	100.7	99.3	**95.8**	**96.3**	100.8	100.6	99.7	100.1	99.5	100.6	[illegible]	**96.4**

（注）ゴシック体の数値は，各系列ごとに各年の５年間移動標準偏差を計算し，それをもとに，±2σの範囲外に出たものであり，不規則要素の中でもとくに例外的なものとみなされるものである。

第２のケースとしては，特定の月でなく，特定の年にかなりの月数にわたって特異項とみられる不規則な変動があらわれる場合であるが，この場合にはこの年の各月の変動があくまでも例外であることから，特異項管理限界を標準オプションより低くすることも考えられる。

第３のケースは，長期にわたって著しく平滑な時系列の場合であり，この場合には，特異項管理限界を高くしてわずかの変動でも特異項とされる危険性を避けることも考えなければならないだろう。

では，労働経済の主要系列はどのような変動をしているのであろうか。いま，センサス局法で計算した不規則要素をもとに特異項と認定される可能性のありそうなデータがどの月に出現するかを試算してみる（第４表，一部の系列のみを表示）。それによると，例えば総実労働時間，出勤日数については１月に，有効求職については３月，４月，11月，12月に特異項が出やすくなっているが，その出方がランダムであり，特異項管理限界に特別の配慮を払う必要はないとみられる。他の系列の場合には特定の月に集中するパターンは観察されない。ただし，現金給与総額の場合，賃上げ率が異常に高かった昭和49年前後で不規則変動の大きい月が続いており，先ほどの特定の年にかなりの月数にわたって特異項とみられる不規則変動があらわれるケースに該当するとみられ，特異項管理限界を低めるといったことを考慮する必要があるかも知れない。

具体的に特異項管理限界を変更したときの効果をみるため，季節調整をした結果から不規則要素のランダム性，予測季節要素の安定性の２点について比較してみよう。

不規則要素のランダム性に関しては，殆んどの系列の不規則要素変動の連の平均月数が1.36～1.75の間にあり，おおむねランダム系列とみることができよう（第５表）。ただし，常用雇用指数の場合，標準ケース及びそれを若干拡張した管理限界の範囲で

第5表　特異項管理限界別不規則要素変動の連の平均月数（ADR）（昭和45～53年）

系　　列	1σ～2σ	1.5σ～2.5σ	2σ～3σ	1σ～3σ	9.8σ～9.9σ
常用雇用指数（F）	1.77	1.77	1.77	1.77	1.66
総実労働時間（F）	1.59	1.50	1.54	1.50	1.54
所定外労働時間（F）	1.59	1.50	1.59	1.59	1.64
現金給与総額（F）	1.44	1.40	1.83	1.40	1.61
出勤日数（F）	1.50	1.50	1.46	1.50	1.46
有効求人	1.64	1.66	1.61	1.64	1.69
有効求職	1.54	1.59	1.57	1.52	1.57

（注）1σ～2σは，下限が1σ，上限が2σのことである。

は有意水準がやや低く，全てのデータをそのまま計算基礎に加えた場合（管理限界の下限9.8ァ～上限9.9ァ）に有意になっており，先にあげた類型によれば，管理限界の変更を配慮した方がよい第3のケースの具体例といえよう。また，現金給与総額に関しては管理限界を2σ～3σとしたときに不規則要素のランダム性が有意ではなくなっているが，先にもみたように，現金給与総額が49年を中心に各月に特異的な不規則要素があらわれていることから，管理限界変更の第2のケースと考えた処理が必要かも知れない。

予測季節要素の安定性に関しては，管理限界の差によって傾向的な差はみられず（第6表），不規則要素のランダム性と併せ考慮すると，常用雇用指数と現金給与総額についてはさらに検討を要するものの，他の系列について標準のケース（1.5σ～2.5σ）を採用しても問題はないといえよう。

2）移動平均項数の選択

季節調整の計算は，原系列（TCSI）からモデ

第6表　特異項管理限界別予測季節要素の安定性（昭和53年の季節要素の予測値と実績値の乖離度）

系　　列	1σ～2σ	1.5σ～2.5σ	2σ～3σ	1σ～3σ	9.8σ～9.9σ
常用雇用指数（F）	1.01	0.97	0.82	0.98	0.80
総実労働時間（F）	2.70	4.88	3.90	2.96	3.41
所定外労働時間（F）	10.00	8.08	8.38	8.55	8.92
現金給与総額（F）	5.88	9.50	8.01	7.81	8.96
出勤日数（F）	2.30	4.58	3.21	2.31	3.08
有効求人	9.25	6.73	6.28	8.97	6.60
有効求職	10.31	10.44	11.97	11.12	9.99

（注）安定性を示す数値の算出方法は第3表の（注）と同じ。

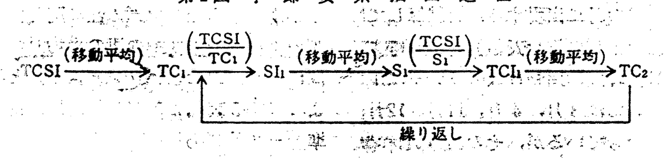
ル的に示した第1図のような過程を経て暫定季節要素（S_1）を作り，それをくり返して，最終的に，すう勢・循環要素（TC），季節要素（S），不規則要素（I）を分離，確定させるという方法をとっている。この場合，移動平均はすう勢・循環要素（TC）を推定する場合と季節要素（S）を推定する場合に使われる。

(ｲ)季節要素推計のための移動平均

まず季節要素を推計する過程での移動平均からみてみる。季節要素の計算は月単位に季節・不規則要素（SI）を移動平均して行われる。この移動平均は暫定季節要素から最終の季節要素の算出まで3回同様な計算過程を経て行われるが，その都度，前段後段2回づつの移動平均が行われる。SI要素から季節要素の変化を取り出す場合，その変化を忠実にとらえようとするなら移動平均項数は短い方がよいがその場合には季節要素の中に不規則要素が入り込む可能性があり，その結果，季節要素がかえって不規則要素でゆがめられるという結果にもなりかねない。そこで，不規則変動をなるべく取り入れないようにすることを重視するなら移動平均の項数は長い方がよいことになり，その兼ね合いが問題になる。標準オプションは，この移動平均の項数を各月とも前段では3項の移動平均（月別に当年を中心に前後3年間の平均）を3回繰り返し（3×3項移動平均），後段では3項の移動平均を5回繰り返す（3×5項移動平均）ことになっている。季節要素の推計に使用する移動平均項数の大きさは，それによって季節要素の抽出に影響を与える度合が大きいことから，標準オプションのほかに，いくつかのパターンが時系列変動の特性に合わせて選択できるようになっている。

したがって，移動平均の項数としていかなるパターンをとるかが問題になるが，その1つの目安として，月毎の不規則変動要素の振幅の程度をあらわす季節性比率MSR（Moving Seasonality Ratio）を使う場合が多い。ここで，季節性比率は，不規則要素の対前年変化率の絶対値平均を季節要素の対前

第1図　季節要素抽出過程



第7表　月別移動季節性比率(ＭＳＲ)

月	常用雇用指数（F）	総実労働時間（F）	所定外労働時間（F）	現金給与総額（F）	出勤日数（F）	有効求人	有効求職
1　月	1.20	4.58	1.26	1.01	4.76	1.42	4.88
2	0.82	3.78	3.01	0.90	4.98	3.85	1.44
3	3.92	4.98	4.17	0.98	5.46	2.36	3.03
4	0.68	5.04	3.71	0.99	5.77	1.67	2.31
5	1.07	3.17	1.68	1.85	3.82	1.08	5.14
6	1.30	4.69	5.79	1.72	4.52	1.68	1.71
7	1.83	3.56	2.27	1.04	4.25	3.23	2.32
8	2.28	2.63	2.45	3.29	2.62	3.88	4.28
9	1.73	6.08	2.56	1.38	6.91	3.66	2.14
10	1.52	6.24	1.90	0.97	6.63	2.48	1.89
11	1.51	5.60	5.16	1.39	4.44	1.87	1.42
12	0.94	[illegible].93	1.83	0.96	6.63	1.19	3.63

（注1）　$\text{ＭＳＲ} = \dfrac{\text{不規則要素の対前年変化率の絶対値平均（30年(注2)～53年）}}{\text{季節要素の対前年変化率の絶対値平均（30年～53年）}}$

（注2）　有効求人，有効求職の計算期間は38年～53年

年変化率の絶対値平均で除した比率（$\overline{I}/\overline{S}$）である。

労働経済の主要系列のＭＳＲは第７表のとおりであり，総実労働時間，所定外労働時間，出勤日数等ではＭＳＲが大きくなる月が多く，一方，常用雇用指数，現金給与総額等では比較的ＭＳＲの小さい月が多いというように系列によってかなり異っている。したがって，それに合わせて月別の移動平均項数を決めるのが望ましいといえそうである。

ＭＳＲと移動平均項数の対応関係は第８表のとおりであるが，この試算では経済企画庁で設定している対応関係をもとに月別の移動平均の項数を指定してみよう。

第８表　移動季節性比率（ＭＳＲ）と移動平均項数の対応関係

ＭＳＲ	センサス局法Ⅱ・Ｘ—10での対応	同左をセンサス局法Ⅱ・Ｘ—11へ読み替えた場合	経企庁で設定している対応
0　～1.49	移動平均せず	1×3項	1×3項
1.50～2.49	3項	1×3	3×3
2.50～4.49	5項	3×3，3×5	3×5
4.50～6.49	9項	3×5，3×9	3×9
6.50～8.49	15項	3×9	3×9
8.50～	全項平均	全項平均	3×9

まず，所定外労働時間及び現金給与総額について月別の移動平均の項数を各月一律に変えて季節調整をしたときの不規則要素変動のＡＤＲ（連の平均月数）が第９表に示してある。これによると，全項平均を指定した場合には不規則要素のランダム性が失われる危険性があるが，他のケースでは，その不規則要素は一応ランダム系列とみなすことができる。次に，季節性比率（ＭＳＲ）に対応して月別の移動平均項数を指定した場合と標準オプション（前段３×３項，後段３×５項）を指定した場合の季節調整結果を比較してみると，不規則要素のランダム性については，有効求人数を除いて各系列ともいずれも確保されているとみられるが，季節要素の安定性については，月ごとにＭＳＲに対応して項数を指定した方の安定性が高くなるケースがみられる（第９表，第10表）。したがって，有効求人数については更に検討が必要であるが，それ以外の系列については，ＭＳＲに対応して項数を指定する方が有効といえそうである。

第９表　月別移動平均項数別不規則要素変動の連の平均月数（ＡＤＲ）（昭和45年～53年）

系　列	各月一律に移動平均の項数を指定する場合						各月別に任意に項数を指定する場合
	標準タイプ（前半3×3 後半3×5）	1×３項	3×３項	3×５項	3×９項	全項平均	
常用雇用指数（F）	1.77	—	—	—	—	—	1.69
総実労働時間（F）	1.50	—	—	—	—	—	1.50
所定外労働時間（F）	1.50	1.57	1.59	1.59	1.74	1.83	1.59
現金給与総額（F）	1.40	1.48	1.57	1.40	1.48	2.08	1.57
出勤日数（F）	1.50	—	—	—	—	—	1.46
有効求人	1.66	—	—	—	—	—	1.80
有効求職	1.59	—	—	—	—	—	1.54

（注）各月別に任意に項数を指定する場合の移動平均項数は，第７表のＭＳＲと第８表の対応関係のうち経企庁で採用されているものを使って決めた。

第10表　月別移動平均項数別季節要素の安定性（昭和53年季節要素の予測値と実績値の乖離度）

系　　列	標準タイプ（前半3×3 後半3×5）	各月別に任意に項数を指定
常用雇用指数（F）	0.97	1.04
総実労働時間（F）	4.88	1.93
所定外労働時間（F）	8.08	8.19
現金給与総額（F）	9.50	5.16
出勤日数（F）	4.58	3.00
有効求人	6.73	7.38
有効求職	10.44	7.97

（注）　安定性を示す数値の算出方法は第３表の（注）と同じ

(ロ)すう勢・循環要素推計のための移動平均（ヘンダーソン移動平均）

すう勢・循環要素を推計するための移動平均はヘンダーソン移動平均といわれる手法によって行われる。これは，移動平均の長さとして９カ月，13カ月23カ月のいずれかの期間を選び，それに，時系列のすう勢・循環要素が３次曲線（山と谷が同時に表現できる曲線）からなっているとの前提のもとにウェートを付けて行う移動平均である。

この移動平均の標準的な計算手順は，その移動平均の月数を暫定的なすう勢・循環要素と，不規則要素の対前月変化率の絶対値平均とを対比（$\overline{\text{I}}/\overline{\text{TC}}$）させた結果を利用し，次の表の基準に従ってプログラム上で自動的に選択して行われるものである。

$\overline{\text{I}}/\overline{\text{TC}}$	移動平均項数の長さ
0.00 ～ 0.99	9 カ月
1.00 ～ 3.49	13 カ月
3.50 ～	23 カ月

しかし，系列の変動の特性や季節調整における計算過程の時系列的な安定性を重視する場合には，この移動平均の長さをオプションとして指定することもできる。その場合，一般の系列には13カ月移動平均が適用されるケースが多く，著しく不規則変動の大きい系列には23カ月移動平均を，非常に滑らかな系列について９カ月移動平均が適用されよう。

労働経済の各系列の $\overline{\text{I}}/\overline{\text{TC}}$ 比率は前掲第2表のとおりであり，系列により適用される移動平均の長さはかなり違うことになる。しかし，現金給与総額について標準オプションによる場合と移動平均の長さを指定する場合（$\overline{\text{I}}/\overline{\text{TC}}$ 比率から13項を指定）でその効果を比較すると，不規則要素のランダム性は確保されており（不規則要素変動のＡＤＲが標準オプションで1.40，13項指定の場合1.66），また季節要素の安定性についても殆んど差はなく（季節要素の予測値（53年）と実績値（53年）の各月の乖離の絶対値の和が前者で10.76，後者で9.69），特別に移動平均の長さを指定するメリットはなさそうである。

3）　曜日変動調整

月により，たとえば土曜日，日曜日が何回あるかがその系列の変動に影響を与えていることが明らかな場合には，そのような要素を除去するのが望ましいことはいうまでもない。センサス局法では，月中の曜日の数の相違と関連している変動を回帰方程式によって推計し，これを季節要素とともに原系列から除去することのできるオプションが用意されている。

労働経済の系列の中にも，出勤日数や総実労働時間などのように，月間の土曜日，日曜日の数によって明らかに変化があらわれると考えられるものもある。そこで，月間の曜日構成が各系列の変動に関係があるかどうかを確認するため，分散分析によるＦ検定を行ってみる。この検定自体，センサス局法のオプションとして自動的に行われるが，それによると，総実労働時間，出勤日数，有効求人，有効求職に関しては１％水準で有意であり，曜日変動が系列の変動に影響を与えていることが明らかである。しかし，これら有意と判断された系列について曜日調整の後季節調整を行った結果は，曜日調整を行わない場合と比べ，不規則要素変動のランダム性，季節要素の安定性からみて改善されたとは必ずしもいえない（第11表）。これは，例えば，１カ月間の出勤日数にしても１カ月という期間そのものが給与計算

第11表　曜日調整の有無別季節要素，不規則要素の特性値

系　　列	曜日構成要素の有意性の検定（○……有意）	予測季節要素の安定性（昭和53年季節要素の予測値と実績値の乖離度） 曜日調整を行わない場合	行った場合	不規則要素変動ランダム性（不規則要素変動の連の平均月数（ＡＤＲ）（45～53年）） 曜日調整を行わない場合	行った場合
常用雇用指数（F）	—	0.97	—	1.77	—
総実労働時間（F）	○	4.88	4.65	1.50	1.52
所定外労働時間（F）	—	8.08	—	1.50	—
現金給与総題（F）	—	9.50	—	1.40	—
出勤日数（F）	○	4.58	—	1.50	1.48
有効求人	○	6.73	—	1.66	1.57
有効求職	○	10.44	10.62	1.59	1.69

（注）　予測季節要素の安定性を示す数値の計算方法は第３表の(注)と同じ。

締切日を基礎とした月次データとなっているため，事業所毎の給与締切日の相違から必ずしも暦月となっていないことや，曜日ウェートがデータの全期間

を通じて固定パラメータとして推定されるため，特に最近のように祝日による休日の増加や週休２日制夏季休暇制等の普及の進展による休日の増加があるような場合には調整しきれなくなるためと考えられる。

曜日変動の存在が有意となっている系列についてそれがはたして曜日調整を必要とするのに十分根拠があるものかどうかについても，さらに吟味する必要があろう。

現段階では，曜日変動の存在が有意であっても，季節調整結果に曜日調整をした効果があらわれていない以上，少くともここで例にあげた系列に関しては曜日調整の必要性は薄いとみざるを得ないだろう。

以上，センサス局法で特徴とされる３つのオプションについて，労働経済の時系列の季節調整にそれを使った場合，どの程度の効果があがるかという観点で検討を加え，その結果，特異項の管理限界の設定方法と季節要素を抽出するための移動平均項数の指定については何んらかの工夫の必要があることを指摘することができた。しかし，それ以外の内容については，本稿の検討の範囲ではオプションを使うことが季節調整結果の信頼性を高める上に必ずしも有効とは認められなかった。なお，センサス局法の選択的計算ルーチンとしては，これら３つのオプション以外に，“月の長さの調整”や“歪み調整”など，かなりきめ細かいオプションが用意されており，それらについても一応の試算を行ってみたが，それらの計算ルーチンを採用する場合と，しない場合で，季節調整結果の改善という点では殆んど差がみられなかったため，この検討では特に言及しなかった。いずれにしても，本稿は今後センサス局法に切り換えていく際に必要な検討の入口を整理しただけであり，これらのオプションの使い方に関する結論は今後本格的に行われる検討をまちたい。

（参考）

情報処理課で開発した季節調整システムは，季節調整済みの公表数値を計算するためのバッチ処理システムと，現在，本省各局に労働情報提供のために設置されているディスプレイ装置より直接指示を与えることによって個別分析用に季節調整ができるオンライン処理システムからなっている。このシステムで利用できる計算オプションは付表のとおりである。表の右欄にある「ＬＯＩＳで可能なオプション」は，オンライン処理システムでディスプレイ装置から直接指示できるセンサス局法の計算オプションである。なお，オンライン処理システムでは，このセンサス局法のほか，EPA 法による季節調整方法も選択できるようになっている。

（梶原，工藤，高井）

付表　センサス局法のオプション

項目		オプション	標準オプション	ＬＯＩＳで可能なオプション
事前曜日調整		①しない，②する（曜日毎にウェイトを与える）	①	①
事前月次調整		①しない，②する（月毎に調整係数を与える）	①	①
曜日調整	回帰計算	①計算しない，②計算するが調整しない，③計算して調整する，④計算しパートＢでは調整し，パートＣでは検定が有意なときに調整する	①	①，④
	計算の開始期(年)	①データの最初の月，②指定年の１月	①	①
	調整の開始期(年)	①データの最初の月，②指定年の１月	①	①
	月の長さの調整	①しない，②する	①	①
	特異項の管理限界	①2.5σ，②0.1σ～9.9σのうちから選択	①	①，②
季節要素，すう勢，循環要素を算出するための特異項の管理限界（下限，上限）		①1.5σ～2.5σ，②0.1σ～9.9σのうちから選択	①	①，②
月別移動平均項数		①前半３×３項，後半３×５項，②３項，③３×３項，④３×５項，⑤３×９項，⑥全項（全平均）	①	①，③
ヘンダーソン移動平均項数		①プログラムで選択，②９項，③13項，④23項	①	①
歪み調整		①しない，②する	①	①

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THE ACCURACY AND UNIFORMITY OF SEASONAL ADJUSTMENT

This paper, which discusses the current methods of seasonal adjustment, problems associated with techniques and applications, and alternative procedures suggested for improving seasonal adjustment, is in response to agenda item G in the basic legislation.

Background

The accuracy and uniformity of seasonal adjustment has been of great concern in recent years. With the advent of electronic computers which enabled the development of sophisticated computational programs for seasonal adjustment, it has become possible to identify seasonal variation in large numbers of series at minimal cost and to publish those series on a current seasonally adjusted basis. The Bureau of Labor Statistics now directly adjusts well over 2,000 labor force, employment and unemployment, hours and earnings series. These are in turn used as the basis for nearly 2,000 additional component or aggregated series, including rates, distributions and the like. Most of the nearly 400 series appearing in the monthly Employment Situation press release, for example, are published in both original and seasonally adjusted form. Many more appear in the monthly Employment and Earnings publication.

This magnitude of the seasonal adjustment effort brings administrative problems, but, more importantly, opens questions of interpretation of the results. The purpose of seasonal adjustment, briefly, is to remove seasonal variation from the raw data, thus facilitating identification of the underlying movement of the series. In the process, information on the timing and magnitude of seasonal changes affecting the level of economic activity for subannual periods is obtained.

While there is now fairly widespread public acceptance of seasonally adjusted data, there is relatively little understanding or discussion of the basis and techniques for seasonal adjustment. Yet the impact of accepting seasonally adjusted indicators of the Nation's economy is appreciable. In the case of the unemployment rate, about 95 percent of the month-to-month change in the raw data is due to the seasonal component, i.e., adjustment determines about nine-tenths of the monthly movement in the series over the course of the year. Thus, for some months, the seasonally adjusted unemployment rate may differ by as much as a full percentage point from the unadjusted rate. The above impact underscores the need to use the best available seasonal adjustment methodology.

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Many technical criteria are used to judge the quality of a good seasonal adjustment; moreover, these technical criteria should be achieved within the framework of conditions imposed by the Bureau's responsibilities to the users of these data: first, the seasonal factors should be announced annually for the forthcoming 12 months to preclude suspicion of tampering, to promote public understanding of the current data, and to remove any doubt that the factors are subject to short-term economic change; second, the revision of these factors should be minimal (within this annual release time frame) and there should be a consistent direction of change (i.e., monotonic) from revision to revision; third, the methodology should be reproducible within the context of the options available; and fourth, the user should, with a fairly limited statistical background, be able to understand the general process.

The last major, systematic evaluation of procedures for seasonal adjustment of employment and unemployment statistics by an outside group was undertaken in 1961-62 by the Gordon Committee, which devoted 1 of the 10 chapters of its final report to the issue. At that time, computer-based methods were in a relatively formative stage of development. Among labor force statistics, only a few establishment survey data series and the unemployment rate were published regularly on a seasonally adjusted basis. The remaining household data and insured unemployment series were published only in unadjusted form, though adjustment factors for some series were maintained by the Bureau. The Gordon Committee proposed, and the Bureau initiated shortly thereafter, regular publication of many more household survey series in seasonally adjusted form.

In addition to expansion of publication of seasonally adjusted series since the early 1960's, the power of the adjustment methods has been significantly enhanced. Two major procedural revisions have been made to the BLS Seasonal Factor Method and the Census X-9 and X-10 (then in trial application) matured into the X-11 method. The ratio-to-moving-average method, upon which both systems are based, has been subjected to intense user scrutiny, and is widely used. While alternative methods have been offered, none have clearly exceeded this approach in terms of reliability, universality, flexibility, relative simplicity, and convenience. These conclusions were once again drawn at the 1976 NBER Conference on Seasonal Adjustment.

Most recent criticism centers not on the basic method (ratio-to-moving-average) used but rather on applications of the methods: that is, selection of options within each of the methods, aggregation, and the components adjusted. Though those issues are covered later in this paper, it should be noted that the BLS shares with other data producers, users, and outside statistical analysts a deep concern over the adequacy and the appropriateness of current practices of seasonal adjustment. In the case of the unemployment rate, the Bureau now provides users with several alternative computations on a regular basis.

Current Status

The Bureau currently uses the X-11 Variant of the Census Method II Seasonal Adjustment Program (X-11) to seasonally adjust all of its labor force series derived from the Current Population Survey (CPS). The BLS Seasonal Factor Method (BLS-SF) is used for the Industry Employment Series.

Using the X-11, the components of the household series are seasonally adjusted and the values derived are combined to provide seasonally adjusted values for many series. For example, to derive the adjusted labor force, each of the three major labor force components--agricultural employment, nonagricultural employment, and unemployment--for four age-sex groups (male and females under and over 20 years of age) are separately adjusted and added to derive seasonally adjusted total. The unemployment rate for all civilian workers is obtained by dividing the estimate for total unemployment (the sum of four seasonally adjusted age-sex components) by the civilian labor force (the sum of 12 seasonally adjusted age-sex components).

The BLS-SF method is used to adjust the Industry Employment Series based upon the establishment survey. When the components of the industry employment aggregates are seasonally adjusted and published, the aggregate levels represent the sum of the seasonally adjusted components.

Both the X-11 and the BLS-SF methods are variants of the ratio-to-moving-average technique. (This technique assumes that the original series (X) is the product of three components--the trend cycle (C), the seasonal factor (S), and the irregular (I). The ultimate purpose of the adjustment process is to identify CI by deriving S.) In both methods, the trend cycle is first approximated by a centered moving average of the original series. This is divided into the original series yielding the first approximation of the seasonal irregular (SI). Both methods then take moving averages of the seasonal irregular across the years for each month yielding seasonal factors. These factors are divided into the original series to give a preliminary seasonally adjusted series, which is divided by the trend cycle to produce an irregular series. Irregulars are used to identify extreme observations and replace or modify the corresponding observations using a weighting scheme prior to the next stage of the processing. Based on the seasonally adjusted series (X-11) or the irregular series (BLS-SF) and the extreme irregular weightings, new trend cycle, seasonal factor and irregular series are created from which an original series modified for extremes is generated. This series serves as the basis for a second cycle of processing similar to the first, followed by a third.

While broad steps of the two are identical, they differ in detail--the most important technical difference being the treatment of extreme values in the series. The BLS-SF method has been used for the industry employment series primarily because it seems to better handle the identification and treatment of periodic strikes. The X-11 does incorporate a strike adjustment procedure which is currently being re-evaluated to determine if alternatives can be made to enhance its usefulness with the industry employment series.

The X-11 has a number of extra features which enhance its usefulness to the analyst. While a number of these have been incorporated in the BLS-SF method, certain features are exclusive to the X-11: trading day factor deviation and application; prior factor adjustments; assorted means; totals, standard deviations, and ratios supplied with the tables; charts of seasonally adjusted series, measures of trend-cycle, seasonal and irregular variation; and the measure of cyclic dominance (MCD) as well as the seasonal factors. The X-11 also permits the selection of different moving averages, extreme ranges as may fit the need, and variable trend-cycle routine.

The Bureau of Labor Statistics now publishes or makes available at annual intervals the seasonal factors to be used for the next 12 months prior to their use for all adjusted series. Both the X-11 and the BLS-SF are designed to produce factors that fulfill this requirement; other methods, such as concurrent regression adjustment, are not. Also, both methods are designed to enable adjustment of many time series in a short time frame, with provisions for professional review.

Finally, both methods are fairly easily understood by the public, and the results are reproducible which leads to public confidence in the data. These methods perform well with respect to revision; that is, they tend to yield minimal monotonic revision to the series in prior years when an added year of unadjusted data is added to the series.

Issues Relating to Methods of Seasonal Adjustment

There are several well known alternatives to the ratio-to-moving-average technique: regression analysis, spectral analysis, X-11/ARIMA, and X-11/amplitude adjustment.

Regression techniques have been considered optimal for series whose seasonality is deterministic. This approach has been investigated at the Bureau in a number of forms; however, the results indicate some problems: the amount of revision is larger than with the X-11 and is not monotonic; the application of the methodology does not lend itself to the seasonal adjustment of several thousand time series in a limited time frame; and these techniques are optimized only with concurrent adjustment (rerunning the adjustment with each added observation).

Spectral analysis is another technique that has been used to seasonally adjust time series. In applying these notions to an economic time series, one is assuming that the series, or a simple transformation of the series, is a sample from a stochastic process which satisfies the theory to the extent that it is useful to look at the spectral estimates.

Essentially, spectral analysis distributes the total variance of a series according to the proportion that is accounted for by each of the cycles of all possible periodicities, in intervals of 2-month and longer cycles. If there is a seasonal pattern, a large part will be accounted for by the 12-month cycle and its harmonics.

Problems with spectral analysis arise when one considers an economic time series from the viewpoint of the frequency domain, rather than the time domain. A number of attempts have been made to perform adjustments in the frequency domain, but questions arise as to how deviations from goals measured in spectral terms, can be translated to the time domain. This issue, combined with the demand for long-time series as input to the spectral program has rendered those techniques impractical.

The combination of X-11 and the Box-Jenkins ARIMA techniques by the Statistics Canada staff has provided an area for potential improvements in seasonal adjustment. This approach enlarges the unadjusted time series with one year of forecasting technique using the ARIMA (auto-regressive integrated moving averages) models which have been identified and fitted to the original series. The addition of another year's data is especially important for years with turning points, since the X-11 program is only able to estimate a trend cycle that follows a straight line and thus, may miss the turning point in the most recent year. The ARIMA forecast "tricks" the program into using the central weights of the Henderson moving average for the last 12 observations of the trend cycle estimate. The X-11 ARIMA weights, applied to the seasonal-irregular ratios/difference, can thus estimate moving seasonality more accurately. This is especially true if the most recent year contains a cyclical turning point; however, this depends on the accuracy of specification of the model.

The X-11 ARIMA method is not without problems. First, it is difficult to identify the models--a process that should be repeated every year. Second, the model does not allow for changing structure of the seasonal in recent years. Also, the method implicitly includes a forecast of the future behavior of the unadjusted series. The fact of the Bureau of Labor Statistics producing implicit forecasts carries policy implications and potential public relations problems, particularly for sensitive series such as unemployment.

Using the X-11 in combination with an adjustment for the amplitude of the series has also been suggested. Over the years it has been observed that, depending upon whether an additive or multiplicative adjustment is used and depending on the level of the series being adjusted, there can be over- and under-adjustment. John Brittain has suggested a season-amplitude variable that would take into consideration the level of the series. This procedure uses regression techniques to identify the amplitude variable. This has not proven a satisfactory answer, because the regression coefficients 12 months into the projection period may be unstable. A similar result may be obtained by choosing the proper relationship, multiplicative or additive. In so doing, the variation between the observed seasonal and the final seasonal (identified after several years of additional experience) is lessened.

Finally, a minor issue that should be explored has to do with the adequacy of the computer language of X-11. Presently, the BLS seasonal factor method and the X-11 seasonal factor method are both written in single precision. The BLS is converting the BLS-SF to double precision this year in order to lend added numerical stability.

Issues Relating to Application of Seasonal Adjustment Methods

Within the framework of generally accepted adjustment methods, there is considerable disagreement on a number of fairly important issues, including: the appropriate unit for analysis; the selection of the appropriate model of behavior of the series; the selection of the optimal time period; and the use of various optional procedures within the methods. Some of these issues remain unsettled 15 years after identification in the Gordon Committee report.

Unit for analysis. As the Gordon Committee report pointed out, "There is considerable discretion as to which series are to be adjusted directly and which, if any, by implication." BLS often seasonally adjusts components of series directly, then combines the values derived to provide seasonally adjusted values for many other series, including summations, rates, and distributions. The results of component aggregation will, of course, differ from results obtained by direct adjustment of totals, rates, and distributions because different seasonal patterns among the components will be captured due to the limitation of the processes. Separate adjustment of components also isolates, over time, changes in the underlying component groups, hopefully enabling identification of the causes of change and increasing the accuracy of the adjustment. Nonetheless, there is an element of professional judgement in selection of series to be directly adjusted and those to be aggregated. For example, the adjusted overall unemployment rate is an aggregation of 12 directly adjusted age/sex/employment/unemployment series. Would it be more reliable if race or more detailed age groups were included among the components? If there were a bias in the existing aggregation pattern, would it be avoided by higher-level direct adjustments, say by adjusting civilian labor force and employment and deriving unemployment as a residual? (The latter suggestion for a "residual method" was considered and rejected by the Gordon Committee. It remains, however, an attractive alternative to some analysts).

Period for analysis. Due to the nature of the moving average, the seasonal adjustment methods used by BLS produce a "best" result with at least 7 years of data for input. In 1976, 10 years of original observations for both household and establishment data were adjusted. When movements in the economy are quick and severe, however, as they were during the 1974-75 period, some analysts have been concerned that the factors might include some cyclical as well as seasonal movement. It has been suggested that factors be developed based not on most recent experience, but on "normal" periods to avoid contamination of the result with purely cyclical developments.

At the opposite extreme, other analysts have noted that once-a-year revision of seasonal factors is not frequent enough. One alternative to the present method of adjusting current year data with last year's factors is to use projected ("year-ahead") factors, which give added weight to recent trends. The second alternative is to update the seasonal adjustments each month, as the current month's data become available. The first alternative was used by the Bureau for household series in 1973. However, based on research which showed that revisions were larger when recent trend was included, the Bureau shifted back to the use of previous year's factors in 1974. The second alternative poses three serious problems: (1) the difficulty in maintaining credibility in the objectivity of the procedure when the seasonal adjustment depends on current data rather than factors published in advance; (2) frequent revisions are confusing, with some analysts suggesting that the present annual revisions (limited to the last 5 years' data) are too often; and, (3) the operational difficulty of adjusting thousands of series by this monthly update method would be substantial. For analytical purposes, both of these suggested procedures are provided in the monthly table of alternative unemployment rates.

A compromise alternative in regard to frequency of revision of seasonal factors is to compute new seasonal factors each 6 months. In tests performed at the Bureau of the Census several years ago, it was found that, by providing factors a year in advance, the size of revisions increase toward the end of the period. The tests showed that the revisions were significantly larger in the last 6 months--as the factors moved farther away from the center of the moving average.

Identification of Moving Seasonality. Most series tend to display a pattern of month-to-month variation that is strikingly similar from one year to the next. This is due obviously to the round of climatic seasons, conventional holidays, buying periods and the like. But seasonal movements are not precisely repetitive from year to year. Climatic conditions do change; the dating of holidays shifts; trade practices (such as model changeovers) are somewhat flexible; school vacation periods change; and technological advances tend to extend the construction year. Finally, as mentioned earlier, the relative importance of the components of a series may change over time, thus affecting the seasonal pattern of the overall series. The impact of these and many other occurrences is that seasonal patterns today may be quite different from those of 5 years ago. This movement in seasonality must be taken into account both in directly adjusting component series and in developing aggregated series from components which change in relative importance over time.

Both seasonal adjustment methods used by the Bureau have options for moving seasonality over time, by allowing different weighting patterns in the final seasonal factors curves. The Bureau now uses, for most series, a weighting schematic which gives approximately equal weights to the past 3 years, a reduced weight to the fourth year, and no weight to the years prior to the fourth. It has been suggested, however, that a stable seasonal factor curve (with equal weight to all years) or an accelerated weight pattern (with larger weights in more recent years) might better represent the movement in seasonality in the current period.

Model of Behavior of the Series. Both the X-11 and the BLS Seasonal Factor Method incorporate a set of assumptions concerning the generating mechanisms of the series which are expected to be met for the final output to yield an appropriately adjusted result. One primary decision which must be made is whether the level of the seasonal component of the series is independent of the trend cycle (additive model), or is the level of the seasonal component proportionate to the trend cycle (multiplicative model). Selection of the appropriate model for the time series' behavior is critical; applying the multiplicative option to a series whose intra-year movements are essentially independent of the level of a series will result in over- or under-adjustment, depending on the phase of the cycle. For a number of years, BLS had used the multiplicative option for adjusting teenage unemployment, although seasonal changes in youth joblessness are not so much related to their overall level of unemployment as to the size of their civilian population and the number attending school. Because this distorted the teenage adjustment in months of large changes in levels, especially June, the adjustment of the overall level changed significantly. Based on those findings, the additive procedure for adjusting teenage unemployment was adopted in January 1976.

There is evidence now to indicate that the additive model--the assumption of independence of seasonal amplitude and trend cycle--may be representative of the behavior of many more labor force series. A major study of the behavior of the labor force series on a month-by-month, year-by-year basis is now underway in the Bureau. The method to be used to compute the 1978 seasonal factors for household data is being examined in these studies.

Recommendations for Commission Studies

Although we are more advanced now in our understanding of the conceptual basis for seasonal adjustment and though techniques have improved dramatically since the Gordon Committee's evaluation, the need for improvement remains. In fulfilling the Congressional mandate, the Commission might wish to assess the following problem areas:

1. The purpose and meaning of seasonal adjustment

   * Are seasonally adjusted labor force data clearly superior for analytical purposes to original observations?

   * Is the model which decomposes original series into seasonal, trend cycle and irregular movements appropriate?

2. The techniques and methods of seasonal adjustment

   * Is there an alternative superior to the traditional ratio-to-moving-average method for identification of seasonality? Is the alternative method practical and flexible enough to encompass a wide variety of economic time series and remain acceptable to a wide range of users without specific technical training or guidance?

   * Should a mixed model approach be developed?

   * Are there any revisions to procedures within the context of the X-11 or the BLS methods which would improve performance?

   * Should the BLS seasonal factor method be continued? If not, how can its attractive features be built into the X-11?

   * Should an alternative weighting schematic be developed for the final seasonal factor curve?

3. The application of adjustment methods

* Has the BLS selected the appropriate unit for analysis in adjusting labor force, earnings and hours series?

* Are there any special problems encountered with application of the methods to State and local area data? Can guidelines be developed for selecting the optimum period of analysis?

* What is the trade-off between the accuracy of current seasonal adjustments and the magnitude of revisions that can be tolerated?

* Which model, multiplicative or additive, best specifies the underlying behavior of employment and unemployment series?

* Are there tests not now used by the BLS that could assist in assessing the adequacy of the adjustment?

4. Future studies for the BLS

* Are there suggested lines of investigation which the BLS could follow in attempting to refine the seasonal adjustment process?

T E C H N I C A L   P A P E R

Tests of Alternative Seasonal Adjustment Methods: Observations and Recommendations

U.S. DEPARTMENT OF LABOR
Bureau of Labor Statistics
January 14, 1976

Summary

. The method of seasonal adjustment which the Bureau has employed for all CPS series assumes that the magnitude of the seasonal increase or decrease is proportional to the level of the series (multiplicative).

. Especially during periods of rapidly changing unemployment, however, it becomes apparent that some time series behave additively, that is, the magnitude of the seasonal increase or decrease is essentially constant and without regard to the level.

. Over the course of 1975, it became apparent that the assumption of proportionality was resulting in difficulties in seasonal adjustment of unemployment, with adjustments for some months (particularly May and June) possibly departing substantially from the "true" rate of unemployment--the objective of deseasonalizing the data.

. This research into the seasonality of major labor force, employment and unemployment series, and our tests of alternative methods of seasonal adjustment of these time series, indicate:

-- Teenage (16-19 years of age) unemployment has behaved preponderantly in an additive pattern since 1967.

-- Adult male (20 years of age and over) unemployment is clearly multiplicative in behavior.

-- Adult female unemployment has elements of both multiplicative and additive behavior, with a slight preference for the multiplicative pattern.

-- Young adult (20-24 years of age) unemployment for each sex does not have a markedly different model of seasonality from that for unemployment of older adults (25 years of age and over) or all adults (20 years of age and over) for the same sex.

-- Employment series, for all age groups, mostly show evidence of multiplicative patterns of seasonality, though teenage nonagricultural employment does have strong elements of additivity.

-- No improvement in the quality of the result is evident when seasonally adjusted unemployment is calculated by subtracting the adjusted employment from the directly adjusted labor force (the "residual" method).

Recommendations

. That the method of seasonal adjustment of the unemployment series be revised to adjust teenage series, and those series predominately influenced by teenage patterns, using the additive option of the X-11 program.

. That all other employment and unemployment series continue to be adjusted by the X-11 multiplicative procedure.

. That appropriate revisions in the historical series be made for the last 5 years, in keeping with normal BLS revision policy.

. That additional research, possibly under the aegis of the "new Gordon Committee," be conducted to explore additional alternative methods of aggregation of series to derive the overall rate and develop procedures for simultaneous multiplicative/additive adjustment to each month.

Introduction to the Study

The Bureau of Labor Statistics has utilized the Bureau of the Census X-11 seasonal adjustment program 1/ to adjust employment and unemployment series from the Current Population Survey (CPS) since 1972. The X-11 seasonal adjustment method is an adaptation of the traditional ratio-to-moving average method, with allowance for changing seasonal patterns. The original data are regarded as a product of a trend-cycle (TC) component, a seasonal (S) component, and an irregular (I) component. The trend-cycle represents the long-run trend and cycle movements of the series. The seasonal component is the annual repetitive pattern which makes certain months consistently higher or lower than others. The irregular component is a residual, including sampling errors and short-term fluctuations due to unusual weather, strikes, etc., which do not follow any consistent pattern. After a satisfactory decomposition is achieved, the seasonally adjusted series is computed by dividing each month's original value by the corresponding seasonal factor.

After the computation of seasonal factors and seasonally adjusted data for independently adjusted series, these series are then combined to yield other aggregate seasonally adjusted estimates. In other words, components of a series are adjusted directly, and these values are combined to provide seasonally adjusted values for many other series.

All civilian labor force and unemployment rate statistics, as well as the major employment and unemployment estimates, are computed by this method of aggregation. For example, for each of the three major labor

1/ For a description of the X-11 method, see Technical Paper No. 15, The X-11 Variant of the Census Method II Seasonal Adjustment Program, Bureau of the Census, 1967.

force components--agricultural and nonagricultural employment, and unemployment--data for four age-sex groups (males and females under and over 20 years of age) are separately adjusted for seasonal variation and are then added to derive seasonally adjusted total figures. In order to provide seasonally adjusted total employment and civilian labor force estimates, the appropriate series are aggregated. The unemployment rate for all civilian workers is derived by dividing the estimate for total unemployment (the sum of 4 seasonally adjusted age-sex components) by the civilian labor force (the sum of 12 seasonally adjusted components).

The underlying assumption in the BLS adaptation of the X-11 method has been that the magnitude of the seasonal increase or decrease is proportional to the level of the series. This so called multiplicative relationship has been used exclusively to adjust employment and unemployment series, since earlier studies have shown that, for most months, the proportional relationship best portrays the character of time series movements for most series.

Another procedure is available with the X-11 which assumes that the magnitude of the seasonal increase or decrease is essentially constant without regard to level. This is called an additive relationship. Most series are known to have elements of both multiplicative and additive relationships, but, generally, the additive relationship less adequately describes most movements for most series than does the multiplicative procedure.

In this context, unemployment has been multiplicatively adjusted, even though it was recognized that in some months, especially the May-June change, unemployment has a nearly constant seasonal relationship which is largely independent of the level.

This limitation in applying the multiplicative method to all months has been well known for some time, but, except in periods when unemployment shifts sharply, the resultant adjusted unemployment level was not seen as sufficiently distorted to engender concern over the adequacy of the procedure. If the level of a series does not change radically from one year to the next, a proportion of that level will be nearly constant. Adjustments by both procedures have been computed monthly as an aid to internal analysis of the significance of between-months movements for some time, and have recently been made generally available in testimony to the Joint Economic Committee. (See table 1). It will be noted that the differences were trivial until unemployment began to rise sharply.

The widening deviation between the results of multiplicative and additive procedures in late 1974 and early 1975 caused concern over the adequacy of the multiplicative procedure in a period of sharply rising unemployment, particularly in those months known to traditionally behave additively. In anticipation of the potential difficulties, the Commissioner of Labor Statistics called for a thorough evaluation of seasonal adjustment procedures in early 1975. The Commissioner discussed the preliminary implications of this research in testimony before the Joint Economic Committee in June and July 1975, warning that the reported rate for May and June would possibly depart from the "true" rate.

In subsequent months, a work group of Bureau technicians has further studied the seasonal properties of published labor force series, and has expanded the project to conduct tests of alternative composities of seasonally adjusted series. The study has included examination of computation of adjusted unemployment by the "residual" method; that is the difference between the directly adjusted labor force and adjusted employment.

(1) Unemployment rate, not seasonally adjusted.

(2) Seasonally adjusted unemployment rate.

This is the rate as published. Each of four unemployed sex-age components--males and females, 16-19 and 20 years and over--are independently seasonally adjusted. The rate is calculated by aggregating the four and dividing them by 12 summed labor force components--these 4 plus 8 employed components, which are the 4 sex-age groups in agriculture and non-agricultural industries. This employment aggregate is also used in the calculation of the labor force base in (3) - (8).

The current "implicit" factors for the total unemployment rate are as follows:

Jan.	--	109.1	July	--	105.5
Feb.	--	111.1	Aug.	--	97.8
Mar.	--	104.2	Sept.	--	98.4
Apr.	--	95.7	Oct.	--	91.0
May	--	89.1	Nov.	--	94.6
June	--	110.7	Dec.	--	93.0

(3) Duration.

Unemployment total is aggregated from 4 independently adjusted unemployment by duration groups (0-4, 5-14, 15-26, 27+).

(4) Full-time and Part-time.

Unemployment total is aggregated from 6 independently seasonally adjusted unemployment groups, by whether the unemployed are seeking full-time or part-time work and men 20 plus, women 20 plus, and teenagers.

(5) Reasons.

Unemployment total is aggregated from 4 independently seasonally adjusted unemployment levels by reason for unemployment--job losers, job leavers, new entrants, and re-entrants.

(6) Occupation.

Unemployment total is aggregated from independently seasonally adjusted unemployment by the occupation of the last job held. There are 11 unemployed components--12 major occupations plus new entrants to the labor force (no previous work experience).

(7) Industry.

Unemployment total is aggregated from 10 independently adjusted industry and class-of-worker categories, again including new entrants to the labor force.

(8) Additive Method.

The basic 4 unemployed sex-age groups--males and females, 16-19 years and 20 years and over--are adjusted by the X-11 additive method rather than the conventional multiplicative method. Employment (8 sex-age groups) is the same, however, as in columns (2) - (7).

(9) Unemployment rate adjusted directly.

(10) Unemployment and labor force levels adjusted directly.

(11) Labor force and employment levels adjusted directly, unemployment as a residual and rate then calculated.

(12) Average of (2), (3), (4), (5), and (11).

(13) Average of (2), (3), (4), (5), (6), (7), and (11).

NOTE: The X-11 method, developed by Julius Shiskin at the Bureau of the Census over the period, 1955-65, was used in computing all the seasonally adjusted series described above.

Table 1.

Unemployment Rate by Alternate Seasonal Adjustment Methods

Month	Unadj. rate	Adj. rate	Other Aggregations					Additive (X-11)	Direct Adjustments			Composite #1	Composite #2	Range (Col.2-13)
			Duration	Ft/pt	Reasons	Occupat.	Industry		Rate	Level	Residual			
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)
1974														
January	5.6	5.2	5.1	5.1	5.1	5.2	5.1	5.1	5.1	5.1	5.1	5.1	5.1	.1
February	5.7	5.2	5.1	5.1	5.1	5.1	5.2	5.2	5.1	5.2	5.1	5.1	5.1	.1
March	5.3	5.1	5.1	5.1	5.0	5.1	5.0	5.1	5.0	5.1	5.1	5.1	5.1	.1
April	4.8	5.0	5.1	5.1	5.1	5.0	5.0	5.0	5.0	5.1	5.1	5.1	5.1	.1
May	4.6	5.2	5.2	5.2	5.3	5.2	5.2	5.1	5.2	5.2	5.2	5.2	5.2	.2
June	5.8	5.2	5.3	5.3	5.2	5.3	5.2	5.3	5.2	5.2	5.3	5.3	5.3	.1
July	5.6	5.3	5.4	5.4	5.4	5.4	5.3	5.4	5.4	5.4	5.4	5.4	5.4	.1
August	5.3	5.4	5.4	5.4	5.3	5.4	5.4	5.4	5.4	5.3	5.5	5.4	5.4	.2
September	5.7	5.8	5.8	5.8	5.8	5.8	5.8	5.8	5.8	5.8	5.8	5.8	5.8	0
October	5.5	6.0	6.0	6.2	6.0	6.0	6.0	6.0	6.1	6.1	5.9	6.0	6.0	.3
November	6.2	6.6	6.6	6.6	6.6	6.6	6.5	6.4	6.6	6.6	6.4	6.6	6.6	.2
December	6.7	7.2	7.0	7.2	7.1	7.1	7.1	7.0	7.3	7.2	7.0	7.1	7.1	.3
1975														
January	9.0	8.2	8.2	8.1	8.0	8.1	8.0	8.4	8.2	8.2	8.4	8.2	8.1	.4
February	9.1	8.2	8.0	8.1	8.0	7.9	8.1	8.5	8.2	8.2	8.6	8.2	8.1	.7
March	9.1	8.7	8.6	8.7	8.5	8.6	8.5	8.9	8.7	8.7	9.0	8.7	8.7	.5
April	8.6	8.9	8.7	8.9	8.8	8.8	8.8	8.8	9.0	9.0	8.8	8.8	8.8	.3
May	8.3	9.2	9.0	9.2	9.3	9.3	9.3	8.8	9.4	9.3	8.9	9.1	9.2	.6
June	9.1	8.6	8.7	8.5	8.8	8.6	8.6	8.7	8.2	8.2	8.7	8.7	8.6	.6
July	8.7	8.4	8.5	8.4	8.4	8.3	8.3	8.5	8.4	8.3	8.4	8.4	8.4	.2
August	8.2	8.4	8.5	8.3	8.4	8.5	8.4	8.3	8.3	8.3	8.3	8.4	8.4	.2
September	8.1	8.3	8.6	8.4	8.6	8.4	8.3	8.2	8.2	8.3	8.2	8.4	3.4	.4
October	7.8	8.6	8.6	8.8	8.6	8.5	8.5	8.3	8.7	8.7	8.1	8.5	8.5	.7
November	7.8	8.3	8.4	8.4	8.0	8.2	8.2	8.0	8.2	8.2	8.1	8.2	8.2	.4
December	7.8	8.3	8.3	8.3	8.2	8.2	8.3	8.1	8.5	8.3	8.1	8.2	8.2	.4

NOTE: An explanation of columns 1-14 appears on the reverse side of this page.

U. S. Department of Labor
Bureau of Labor Statistics
January 9, 1976

(1) Unemployment rate, not seasonally adjusted.

(2) Seasonally adjusted unemployment rate.
This is the rate as published. Each of four unemployed sex-age components--males and females, 16-19 and 20 years and over--are independently seasonally adjusted. The rate is calculated by aggregating the four and dividing them by 12 summed labor force components--these 4 plus 8 employed components, which are the 4 sex-age groups in agriculture and non-agricultural industries. This employment aggregate is also used in the calculation of the labor force base in (3) - (8).
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The basic 4 unemployed sex-age groups--males and females, 16-19 years and 20 years and over--are adjusted by the X-11 additive method rather than the conventional multiplicative method. Employment (8 sex-age groups) is the same, however, as in columns (2) - (7).

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NOTE: The X-11 method, developed by Julius Shiskin at the Bureau of the Census over the period, 1955-65, was used in computing all the seasonally adjusted series described above.

Due to resource and time limitations, the study group confined itself to working within the context of the X-11 program. Other potential methods of adjustment, such as the spectral analysis and least squares approaches, were not considered. The study group, however, benefited from previous studies of the efficacy of these methods by Rosenblatt 2/ and by Lovell and Young, 3/ respectively.

The study group independently adjusted 64 major employment and unemployment series to derive measures of seasonality, regressed 16 major series using 4 different equations to identify monthly seasonal components and their significance, tested the results of 6 possible aggregation composites, and calculated 16 multiplicative and additive residual options. This final report and the recommendations have been delayed for receipt of all months of 1975 original data so that findings based on previous data could be fully confirmed.

Seasonality of Independently Adjusted Series

Seasonally adjusted total unemployment is an aggregation of four independently adjusted major age/sex groups (males 16-19, females 16-19, males 20 years and over, and females 20 years and over). When the overall level assumes either multiplicative or additive relationship, it is only reflecting the strength of the multiplicativity or additivity of the component series. This, of course, is also true of total employment--the aggregation of the four age/sex groups for agriculture and nonagricultural employment.

2/ Rosenblatt, Harvey M., "Spectral Evaluation of BLS and Census Revised Seasonal Adjustment Procedures," American Statistical Association Journal, June 1968, pp. 472-501.

3/ Lovell, Michael C., "Seasonal Adjustment of Economic Time Series and Multiple Regression Analysis," American Statistical Association Journal, December 1963, pp. 993-1010; Young, Allan H., "Linear Approximations to the Census and BLS Seasonal Adjustment Methods," American Statistical Association Journal, June 1968, pp. 445-471.

Thus, to understand the seasonality of the overall level and rates, it is necessary to study the seasonality of the component series. This is an end best served by study of not only the four standard age/sex groups, but also additional age/sex groups which might be expected to behave differently. For purposes of this study, four such groups (males 20-24, females 20-24, males 25 and over, and females 25 and over) were selected.

Two principal tests were applied to thse series: first, the series were adjusted by both multiplicative and additive X-11 procedures with summary measures which indicate the qualties of the seasonally adjusted series extracted, and second, the resultant data were regressed in a series of equations designed to test the interaction of the multiplicative and additive components of the series.

Results of Tests of Seasonal Quality of Component Series

The standard tests indicated that each of the component series exhibited sufficient stable seasonality (by both multiplicative and additive procedures) to warrant monthly adjustment. However, there were important differences within and between the component series when adjusted by different procedures. An indication of the amount of this difference may be inferred from the F-statistic, a computation of the ratio of the between-months variance to the residual variance of the seasonal irregular. Generally, the larger the F-statistic the more seasonal stability the series exhibits. The results of this test for the 1967-75 period are summarized in table 2.

Table 2. F-Statistics of Component Series, 1967-75

Series	Procedure: Multiplicative adjustment	Procedure: Additive adjustment
Employment		
Agriculture, male 16-19	223.9	242.3
" female 16-19	126.0	138.3
" male 20+	116.5	114.4
" female 20+	150.2	135.4
Nonagriculture, male 16-19	397.8	910.9
" female 16-19	266.6	423.0
" male 20+	112.4	92.9
" female 20+	141.0	163.3
Unemployment		
Male, 16-19	72.2	138.6
Male, 20-24	36.2	25.9
Male, 20+	168.6	53.8
Male, 25+	214.5	68.6
Female, 16-19	82.4	160.7
Female, 20-24	30.7	34.3
Female, 20+	30.9	28.7
Female, 25+	28.0	20.5

Source: X-11 Seasonal Adjustment Program

As can be seen, there is little difference in the F-statistic between multiplicative and additive adjustment for a number of component series: all agricultural employment series, adult nonagricultural employment, and unemployment for males 20-24 and all adult women. For these series, it cannot be said with any certainty from this test that either of the procedures would improve the stability of seasonality over the other.

The F-statistic under the multiplicative procedure was clearly higher for unemployment of adult males, especially those 25 years of age and over, while the additive option resulted in improved stability for teenage unemployment and nonagricultural employment. About these series, it can be said that selection of one option over the other will improve the performance of the adjustment process. The F-test results indicate that we may be sacrificing precision in the adjustment process by adjusting all series multiplicatively, as is presently the practice. However, the probable loss in the employment series is not as important as the unemployment since year-to-year changes in employment are relatively too small to make much difference in the magnitude of multiplicative and additive adjustments.

Results of Tests of Seasonal Characteristics of Component Series Using Regression Analysis

If one subtracts the trend-cycle curve of a series (TC) from the original series, the remainder contains all of the seasonal and irregular variation (SI). If the seasonal behavior of the series is multiplicative, then the magnitude of SI for a given month should be proportional to TC. If the seasonal is additive, then the SI for a given month should be constant and unrelated to TC. To test for these relationships, the SI and TC were calculated using the X-11 method for each of the eight component unemployment series being studied. Then for each series the following regression equation was fitted for each month of the year over the years 1967-75.

$$SI = a + b(TC)$$

If coefficient a is significant and b is not, then the relationship for that month and that series is probably additive. Conversely, if b is significant and a is not, then the relationship is probably multiplicative.

Tables 3 - 10 summarize the results of these regressions for the unemployment series. Coefficients a and b correspond to those in the above equation. Each is accompanied by its t-score. Significance of the coefficients at 5 percent and 10 percent two-tailed t-test levels are indicated. The Dubin-Watson (D.W.) statistic is presented as a guide to possible autocorrelation of the residuals. The average SI is presented for each month. This value can be used to identify the most highly seasonal months, which would in general suffer most from a misspecification of the model. It can also serve to identify the relative importance of the a and b coefficients in determining the size of the adjustment. The error of the estimate is included as a guide to the goodness of the fit.

Even a quick glance at the 3 adult male series (ages 20-24; 25 and over; 20 and over) shows that they are all clearly multiplicative cases in the overwhelming majority of months. (Tables 5, 7, and 9.) The other series provide somewhat more complex sets of choices.

The males 16-19 years series has significant or marginally significant additive coefficients in five months and multiplicative in three with two of these months (January and May) showing both characteristics. The months with the largest seasonal--June and July--are clearly additive.

January has a very small seasonal and in May more than half the seasonal, on the average, is additive. The results point to the appropriateness of an additive model. (Table 3)

For the females 16-19 years, six months have significant or marginally significant additive coefficients. Three have multiplicative--all also possessing a significant additive one. In the cases of March and June, the additive factor seems to be controlling, but for December there appears to be greater multiplicativity. The preponderance of the evidence favors the additive model. (Table 4.)

The females 20-24 years series has four months with significant additive factors and five with multiplicative. Two of these (May and December) have both coefficients significant. The most seasonal month (December) has about equal additive and multiplicative effects. Of the next four largest seasonals, two are multiplicative and two additive. In sum, there is no clear evidence on this series, although a possible preference for the multiplicative exists if a choice must be made. (Table 8).

Females 25 years and over are multiplicative in four months and additive in two, one of which has both coefficients significant. Although there is no especially clear case here, the multiplicative model is preferable. (Table 10.)

For the series on females 20 years and over, there are five significant or nearly significant additive coefficients and four multiplicative. Of these, three occur in the same months. The magnitude of the seasonal in the two exclusively additive months are larger than that of the exclusively multiplicative. Of the three months with both coefficients significant, two exhibit no preference for one or the other, but January

Table 3. Analysis . Seasonal Characteristics of U. .ployment, Males 16-19, 1967-75

Month	a	b	Standard Error	D.W.	Average SI
January	-86 (-2.3)*	.142 (2.4)**	30	2.17	0
February	-45 (-1.1)	.087 (1.3)	33	3.09	8
* March	-.32 (-1.0)	-.003 (-.1)	24	2.56	-33
April	-57 (-2.5)**	-.055 (-1.5)	19	2.95	-91
May	-76 (-6.2)**	-.100 (-5.3)**	10	3.28	-139
June	271 (4.2)**	.041 (.4)	51	2.04	297
July	182 (3.6)**	.016 (.2)	39	2.33	192
August	-21 (-.4)	-.015 (-.2)	36	2.23	-30
September	-53 (-1.6)	-.028 (-.6)	24	2.21	-71
October	30 (1.3)	-.165 (-4.7)**	18	1.83	-77
November	9 (.3)	-.066 (-1.6)	21	1.19	-34
December	-23 (-.7)	-.011 (-.2)	21	2.32	-29

** Significant (5%)
* Marginally Significant (10%)
(T-Statistic in Parenthesis)

Table 4. Analysis of Seasonal Characteristics of Unemployment, Females 16-19, 1967-75

Month	a	b	Standard Error	D.W.	Average SI
January	-162 (-3.0)**	.142 (1.4)	39	2.23	-87
February	5 (.1)	-.128 (-1.9)	27	1.56	-62
March	-105 (-8.0)**	.070 (2.9)**	10	1.81	-68
April	-15 (-.3)	-.167 (-1.7)	40	2.00	-105
May	-112 (-1.9)*	-.007 (-.1)	42	1.61	-116
June	414 (11.3)**	-.130 (-2.0)*	25	3.08	344
July	211 (3.3)**	-.022 (-.2)	43	2.75	199
August	49 (.8)	-.052 (-.5)	39	2.24	20
September	-23 (-.9)	.067 (1.6)	16	1.68	15
October	-15 (-.6)	-.022 (-.5)	17	2.29	-27
November	45 (1.0)	-.107 (-1.4)	30	2.24	-15
December	-37 (-1.9)*	-.128 (-3.7)**	12	2.65	-106

** Significant (5%)
* Marginally Significant (10%)
(T-Statistic in Parenthesis)

Table 5. Analysis of Seasonal Characteristics of Unemployment, Males 20+, 1967-75

Month	a	b	Standard Error	D.W.	Average SI
January	-32 (-.7)	.254 (10.0)**	46	1.40	380
February	52 (1.0)	.234 (8.3)**	55	2.12	436
March	-53 (-2.1)*	.212 (15.2)**	29	0.85	301
April	-126 (-3.2)**	.113 (5.3)**	46	1.50	64
May	-165 (-6.3)**	.029 (2.1)*	32	2.21	-115
June	37 (4.0)**	-.049 (-9.9)**	11	1.76	-47
July	46 (1.5)	-.084 (-5.4)**	37	1.01	-102
August	124 (3.0)**	-.166 (-7.8)**	51	1.39	-173
September	21 (.5)	-.171 (-8.9)**	46	1.78	-288
October	67 (1.4)	-.205 (-8.6)**	57	2.45	-308
November	64 (1.6)	-.161 (-8.1)**	49	1.42	-235
December	-49 (-1.0)	.007 (.3)	46	1.86	-37

** Significant (5%)
* Marginally Significant (10%)
(T-Statistic in Parenthesis)

Table 6. Analysis of Seasonal Characteristics of [illegible]mployment, Females 20+, 1967-75

Month	a	b	Standard Error	D.W.	Average SI
January	-127 (-1.9)*	.158 (3.5)**	60	0.98	101
February	25 (.4)	.041 (1.1)	53	1.69	84
March	-26 (-.7)	.020 (.9)	33	2.49	4
April	-39 (-1.4)	-.034 (-2.0)*	26	1.92	-89
May	-132 (-3.3)**	-.012 (-.05)	39	3.00	-134
June	75 (3.4)**	-.048 (-3.5)**	21	0.92	3
July	4 (.1)	.0003 (.01)	31	3.30	5
August	17 (.3)	.039 (1.1)	51	1.40	77
September	141 (4.3)**	.004 (.2)	29	2.54	147
October	99 (.9)	-.066 (-.9)	102	1.96	-4
November	5 (.1)	-.035 (-1.3)	39	1.87	-50
December	-90 (-2.2)*	-.076 (-2.9)**	31	2.71	-202

** Significant (5%)
* Marginally Significant (10%)
(T-Statistic in Parenthesis)

Table 7. Analysi of Seasonal Characteristics of employment, Males 20-24, 1967-75

Month	a	b	Standard Error	D.W.	Average SI
January	-4 (-.2)	.162 (5.1)**	21	2.57	74
February	18 (.6)	.161 (3.1)**	37	1.81	97
March	-19 (-1.4)	.136 (5.7)**	18	1.89	49
April	-60 (-4.1)**	.097 (3.8)**	19	2.25	-11
May	-41 (-2.8)**	.015 (.6)	20	1.58	-33
June	83 (5.6)**	-.032 (-1.3)	19	2.63	66
July	6 (.3)	.003 (.1)	22	2.31	7
August	20 (1.2)	-.104 (-3.6)**	22	2.50	-36
September	-10 (-.7)	-.070 (-3.0)**	18	1.92	-49
October	27 (1.2)	-.204 (-5.7)**	28	1.90	-86
November	18 (1.4)	-.141 (-6.6)**	17	1.52	-61
December	-7 (-.6)	-.077 (-3.9)**	11	1.68	-46

** Significant (5%)
* Marginally Significant (10%)
(T-Statistic in Parenthesis)

Table 8. Analys of Seasonal Characteristics of nemployment, Females 20-24, 1967-75

Month	a	b	Standard Error	D.W.	Average SI
January	-9 (-.6)	-.042 (1.2)	16	2.35	9
February	21 (1.0)	-.009 (.2)	21	2.01	16
March	-9 (-1.3)	-.016 (-1.1)	7	2.91	-16
April	-4 (-.3)	-.119 (-4.3)**	13	3.05	-56
May	-97 (-9.1)**	.149 (6.5)**	11	2.95	-31
June	97 (3.9)**	-.046 (-.9)	24	2.69	77
July	-7 (-.3)	.125 (2.4)**	23	1.52	49
August	-24 (-1.1)	.170 (3.8)**	19	2.05	53
September	45 (2.4)**	.019 (.3)	18	2.30	54
October	37 (.8)	-.107 (-1.1)	44	2.19	-12
November	-7 (-.2)	-.080 (-1.0)	35	1.66	-44
December	-52 (-3.5)**	-.091 (-2.8)**	12	1.35	-92

** Significant (5%)
* Marginally Significant (10%)
(T-Statistic in Parenthesis)

Table 9. Analysis of Seasonal Characteristics of [illegible] mployment, Males 25+

Month	a	b	Standard Error	D.W.	Average SI
January	-60 (-1.5)	.322 (9.6)**	39	0.85	306
February	5 (.2)	.290 (13.1)**	28	1.70	339
March	-42 (-2.0)*	.251 (15.2)**	22	1.32	250
April	-44 (-1.2)	.098 (3.4)**	41	1.84	73
May	-94 (-4.5)**	.007 (.4)	24	1.87	-86
June	-23 (-1.5)	-.077 (-6.6)**	18	2.61	-116
July	56 (2.9)**	-.135 (-9.1)**	24	0.99	-109
August	106 (3.0)**	-.194 (-7.4)**	42	0.87	-135
September	31 (1.2)	-.213 (-10.8)**	32	1.88	-237
October	37 (1.3)	-.201 (-10.2)**	32	2.77	-221
November	50 (1.7)	-.173 (-8.4)**	35	1.62	-175
December	-67 (-1.9)*	.066 (2.3)*	29	2.01	9

** Significant (5%)
* Marginally Significant (10%)
(T-Statistic in Parenthesis)

Table 10. Analysi' of Seasonal Characteristics of 'employment, ..male 25+, 1967-75

Month	a	b	Swinman Freq	D.W.	Average SI
January	-159 (-2.4)**	.251 (4.1)**	54	1.47	95
February	-14 (-.3)	.083 (1.9)*	41	1.78	71
March	-21 (-.6)	.042 (1.3)	33	2.56	23
April	-28 (-1.7)	-.003 (-.2)	16	1.70	-31
May	-19 (-.7)	-.078 (-3.3)**	26	2.66	-101
June	-4 (-.1)	-.063 (-1.6)	43	1.83	-71
July	30 (1.1)	-.067 (-2.8)**	25	3.17	-42
August	54 (1.1)	-.024 (-.6)	43	2.00	28
September	100 (3.4)**	-.0006 (-.03)	26	2.21	99
October	57 (.8)	-.038 (-.6)	64	1.80	16
November	2 (.1)	.002 (.1)	22	2.09	4
December	-53 (-1.5)	-.049 (-1.5)	26	2.50	-104

** Significant (5%)
* Marginally Significant (10%)
(T-Statistic in Parenthesis)

is clearly predominantly multiplicative. This series is certainly a very mixed situation with possibly some indications of additivity. But this evidence, when combined with other parts of the study is not sufficient to convince us that the current multiplicative procedure should be abandoned. (Table 6.)

In addition to these primary equations, three additional estimation forms were used incorporating a time trend variable (t) to test for a linear trend in either or both coefficients. The three equations were of the form:

$$SI = a + b(TC) + ct$$

$$SI = a + b(TC) + dt(TC)$$

$$SI = a + b(TC) + ct + dt(TC)$$

With so few observations, these should not be too heavily regarded and in fact played only a supplementary role in our deliberations.

Some of the foregoing results suggest that simultaneous additive and multiplicative factors may be operating in a single month. This possibility suggests that some sort of new combination technique should be investigated. In this regard, it should be noted that early work in seasonal adjustment at both the BLS and the Bureau of the Cenus (utilizing equations similar to those we have used for testing here) did experiment with combination approaches, but these were abandoned for a variety of reasons.

Results of Tests of Alternate Aggregation Composites

In view of the above findings, individual components of the total unemployment levels were alternately seasonally adjusted by the additive and the multiplicative methods of the X-11 program. Tests of seasonal stability and intensity of seasonal fluctuations of the total unemployment levels of various mixes of demographic groups (16-19, 20-24, 20 plus and 25 plus) by sex and type of adjustment were obtained.

Six aggregations of the various age/sex mixes selected for these tests:

1. Both sexes 16-19 - additive; both sexes 20+ - multiplicative.
2. Both sexes 16-19 - multiplicative; both sexes 20+ - multiplicative.
3. Both sexes 16-19 - additive; both sexes 20+ - additive.
4. Both sexes 16-24 - additive; both sexes 25+ - multiplicative.
5. Both sexes 16-19 - additive; males 20+ - multiplicative; females 20+ - additive.
6. Both sexes 16-19 - additive; males 20-24 - multiplicative; females 20-24 - additive; both sexes 25+ - multiplicative.

To determine which of the above mixes did a better job of filtering seasonal fluctuations from the various aggregations of series, certain measures were computed and examined. These included the F-statistic, the $\bar{I}/\bar{C}$ ratio, the Average Duration of Run (ADR) of the seasonal components and months for Cyclical Dominance (MCD). 4/

4/ The $\bar{I}/\bar{C}$ value is the ratio of the average month-to-month change in the cyclical component ($\bar{C}$) to the irregular component ($\bar{I}$), and as such, is a measure of the relative smoothness of the seasonally adjusted series--a small value indicating considerable smoothness and a large value indicating erratic month-to-month behavior. The ADR indicates the average number of months in which the component moves in the same direction, To test the quality of the result of these aggregation composites, the aggregation of the seasonally adjusted components were again subjected to the multiplicative option of the seasonal adjustment procedure. In a sense, the test series are adjustments of adjusted data, with the value of the F-statistics and the smoothness of the other measures indicating the relative presence of residual seasonality. (Because the tests for residual seasonality used the multiplicative procedure, option 2 would be expected to have the lowest F-statistic value.)

The results of these tests are shown in table 11. Certain observations can be drawn from the results:

(1) The F-statistics of options 4 and 6 indicate the presence of unsatisfactorily large residual seasonality, while the residual seasonality of the remaining options is fairly consistant at a lower level.

(2) The relatively larger $\overline{I}/\overline{C}$ value of the current procedure (option 2) indicates that pure multiplictive adjustment results in a less smooth seasonal adjusted series and the presence of a larger irregular component.

(3) The purely additive model (option 3) is the smoothest series, but this may be at the expense of passing some of the irregular to the seasonal since the low ADR of the irregular suggests the presence of a non-random irregular. This series has greater residual seasonal than our proposed (option 1).

(4) Although the relative stability of option 5 (with additive adjustment of adult women as well as teenagers) would indicate approximately the same result as option 1, the results of our tests of this option in previous time periods showed much more residual seasonality. The findings for this option are not consistant over time.

4/ - continued
with a larger value indicating a more smooth series. The MCD represents the minimum number of months needed for the average change, without regard to sign, in C to exceed the average percent change, without regard to sign, in I.

In summary, the findings relating to the quality of aggregation composites are largely consistent with our observations of the quality of the independendently adjusted component series. The stability of seasonality is enhanced by use of composites of multiplicative and additive adjustment. The series which seems to perform best in terms of our tests, for the 1967-75 period, was that obtained when teenagers (both sexes, 16-19 years of age) are adjusted by the additive procedure and adults (both sexes, 20 years of age and over) are adjusted by the multiplicative method. While the aggregation composite tests do not, in and of themselves, establish a clear case for preference of one option over others, the results are consistant with our preference for option 1.

Residual Derivation of Adjusted Unemployment

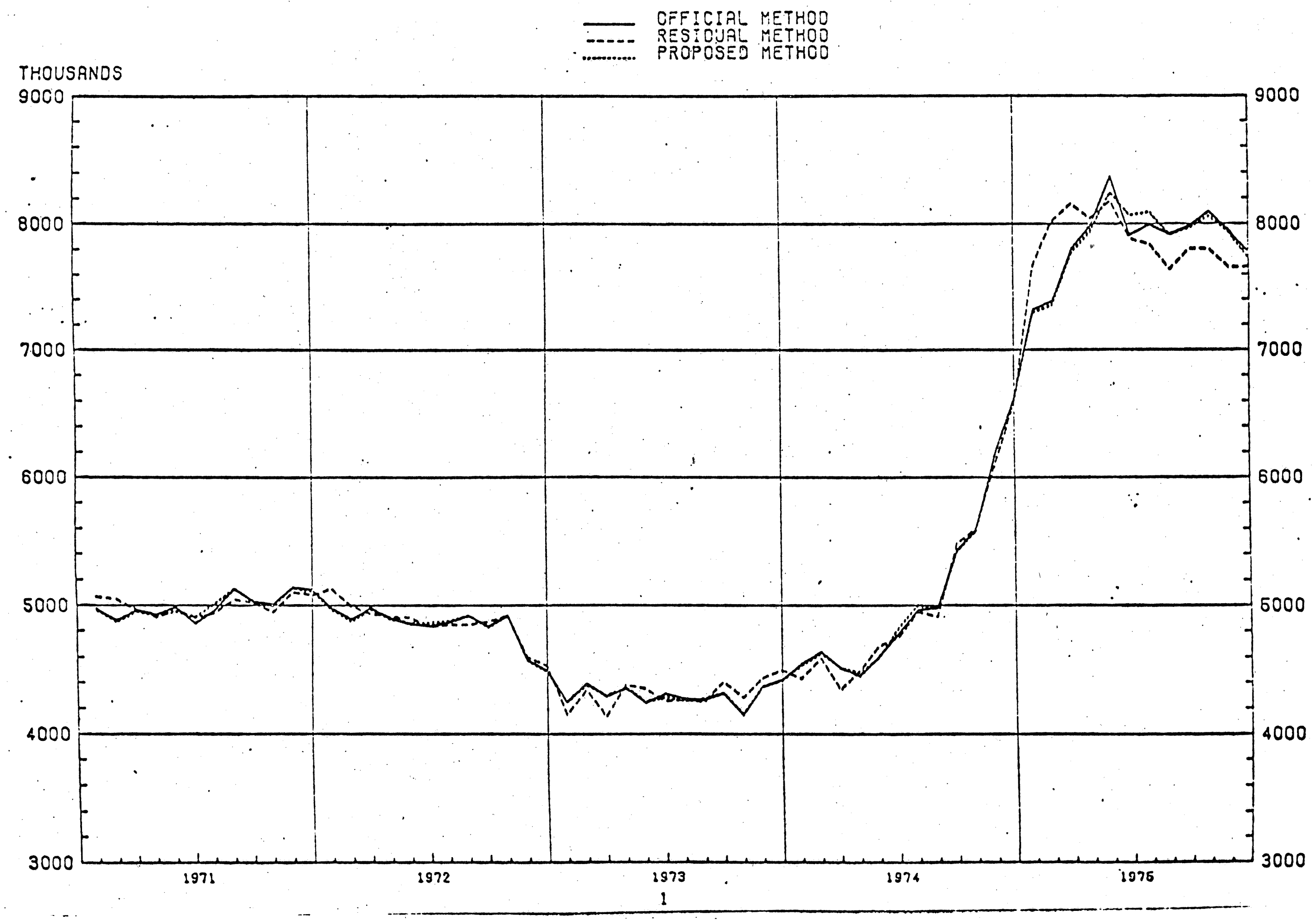
All of the above tests have assumed a normal aggregation procedure. However, some economic analysts have suggested that the seasonal fluctuations of the unemployment level would be reduced if adjusted unemployment were calculated by subtracting adjusted employment from directly adjusted labor force (the "residual" method). The logic behind this method holds that the extreme fluctuations obtained by independently adjusting the unemployment series would be minimized if the dominant influence of the larger and more stable employment and labor force series governed the result.

Table 11. Summary measures for selected aggregations, 1967 - 1975

Aggregations 1/	F test	I/C	I	C	Average duration of run			MED
					I	C	C I	
1974 - 1975								
1. BS16-19(A); BS20+(M)..........	.317	1.15	1.81	1.57	1.55	8.23	2.10	2
2. BS16-19(M); BS20+(M)..........	.298	1.25	1.97	1.58	1.55	5.63	2.10	2
3. BS16-19(A); BS20+(A)..........	.362	1.06	1.78	1.68	1.37	8.23	2.38	2
4. BS16-24(A); BS25+(M)..........	.716	1.12	1.84	1.63	1.55	6.29	2.28	2
5. BS16-19(A); M20+(M); F20+(A)..	.322	1.14	1.82	1.59	1.51	7.13	2.10	2
BS16-19, M20-24(M); F20-24(A); BS25+(M)....................	.671	1.15	1.85	1.61	1.43	8.23	2.10	2
sidual method....................	1.282	1.26	1.97	1.56	1.70	9.50	2.50	2

1/ (A) = additive method; (M) = multiplicative method; BS = both sexes.

UNEMPLOYMENT LEVELS AGGREGATED BY THREE DIFFERENT METHODS



A comparison of the results of residual calculation of the adjusted unemployment level and rate with the present procedures was conducted as part of the study. The residual level was level was calculated by aggregating directly adjusted civilian labor force for the four major age/sex groups and subtracting aggregated adjusted employment. Both labor force and employment for the groups were adjusted by the multiplicative procedure of the X-11 program. 5/

The residual and present method track each other closely, though not as closely as the present and proposed change methods. (See chart.) The major exception is 1975. In that year the residual method rose more rapidly to its high and then fell faster than the official or proposed series. Paul Samuelson noted a similar occurrence in the 1961 recession 6/ and a hint of that pattern exists in 1971. In light of the later turning points of other series and propensity of unemployment to lage at throughs, the 1975 behavior of the residual method seems not as reasonable as the other. (Table 12.) The dip of the residual series in early 1969 also seems unlikely. The more erratic behavior of the official method in 1971 is in part owing to the presence of some additive seasonal behavior which will be at least partly expunged as the result of this proposal.

5/ Both total employment and labor force for the major age/sex groups were found to have multiplicatively.

6/ Paul Samuelson, letter to the editor of the Sunday New York Times, November 12, 1961. Noted in Lovell, p. 993.

Table 12. Adjusted unemployment level and rate residual and official methods, 1975 1/

Month	Level		Rate	
	Residual	Official	Residual	Official
January..............	7,692	7,324	8.3	8.0
February.............	8,022	7,391	8.7	8.1
March................	8,163	7,804	8.8	8.5
April................	8,046	7,984	8.7	8.7
May..................	8,186	8,373	8.8	9.0
June.................	7,891	7,918	8.5	8.6
July.................	7,844	7,997	8.5	8.6
August...............	7,648	7,923	8.3	8.5
September............	7,809	7,984	8.4	8.6
October..............	7,807	8,098	8.4	8.7
November.............	7,659	7,950	8.5	8.5
December.............	7,668	7,783	8.2	8.4
Average 2/...........	7,869	7,877	8.5	8.5

1/ Adjustment by multiplicative option of X-11 method. Data adjusted for the period, January 1967-December 1975.

2/ Unadjusted annual average level = 7,830; rate = 8.5 percent.

The "bottom line" test of whether or not the residual method works better is whether it more satisfactorily filters out seasonal movements. To test this property of residual adjustment, the adjusted unemployment level derived from residual calculation (presumably free of seasonal fluctuations) was subjected once more to the multiplicative seasonal adjustment process. The results of this test indicated that there was more remaining seasonality after residual adjustment than there was for any of the composite methods (F-statistic + 1.28). Indeed, the seasonal component still was found to contribute about one-fourth of the variance in the presumably "fluctuation free" series over a one month span. The residual method evidently fails to adequately deseasonalize the series by the widest margin in June.

There are good theoretical reasons why the residual method may not apply as good a seasonal adjustment. In any seasonal adjustment there is an element of error or uncertainty in the derivation of the seasonal factor. This error of the adjustment is proportional to the irregularity of the series. This irregularity is the result of both erratic economic occurences and changes which arise only from sampling variability. Erratic economic movements are not likely to affect one method of adjustment more than the other. However, sampling error may.

The standard error of the monthly change in unemployment is 90,000 so the error of adjustment of that series would be proportional to 90,000. The standard error of total employment is 210,000 and of civilian labor force is 205,000. Thus, the error of adjustment in those series would be proportional to 210,000 and 205,000. Now if we assume the error in the two adjustments are independent (which seems to be a reasonable assumption) then the error in the difference of the two series will be in proportion to somewhat more than 200,000--or more than twice the error of adjustment in the direct case.

Recommendations and Implications

Present procedures have been incorrectly assuming a proportionality in the relationship between the magnitude of the seasonal change and the level of all adjusted series. While it is true that the multiplicative assumption best fits the data for the large majority of adjusted series, some series, particularly teenage unemployment, are better portrayed by an assumption of constant seasonal fluctuation without regard to level. These series would be more appropriately deseasonalized if additivity were assumed.

Therefore, our principal recommendation is that teenage unemployment series, and those few other unemployment series of which teenagers are the exclusive or major part, be adjusted using the additive procedure of the X-11 program. Other series which exhibit strong multiplicative relationships or for which neither option was found superior will continue to be adjusted using the multiplicative procedure.

A preliminary indication of the impact of the both the routine annual revision and the proposed new procedure on the monthly unemployment levels and rates is shown in tables 13, 14, and 15.

Table 13. "Implicit" multiplicative seasonal factors 1/ for unemployment rate, 3 procedures of computation, by month, 1976

Month (1)	Implicit factors: Initial computation (2)	Implicit factors: Routine revision (3)	Implicit factors: Proposed revision (4)	Change in factors: (3 - 2)	Change in factors: (4 - 3)	Change in factors: (4 - 2)
January	109.8	112.5	113.9	+2.7	+1.4	+4.1
February	111.0	112.3	113.8	+1.3	+1.4	+2.8
March	104.6	107.1	107.1	+2.5	0	+2.5
April	96.6	98.9	100.0	+2.3	+1.1	+3.4
May	90.2	92.2	93.3	+2.0	+1.1	+3.1
June	105.8	105.8	104.6	0	-1.2	-1.2
July	103.6	101.2	100.0	-2.4	-1.2	-3.6
August	97.6	96.5	96.5	-1.1	0	-1.1
September	97.6	94.3	94.2	-3.3	-.1	-3.4
October	90.7	89.7	90.7	-1.0	-1.0	0
November	94.0	91.8	91.8	-2.2	0	-2.2
December	94.0	92.9	94.0	-1.1	+1.1	0
Average change without regard to sign				1.8	0.8	2.3

1/ "Implicit" factors are derived by dividing original rate by adjusted rate, and are provided for purposes of illustration only, since adjusted rate is calculated by independent adjustment of 12 major age/sex components. Due to rounding of rate, factors are approximate.

Explanation of Columns:

(2) Implicit factors for 1976 assuming no revision. Based on experience through 1974.

(3) Implicit factors for 1976 assuming routine revision (all series multiplicatively adjusted). Based on experience through 1975.

(4) Implicit factors for 1976 assuming proposed revision (teenage unemployment additively adjusted). Based on experience through 1975.

Source: Bureau of Labor Statistics
January 1976

NOT FOR PUBLICATION

Table 14. Impact of proposed procedure on 1975 unemployment rates, published and revised

Month	Current method			Proposed method			Difference between 1975 published and proposed revision (5) - (1)
	Initial computation (1)	Routine revision (2)	Difference (2) - (1)	Initial computation (4)	Routine revision (5)	Difference (5) - (4)	
January..............	8.2	8.0	-.2	8.1	7.9	-.2	-.3
February.............	8.2	8.1	-.1	8.1	8.0	-.1	-.2
March................	8.7	8.5	-.2	8.7	8.5	-.2	-.2
April................	8.9	8.7	-.2	8.8	8.6	-.2	-.3
May..................	9.2	9.0	-.2	9.0	8.9	-.1	-.3
June.................	8.6	8.6	-	8.7	8.7	-	+.1
July.................	8.4	8.6	+.2	8.6	8.7	+.1	+.3
August...............	8.4	8.5	+.1	8.3	8.5	+.2	+.1
September............	8.3	8.6	+.3	8.3	8.6	+.3	+.3
October..............	8.6	8.7	+.1	8.5	8.6	+.1	-
November.............	8.3	8.5	+.2	8.3	8.5	+.2	+.2
December.............	8.3	8.4	+.1	8.2	8.3	+.1	-

Explanation of Columns

(1) Employment and unemployment for major age/sex groups adjusted multiplicatively applying 1974 factors to 1975 data. These data were published monthly during 1975.

(2) Seasonality revised to incorporate 1975 experience.

(4) Employment and adult unemployment adjusted multiplicatively; teenage unemployment adjusted by additive procedure. Calculated by applying 1974 factors to 1975 data. Would have been published in 1975 if the new procedure were in effect.

(5) Seasonally adjusted to incorporate 1975 experience.

SOURCE: Bureau of Labor Statistics, January 1976

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Table 15. Impact of proposed procedure on 1975 adjusted unemployment level

(In thousands)

Month	Current method			Proposed method		
	Initial computation (1)	Routine revision (2)	Difference (2) - (1)	Initial computation (4)	End-of-year revision (5)	Difference (5) - (4)
January..........	7,529	7,324	-205	7,450	7,297	-153
February..........	7,484	7,391	-93	7,449	7,360	-89
March..........	7,980	7,804	-176	7,947	7,770	-177
April..........	8,175	7,984	-191	8,099	7,941	-158
May..........	8,538	8,373	-165	8,386	8,250	-136
June..........	7,896	7,918	+22	8,09	8,071	-21
July..........	7,838	7,997	+159	7,977	8,096	+119
August..........	7,794	7,923	+129	7,765	7,924	+159
September..........	7,773	7,984	+211	7,764	7,970	+206
October..........	8,003	8,098	+95	7,961	8,062	+101
November..........	7,701	7,950	+249	7,693	7,939	+246
December..........	7,768	7,783	+15	7,685	7,733	+48

Explanation of Columns

(1) Unemployment levels for major age/sex groups adjusted multiplicatively applying 1974 factors to 1975 data (as published).

(2) Seasonality revised to incorporate 1975 experience.

(4) Adult unemployment level adjusted multiplicatively; teenage unemployment by the additive method. Calculated by applying 1974 factors to 1975 data. Would have been published in 1975 if new procedure were in effect.

(5) Seasonality revised to incorporate 1975 experience.

SOURCE: Bureau of Labor Statistics, January 1976

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Table 13 contains the implicit multiplicative seasonal factors for the unemployment rate as originally published and as they would have been revised through routine revisions and by the introduction of the new procedures. It will be noted that the routine revision is responsible for about twice as much revision as the introduction of the new techniques. It will also be noted that while individual months show small differences, the difference in the overall pattern between months is larger, especially in the critical May-June period. The importance of this difference can be seen in table 14. When the June data were published in 1975, they showed a 0.6 drop in the rate which the routine revision would change to a 0.4 drop. However, had the revised procedure been in use, a drop of 0.3 would have first been reported, and been revised only to 0.2.

The revision should apply to adjustment procedures for 14 independently adjusted series, and would require minor modification in the end-of-year processing program. The series that would be additively adjusted are:

	NIH Code	BLS Code
Unemployed, male 16 - 19..................	030	059520 10
" female 16 - 19.................	041	059520 20
" male 16 - 17..................	031	060520 10
" female 16 - 17................	042	060520 20
" male 18 - 19..................	032	067520 10
" female 18 - 19................	043	067520 20
Male 16 - 19 white............	115	059520 11
" " nonwhite.........	117	059520 12
Female 16 - 19 white..........	119	059520 21
" " nonwhite.......	121	059520 22
Full time 16 - 19.............	166	059527 00
Part time 16 - 19.............	172	059529 00
New entrants..................	181	153 00
Never worked before...........	082	524 00

In considering the optimum period for the revision of historical data, four options are considered: (1) to inception of series, usually 1948; (2) to 1967, when the break in series occurred; (3) to 1971, the normal revision period; (4) to January 1975 only.

The study determined that there are major differences in the pre- and post-1967 period, perhaps because of the incorporative of the availability test in the 1967 unemployment definition change. Prior to 1967, both teenage series behaved predominately multiplicatively; since 1967, the behavior has been additive. Thus, revisions should be carried back no further than 1967.

Moreover, the additivity of the teenage series exists over the entire post-1967 period, although certainly it was strengthened by inclusion of 1975 data. Revision of only 1975 data would mask these relationships, and possibly raise more questions than it would solve.

Thus, the series should be revised for more than one, but no more than 9 years. Here practicality suggests the optimum revision period: the BLS policy has been to revise only the last 5 years of data, and end-of year processing programs are not written to provide this function. Although 1971-74 revisions will be minimal, <u>it is recommended that the policy of revising the last 5 years be followed in revising the seasonal adjustment procedure</u>. The major impact of this revision period, as might be expected will be seen in 1975, both periods of rapidly rising unemployment. Other years are little affected.

Finally, some of the results of this study suggest that simultaneous additive and multiplicative factors may be operating in a single month. This possibility suggests that some sort of new combination technique should be investigated. In this regard, it should be noted that early work in seasonal adjustment at both the BLS and the Bureau of the Census (utilizing equations similar to those we have used for testing here) did experiment with combination approaches, but these were abandoned for a variety of reasons. These should be explored once again.

The British Central Statistical Office experienced similar difficulties with seasonally adjusting their unemployment, as have other European countries and Canada. Tests performed by the British on their own data and that of Canada and Holland showed that since 1966-67 there has been a radical shift in the seasonal behavior of unemployment from a multiplicative model to an additive one. The British experience appears to be purely additive and they have adopted the additive X-11. Statistics Canada has been considering a similar move. The Dutch case is more complex and still under study. The German statistical agency has recently adopted a new procedure whereby some components are adjusted additively and some multiplicatively. The British have kindly offered to test our data with some of their techniques, but those results are not yet available.

예컨대, 아래의 표 19-1 에서 첫째의 趨勢變動은 直線傾向으로 가정하여 400 에서 시작하여 매월 1 單位씩 증가한다. 둘째의 循環變動은 4 年간에 걸쳐

표 19-1 加法모델의 時系列資料

年 度	月	T 趨勢變動	C 循環變動	S 季節變動	I 不規則變動	實際系列
1次年度	1	401	−137	−100	5	169
	2	402	−126	−150	1	127
	3	403	−114	−50	−4	235
	4	404	−101	20	−9	314
	5	405	−87	20	5	343
	6	406	−72	20	−9	345
	7	407	−56	20	6	377
	8	408	−39	−20	−3	346
	9	409	−21	−40	0	348
	10	410	−4	30	6	442
	11	411	12	20	−3	440
	12	412	27	230	5	674
2	1	413	41	−100	−3	351
	2	414	54	−150	−8	310
	3	415	66	−50	3	434
	4	416	77	20	4	517
	5	417	87	20	−1	523
	6	418	96	20	1	535
	7	419	104	20	6	549
	8	420	111	−20	−1	510
	9	421	117	−40	5	503
	10	422	122	30	0	574
	11	423	126	20	−2	567
	12	424	129	230	−6	777
3	1	425	131	−100	1	457
	2	426	132	−150	−5	403
	3	427	133	−50	9	519
	4	428	130	20	1	579
	5	429	125	20	−3	571
	6	430	116	20	−3	563
	7	431	102	20	−3	550
	8	432	82	−20	−9	485
	9	433	57	−40	−2	448
	10	434	27	30	2	493
	11	435	−1	20	−7	447
	12	436	−27	230	7	646
⋮	⋮	⋮	⋮	⋮	⋮	⋮
8	1	485	−51	−100	8	342
	2	486	−73	−150	1	264
	3	487	−93	−50	−8	336
	4	488	−109	20	−4	395
	5	489	−121	20	−1	387
	6	490	−129	20	9	390
	7	491	−135	20	2	378
	8	492	−139	−20	−3	330
	9	493	−141	−40	4	316
	10	494	−142	30	−8	374
	11	495	−143	20	0	372
	12	496	−143	230	−5	578

(註) John C.G. Boot and Edwin B. Cox, *ibid.*, pp. 443～445.

그림 19-1 時系列의 構成要素(加法모델)

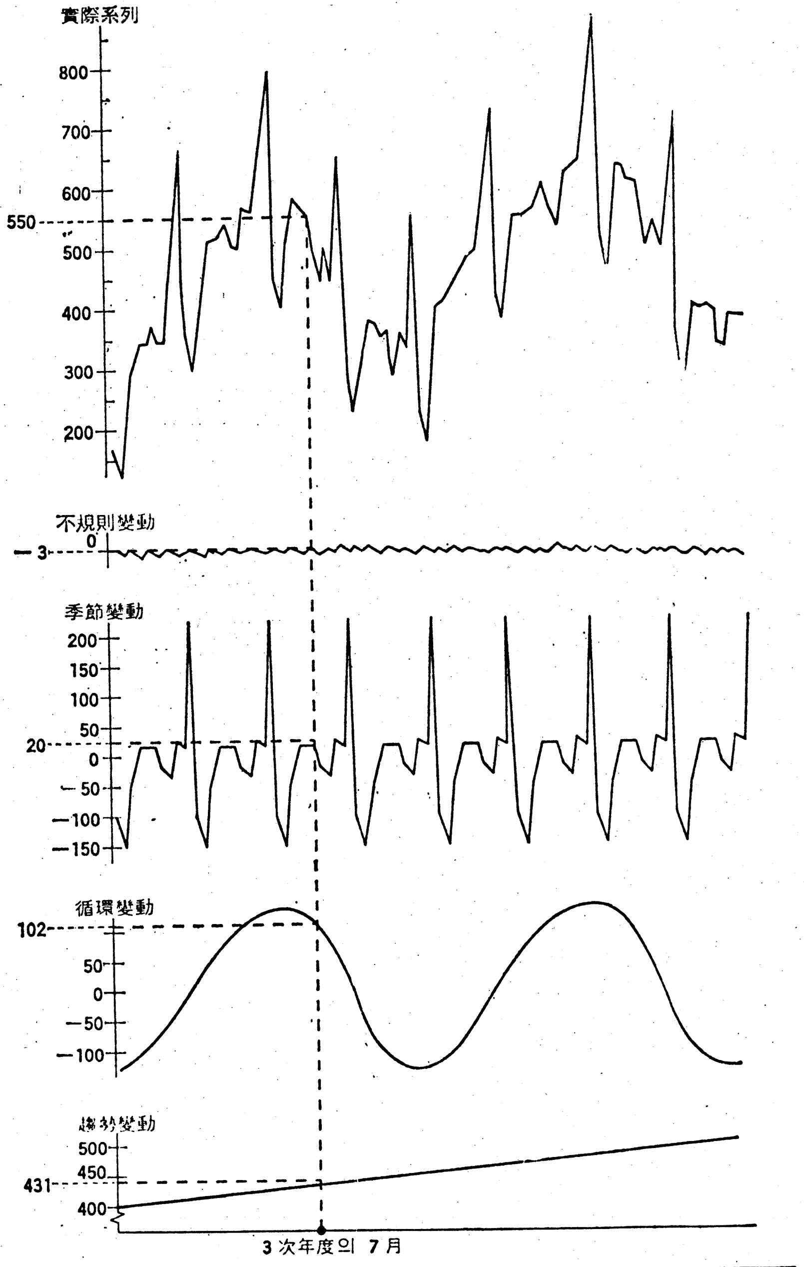
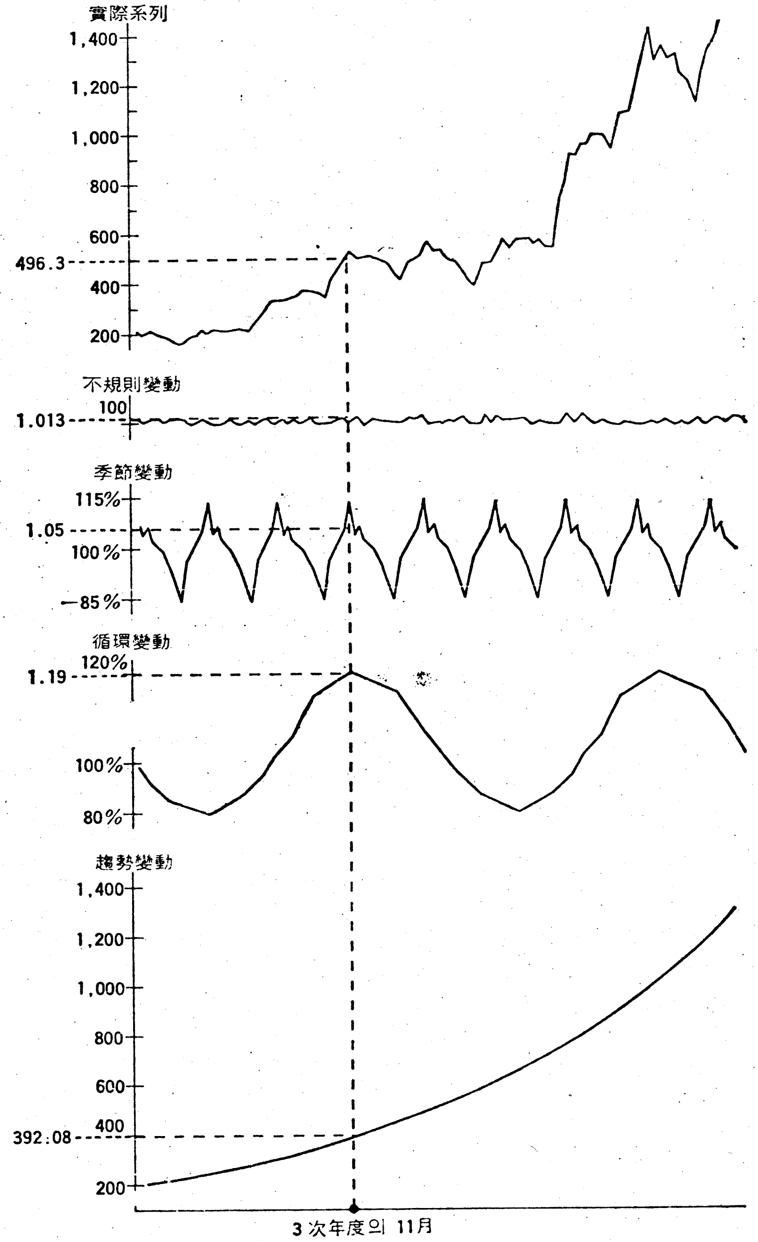


表 19-2

年度	月	*T* 趨勢變動	*C* 循環變動	*S* 季節變動	*I* 不規則變動	實際系列
1次年度	1	200	98%	105%	100.5%	206.8
	2	204	95	107	100.1	207.6
	3	208.08	92	103	101.5	200.1
	4	212.24	90	101	101.0	194.9
	5	216.49	88	98	101.4	189.3
	6	220.81	86	95	100.9	182.0
	7	225.23	85	90	101.2	174.4
	8	229.73	84	85	98.5	161.6
	9	234.32	83	97	99.3	187.3
	10	239.01	82	101	101.3	200.5
	11	243.79	81	105	99.5	206.3
	12	248.67	80	113	98.6	221.7
2	1	253.64	81	105	99.0	213.6
	2	258.71	82	107	100.0	227.0
	3	263.88	83	103	101.4	228.7
	4	269.16	84	101	98.2	224.2
	5	274.54	85	98	99.9	228.5
	6	280.03	86	95	98.7	225.8
	7	285.63	88	90	101.0	228.5
	8	291.34	90	85	100.3	223.5
	9	297.17	92	97	98.2	260.4
	10	303.11	95	101	98.8	287.3
	11	309.17	98	105	100.7	319.1
	12	315.35	100	113	99.6	354.9
3	1	321.65	102	105	101.4	349.3
	2	328.08	107	107	100.2	369.3
	3	334.64	108	103	100.1	372.6
	4	341.33	110	101	102.3	387.9
	5	348.16	112	98	100.0	382.1
	6	355.12	114	95	98.9	380.4
	7	362.22	115	90	99.2	371.9
	8	369.46	116	85	99.6	362.8
	9	376.85	117	97	99.5	425.5
	10	384.39	118	101	100.3	459.5
	11	392.08	119	105	101.3	496.3
	12	399.92	120	113	98.6	534.7
⋮	⋮	⋮	⋮	⋮	⋮	⋮
8	1	1,055.31	119	105	99.1	1,306.7
	2	1,076.42	118	107	100.0	1,359.1
	3	1,097.95	117	103	99.1	1,311.2
	4	1,119.91	116	101	101.2	1,327.8
	5	1,142.31	115	98	98.6	1,269.4
	6	1,165.16	114	98	99.0	1,288.7
	7	1,188.46	112	90	100.7	1,206.4
	8	1,212.23	110	85	100.7	1,141.4
	9	1,236.47	108	97	98.4	1,274.6
	10	1,261.20	105	101	101.1	1,352.2
	11	1,286.42	102	105	101.2	1,394.3
	12	1,312.15	100	113	98.1	1,454.6

(註) John C.G. Boot and Edwin B. Cox. *ibid.*, pp.449~451.

그림 19-2 時系列의 構成要素(比例모델)



Cross-Classification of Cyclical Indicators by Economic Process and Cyclical Timing

A. Timing at Business Cycle Peaks

Economic Process / Cyclical Timing	I. EMPLOYMENT AND UNEMPLOYMENT (18 series)	II. PRODUCTION AND INCOME (10 series)	III. CONSUMPTION, TRADE, ORDERS, AND DELIVERIES (13 series)	IV. FIXED CAPITAL INVESTMENT (18 series)	V. INVENTORIES AND INVENTORY INVESTMENT (9 series)	VI. PRICES, COSTS, AND PROFITS (17 series)	VII. MONEY AND CREDIT (26 series)
LEADING (L) INDICATORS (62 series)	Marginal employment adjustments (6 series) Job vacancies (2 series) Comprehensive employment (1 series) Comprehensive unemployment (3 series)	Capacity utilization (2 series)	New and unfilled orders and deliveries (6 series) Consumption (2 series)	Formation of business enterprises (2 series) Business investment commitments (5 series) Residential construction (3 series)	Inventory investment (4 series) Inventories on hand and on order (1 series)	Stock prices (1 series) Commodity prices (1 series) Profits and profit margins (7 series) Cash flows (2 series)	Money flows (3 series) Real money supply (2 series) Credit flows (4 series) Credit difficulties (2 series) Bank reserves (2 series) Interest rates (1 series)
ROUGHLY COINCIDENT (C) INDICATORS (23 series)	Comprehensive employment (1 series)	Comprehensive output and real income (4 series) Industrial production (4 series)	Consumption and trade (4 series)	Backlog of investment commitments (1 series) Business investment expenditures (5 series)			Velocity of money (2 series) Interest rates (2 series)
LAGGING (Lg) INDICATORS (18 series)	Duration of unemployment (2 series)			Business investment expenditures (1 series)	Inventories on hand and on order (4 series)	Unit labor costs and labor share (4 series)	Interest rates (4 series) Outstanding debt (3 series)
TIMING UNCLASSIFIED (U) (8 series)	Comprehensive employment (3 series)		Trade (1 series)	Business investment commitments (1 series)		Commodity prices (1 series) Profit share (1 series)	Interest rates (1 series)

B. Timing at Business Cycle Troughs

Economic Process / Cyclical Timing	I. EMPLOYMENT AND UNEMPLOYMENT (18 series)	II. PRODUCTION AND INCOME (10 series)	III. CONSUMPTION, TRADE, ORDERS, AND DELIVERIES (13 series)	IV. FIXED CAPITAL INVESTMENT (18 series)	V. INVENTORIES AND INVENTORY INVESTMENT (9 series)	VI. PRICES, COSTS, AND PROFITS (17 series)	VII. MONEY AND CREDIT (26 series)
LEADING (L) INDICATORS (47 series)	Marginal employment adjustments (3 series)	Industrial production (1 series)	New and unfilled orders and deliveries (5 series) Consumption and trade (4 series)	Formation of business enterprises (2 series) Business investment commitments (4 series) Residential construction (3 series)	Inventory investment (4 series)	Stock prices (1 series) Commodity prices (2 series) Profits and profit margins (6 series) Cash flows (2 series)	Money flows (2 series) Real money supply (2 series) Credit flows (4 series) Credit difficulties (2 series)
ROUGHLY COINCIDENT (C) INDICATORS (23 series)	Marginal employment adjustments (2 series) Comprehensive employment (4 series)	Comprehensive output and real income (4 series) Industrial production (3 series) Capacity utilization (2 series)	Consumption and trade (3 series)	Business investment commitments (1 series)		Profits (2 series)	Money flow (1 series) Velocity of money (1 series)
LAGGING (Lg) INDICATORS (40 series)	Marginal employment adjustments (1 series) Job vacancies (2 series) Comprehensive employment (1 series) Comprehensive and duration of unemployment (5 series)		Unfilled orders (1 series)	Business investment commitments (2 series) Business investment expenditures (6 series)	Inventories on hand and on order (5 series)	Unit labor costs and labor share (4 series)	Velocity of money (1 series) Bank reserves (1 series) Interest rates (8 series) Outstanding debt (3 series)
TIMING UNCLASSIFIED (U) (1 series)							Bank reserves (1 series)

Comments on the Paper

"A SURVEY AND COMPARATIVE ANALYSIS OF

VARIOUS METHODS OF SEASONAL ADJUSTMENT"

Estela Bee Dagum

Statistics Canada

(For presentation at the NBER/Bureau of the Census Conference on Seasonal Analysis of Economic Time Series, Washington, D.C., September 9-10, 1976)

Comments on the paper

"A SURVEY AND COMPARATIVE ANALYSIS OF VARIOUS METHODS OF SEASONAL ADJUSTMENT"

Estela Bee Dagum*

In his thorough and interesting study on the comparison of various methods of seasonal adjustment officially adopted by Statistical Bureaux(1), Professor Kuiper reaches the following conclusions:

(i) There are no significant differences among the seasonally adjusted values obtained by each method for the total period of the series analysed (1953-75). This is shown in the corresponding Tables of Inequality Coefficients, Correlation Coefficients and Summary Measures.

(ii) There are significant differences in the current seasonally adjusted values (year 1975) produced by the various methods as shown in the Tables of Inequality Coefficients and Correlation Coefficients(2). To a lesser degree this is also found to be true for the seasonally adjusted figures of the last three years of observations.

(iii) The smallest mean algebraic error and mean absolute error in the current seasonal factors is obtained by the Statistics Canada X-11-ARIMA method(3) as shown in the Table of Stability Indicators.

I will comment on these three points and show that they are not exclusive of the series considered but are the results of the underlying basic assumptions of the methods surveyed. These methods belong to the class that estimates the seasonal component by purely mechanical procedures and not on the basis of a causal explanation of the seasonal variation.

The time series probabilistic model of these methods is the classical one known in the Theory of Stochastic Processes as "error model". (Anderson, T.W. [1], Dagum, E.B. [3]).

In an error model, the generating mechanism of a time series is assumed to be composed of a systematic component (sometimes called "signal") which is a completely determined function of time f(t), and a random component (the noise) u_t which obeys a probability law. The random element is supposed to be purely random, i.e. identically distributed with constant mean, constant variance and zero autocorrelation.

The signal of the observed time series, like the random element, is not observable and assumptions must be made concerning its behaviour.

In general, two types of functions of time are assumed by these methods. One is a polynomial of fairly low degree which fulfills the assumption that the economic phenomenon moves slowly, smoothly and progressively through time (the trend). The other is a linear combination of sines and cosines of different amplitudes and frequencies, which stands for cyclical oscillations, strictly periodic or not (the cycle and the seasonality).

When the systematic part is assumed to be approximated closely by simple functions of time over the entire range of the series, the statistical technique used is that of regression analysis (classical least squares theory).

The methods surveyed, however, make the assumption that although the signal is a smooth function of time, it cannot be approximated well by simple functions over the entire range. Therefore, they use the statistical technique of smoothing.

The general basis for most smoothing procedures is to fit a polynomial to 2n+1 successive observations and use this fitted polynomial to estimate the trend-cycle at the middle value. Since the estimates of the parameters of the polynomial are linear in the observed values, say X_{t+k}, the smoothed series has the form,

$$(1)\ X_t^* = \sum_{k=-n}^{n} c_k X_{t+k},\quad t = n+1, \ldots, T-n$$

The (1) is a moving weighted average of the observed values where the c_k 's are constant weights, n is a positive integer and 2n+1 is the span of the average. It is called "moving" because the weights are moved one position to the right relative to the X_t to obtain successive smoothed values.

The process of fitting a polynomial by the moving average technique consists of determining the weights c which are functions of the length of the moving average, 2n+1, and the degree of the polynomial to be fitted, say, p. For a given p, the variance of the smoothed series decreases with increasing n, and for a given n, the variance goes up with increasing p (Anderson [1, p. 54]). The methods surveyed fix p for each systematic component and let the n vary. Therefore, depending on the n, some methods are more flexible than others. Although this does not affect their historical performance, it indeed introduces differences in their current performance.

The basic properties of moving averages are: (a) scale preservation, (b) superposition principle, and (c) time invariance.

The property of scale preservation means that if the original series X_t is amplified by a given constant, the smoothed series X_t^* will be amplified by the same factor.

The superposition principle means that if two time series are added together and presented as the input to the given moving average, then the output will be the sum of the two smoothed time series which would have resulted from using the original series as inputs to the moving average separately. That is $(X_t + Y_t)^a = X_t^a + Y_t^a$, where the superscript a indicates that a moving average has been applied to the original series(4).

Properties (a) and (b) are a consequence of the fact that moving averages are linear transformations (often called smoothing linear filters).

The time invariant property means that if two inputs to the moving average are the same except for a relative time displacement then the outputs will also be the same except for the time displacement, i.e. if $(X_t)^a = Z_t$ then $(X_{t+h})^a = Z_{t+h}$. In other words, no matter what time in history a given input is presented to the filter, it will always respond in the same way. Its "behaviour" does not change with time.

The methods surveyed apply symmetric filters to estimate the components that fall in the middle of their span, say, 2n+1, and asymmetric filters to the n- first and last observations(5).

The sum of the weights of both kinds of filters equals one, therefore, the mean of the original series is unchanged in the filtering process(6).

It is desirable in filter design that the filter does not displace in time the components of the output relative to the input, i.e., the filter should not introduce phase shifts.

The symmetric moving averages have a phase shift function that is equal to zero or $\pm\Pi$. A phase shift of $\pm\Pi$ is interpreted as a reversal of polarity of a sinusoid which means that its maxima are turned into minima and vice versa.

For practical purposes, however, symmetric moving averages act as though the phase shift is null. This is because the sinusoids that have a phase shift of $\pm 180^0$ in the filtering process are cycles of short periodicities (annual or less) and moving averages tend to suppress or significantly reduce their presence in the output.

On the other hand, the asymmetric filters introduce phase shifts for most of the components of the original time series(7).

Aside from the fact that the asymmetric filters of these methods are bound to introduce phase shifts, the functions not affected by these filters are different from those corresponding to the symmetric filters.

In effect, the symmetric moving averages that are applied to estimate the trend-cycle component reproduce the middle observation of a third degree polynomial within the span of the filter. The fact that the trend is assumed to follow a cubic over an interval of short duration (one or two years approximately) makes the assumptions of these methods quite adequate for the historical adjustment of a large class of economic time series.

The same conclusions are valid for the symmetric filters that estimate the seasonal component; they can fit closely a local linearly moving seasonality. These methods assume that the seasonal pattern changes gradually with only occasional reversals in direction(8).

For current estimation, however, more rigid patterns of behaviour are assumed, namely, a straight line representing the trend-cycle and a stable seasonality. These more restrictive assumptions generally produce systematic errors in the current seasonally adjusted values which are gradually corrected as the series is enlarged by inserting more years of observations.

Since the implicit functions calculated by the symmetric filters of these methods are different from those corresponding to the asymmetric filters, the historical seasonal adjustment will always differ significantly from the current one, except for the trivial (non-existent) case of series with a constant trend-cycle and a stable seasonality.

There will also be significant differences among the current estimates obtained by the various methods. These differences will be more apparent for those series that are highly irregular or have extreme values present in the most recent years. This is due to the short length of the asymmetric filters which does not allow a significant reduction in the variance of the smoothed series and to the fact that these methods use different procedures and sigma limits for the replacement of the outliers.

The third finding of Kuiper's study, i.e. the smallest total error in the current seasonal factors is produced by the X-11-ARIMA method, is also explainable by the basic properties of this method.

The X-11-ARIMA was mainly developed to generate seasonal factor forecasts from the combination of two filters: (1) the filters of Autoregressive Integrated Moving Averages (ARIMA) models to forecast raw data; and (2) the filters of Census II-X-11 variant to seasonally adjust current observations (Dagum, [4]).

This procedure proved to be superior to the X-11 program in the sense that the size of the total error in the monthly forecasts and also in the current seasonal factors (measured by the monthly absolute means) was significantly smaller for the twelve months and the same happened for the bias (measured by the monthly algebraic means).

The main reasons for significant reductions in the total error of the seasonal factor forecasts and current seasonal factors are:

(1) The seasonal factor forecasts of X-11-ARIMA are obtained from forecasted raw data whereas the X-11 method forecasts from estimated seasonal factors and it is well known that the seasonal factors for the last three years are less reliable.

(2) The forecasting filter of the X-11 method is the same for all series while in the ARIMA models the forecasting filters depend on the model chosen and the parameter estimates. The ARIMA filters are very flexible and are able to pick up the most recent movements of the series.

(3) The trend-cycle estimate for the last observation is made with the central weights of the Henderson's moving averages (the same for the centered 12-term moving average) which are capable of reproducing a cubic in their time interval. This is very important for years with turning points since the X-11 program applies the asymmetric weights of the Henderson which only estimate well a linear trend-cycle.

(4) The replacement of the extreme values for the last two years of data is improved. In effect, by adding one more year of data (with no extremes since they are forecasts) a better estimate of the variance of the irregulars is obtained.

(5) The sets of weights applied to the seasonal-irregular ratios (differences) are closer to the central weights, and thus, the moving seasonality can be estimated with more accuracy.

Although Professor Kuiper's analysis was made for the current seasonal factors, I obtained similar results for the seasonal factor forecasts. The mean algebraic error and the mean absolute error were reduced by approximately 40% and 20% with respect to those of X-11. Moreover, the Wilcoxon Signed Rank test indicated that the differences for each month were significant and in favor of X-11-ARIMA.

This is a very important conclusion, especially if one takes into account that producers of current seasonally adjusted data tend to use the seasonal factor forecasts more often than to rerun the series each time that a new observation is added to it.

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Footnotes

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(1) The methods analysed are: (1) the U.S. Bureau of the Census Method II-X-11 variant; (2) Statistics Canada X-11-ARIMA Method; (3) Burman Method of the Bank of England; (4) Berlin Method, ASA-II; (5) the Method of the Statistical Office of the European Economic Communities of Brussels, and (6) the Method of the Dutch Central Planning Bureau.

(2) The current seasonally adjusted values were obtained by applying current seasonal factors from data to December 1975 and not seasonal factor forecasts.

(3) The total error is defined as the difference between the current seasonal factor S_{ij}^{i} and the estimate of the same seasonal factor when the series is enlarged with three more years of observations, S_{ij}^{i+3}.

The two statistics chosen to determine which of the methods generates better current seasonal factors are (a) the mean algebraic error, and (b) the mean absolute error.

(4) In practise, however, the equality is not fulfilled by the methods analysed because of non-linearities introduced at different stages of the calculations, as for example, in the replacement of the extreme values.

(5) The only exception being the Berlin method ASA-II that applies an asymmetric filter for trend-cycle removal but with a weighting scheme based on a third degree polynomial.

(6) The sum of the weights of a filter determines the ratio of the mean of the smoothed series to the mean of the original series assuming that these means are computed over periods long enough to insure stable results.

(7) The necessary and sufficient condition for a linear filter to have a phase shift function $\Phi(\lambda)=0$ for all λ is that its transfer function be real valued and non-negative definite for all λ. Symmetric filters have real valued but not necessarily non-negative transfer functions which leads to the possibility that $\Phi(\lambda)=\pm\Pi$ for some frequencies. A digital filter with (real valued) weights $\{c_k: k=o,\pm1, \ldots\}$ is said to be non-negative definite if for every positive integer n and complex number a_k, $k=o,\pm1, \ldots\pm n$, we have $\Sigma^n_{j=-n} \Sigma^n_{k=-n} a_j \bar{a}_k c_{jk} > o$. (Koopmans, L.H. [4, p. 206-207]).

(8) Series with abrupt or rapid changes in the seasonal variation cannot be seasonally adjusted properly. Sudden changes in the seasonal amplitude can be found, for example, in agricultural series where the level varies considerably from year to year and in series such as unemployment, which undergo rapid changes in composition when the economy changes from expansion to recession and back to expansion.

(9) It is assumed that the relationship among the time series components is multiplicative. For an additive model, the words "difference" and "subtracting" are subtituted for "ratio" and "dividing".

APPENDIX

STATISTICS CANADA X-11-ARIMA METHOD OF SEASONAL ADJUSTMENT

1. Introduction

The Statistics Canada X-11-ARIMA method of seasonal adjustment is a modified version of the Bureau of the Census Method II-X-11 variant that consists of enlarging unadjusted series with one year of forecasted raw data and then seasonally adjusting the enlarged series with the X-11 program. The forecasts of the raw data are made by ARIMA (autoregressive integrated moving averages) models of the Box and Jenkins type which have been identified and fitted to the original series.

The seasonal factor forecasts are thus obtained from the forecasted raw data and their estimation results from the combination of two filters: (1) the filters of ARIMA models to forecast raw data, and (2) the filters of the X-11 program to seasonally adjust current observations.

This new technique produces seasonal factor forecasts and current seasonal factor superior to those of the Census Method II-X-11 in the sense that the mean absolute error and the mean algebraic error of the seasonal factors is significantly smaller for the twelve months.

When applied to Canadian and U.S. series, the reduction found was about 40% in the bias and 20% in the absolute value of the total error.

Another advantage of the X-11-ARIMA is that if current seasonal factors are used to obtain current seasonally adjusted data, there is no need to revise the series more than twice. For many series just one revision will produce seasonal factors that are "final" in a statistical sense.

The X-11-ARIMA also provides a univariate time series model that describes the behaviour of the unadjusted series. Confidence intervals can be constructed for the original observations and since the one-step forecast is an unbiased minimum mean square error forecast it can be used by producers of raw data as a benchmark for the last available figure.

2. The Forecasting Filters of ARIMA Models and their Properties

The ARIMA models used for forecasting the unadjusted series are of the general multiplicative type (Box, G.E.P. and Jenkins, G.M., [2], that is:

$$(1)\quad \phi_p(B)\Phi_P(B^s)\Delta^d\Delta_s^D Z_t=\Theta_q(B)\Theta_Q(B^s)a_t$$

where s denotes the periodicity of the seasonal component (equal to 12 for monthly series); B denotes the backward operator, i.e. $BZ_t=Z_{t-1}$; $B^sZ_t=Z_{t-s}$; $\nabla^d=(1-B)^d$ is the ordinary difference operator of order d; $\nabla_s^D=(1-B^s)^D$ is the seasonal difference operator of order D; $\phi_p(B)$ and $\Phi_P(B^s)$ are stationary autoregressive operators (they are polynomials in B of degree p and in B^s of degree P, respectively); $\Theta_q(B)$ and $\Theta_Q(B^s)$ are invertible moving average operators (they are polynomials in B of degree q and in B^s of degree Q, respectively); a_t is a purely random process.

The general multiplicative model (1) is said to be of order (p,d,q) $(P,D,Q)_s$. Its forecasting function can be expressed in different forms.

For computational purpose, the difference equation form is the most useful. Thus, at time t+l the ARIMA model (1) may be written:

$$(2)\quad Z_{t+\ell}=\psi_1 Z_{t+\ell-1}+\ldots+\psi_m Z_{t+\ell-m}-a_{t+\ell}-\pi_\ell a_{t+\ell-1}-\ldots-\pi_n a_{t+\ell-n}$$

where m=p+s.P+d+s.D and n=q+s.Q; $\psi(B)=\phi_p(B)\Phi_p(B^s)\nabla^d\nabla_s^D$ is the general autoregressive operator; $\pi(B)=\Theta_q(B)\Theta_Q(B)$ is the general moving average operator. For example, if the ARIMA model is of order $(2,1,1)(0,1,1)_{12}$ the difference equation form that generates the observations $Z_{t+\ell}$, is:

$$(3)\quad Z_{t+\ell}=(1+\phi_1)Z_{t+\ell-1}+(\phi_2-\phi_1)Z_{t+\ell-2}-\phi_2Z_{t+\ell-3}+Z_{t+\ell-12}-(1+\phi_1)Z_{t+\ell-13}$$
$$+(\phi_1-\phi_2)Z_{t+\ell-14}+\phi_2Z_{t+\ell-15}+a_{t+\ell}-\Theta a_{t+\ell-1}-\Theta a_{t+\ell-12}+\Theta\Theta a_{t+\ell-13}$$

Standing at origin t, to make a forecast $\hat{Z}_t(\ell)$ of $Z_{t+\ell}$, the conditional expectation of (2) is taken at time t with the following assumptions:

$$(4)\quad E_t(Z_{t+j})=Z_{t+j},\ j\leq 0 \quad ; \quad E_t(Z_{t+j})=Z_t(j),\ j>0$$

$$(5)\quad E_t(a_{t+j})=a_{t+j},\ j\leq 0 \quad ; \quad E_t(a_{t+j})=0,\ j>0$$

where $E_t(Z_{t+j})$ is the conditional expectation of Z_{t+j} taken at origin t. Thus, the forecasts $\hat{Z}_t(\ell)$ for each lead time are computed from previous observed Z's, previous forecasts of Z's and current and previous random shocks a's. The unknown a's are replaced by zeroes.

In general, if the moving average operator $\pi(B)=\Theta(B)\ \Theta(B^s)$ is of degree q+s.Q, the forecast equations for $\hat{Z}_t(1)$, $\hat{Z}_t(2)$,..., $\hat{Z}_t(q+s.Q)$ will depend directly on the a's but forecasts at longer lead times will not. The latter will receive indirectly, the impact of the a's by means of the previous forecasts. In effect, $\hat{Z}_t(q+s.Q+1)$ will depend on the q+s.Q previous $\hat{Z}_t$ which in turn will depend on the a's.

From the point of view of studying the nature of the forecasts, it is important to consider the explicit form of the forecasting function. For $\ell>n=q+s.Q$, the conditional expectation of (2) at time t is:

$$(6)\quad \hat{Z}_t(\ell)-\psi_1\hat{Z}_t(\ell-1)-\ldots-\psi_m\hat{Z}_t(\ell-m)=0 \qquad \ell>m$$

and the solution of this difference equation is:

$$(7)\quad \hat{Z}_t(\ell)=b_0^{(t)}f_0(\ell)+b_1^{(t)}f_1(\ell)+\ldots+b_{m-1}^{(t)}f_{m-1}^{(\ell)} \qquad \ell>n-m$$

This function is called the "eventual forecast function", eventual because when n>m, it supplies the forecasts only for lead times $\ell>n-m$. In (7), $f_0(\ell)$, $f_1(\ell)$, $\ldots f_{m-1}(\ell)$ are functions of the lead time ℓ and in general they include polynomials, exponentials, sines and cosines, and products of these functions. For a given origin t, the coefficients $b_j^{(t)}$ are constants applying for all lead time ℓ but they change from one origin to the next, adapting themselves to the particular part of the series being considered. It is important to point out that it is the general autoregressive operator $\psi(B)$ defined above, which determines the mathematical form of the forecasts function, i.e., the nature of the f's. In other words, it determines whether the forecasting function is to be a polynomial, a mixture of sines and cosines, a mixture of exponentials or some combinations of these functions.

The ARIMA forecasts are minimum mean square error forecasts and can be easily updated as new raw values become available.

In the context of the X-11-ARIMA, the forecasts should follow the general movement of the series.

In our experience with Canadian and U.S. economic time series, we concluded that the ARIMA models chosen must fit the data well and produce forecasts for each of the last three years with a mean absolute error smaller than 5% for well behaved series (e.g. employment men, over 20) and smaller than 10% for volatile series (e.g. unemployment women – 16-19). The smaller the forecasting error, the better. This is particularly true for the forecasting error of the first six months, given the way they will be treated by the X-11 filters.

Since ARIMA models are robust, the identification is often good for several years. However, they should be checked when an extra year of data becomes available to insure that the most recent movements of the series are properly followed by the model.

The seasonal adjustment filters of the U.S. Bureau of Census Method II-X-11 program

The Bureau of the Census program is summarized in Shiskin, J. and Eisenpress. H [6], and described fully in Shiskin, J., Young, A.H. and Musgrave, J.C. [7].

The main steps of this method for obtaining the seasonally adjusted series are as follows (9).

(1) Compute the ratios between the original series and a centered 12-term moving average (2 X 12 m.a., that is, 2-term average of a 12-term average) as a first estimate of the seasonal and irregular components.

(2) Apply a weighted 5-term moving average to each month separately (a 3 X 3 m.a.) to obtain an estimate of the seasonal factors.

(3) Compute a centered 12-term moving average of the preliminary factors in (2) for the entire series. To obtain the six missing values at either end of this average, repeat the first (last) available moving average value six times. Adjust the factors to add to 12(approximately) over any 12-month period by dividing the centered 12-term average into the factors.

(4) Divide the seasonal factor estimates into the seasonal irregular (SI) ratios to obtain an estimate of the irregular component.

(5) Compute a moving 5-year standard deviation (σ) of the estimates of the irregular component and test the irregulars in the central year of the 5-year period against $2.\sigma$. Remove values beyond 2.5σ as extreme and recompute the moving 5-year σ.

Assign a zero weight to irregulars beyond 2.5σ and a weight of 1 (full weight) to irregulars within 1.5σ. Assign a linearly graduated weight between 0 and 1 to irregulars between 2.5σ and 1.5σ.

(6) For the first two years, the σ limits computed for the third year are used; and for the last two years, the σ limits computed for the third-from-end year are used. To replace an extreme ratio in either of the two beginning or ending years, the average of the ratio times its weight and the three nearest full-weight ratios for that month is taken.

(7) Apply a weighted 7-term moving average to the SI ratios with extreme values replaced for each month separately, to estimate preliminary seasonal factors.

(8) Repeat step (3).

(9) To obtain a preliminary seasonally adjusted series divide (8) into the original series.

(10) Apply a 9-, 13-, or 23-term Henderson moving average to the seasonally adjusted series and divide the resulting trend-cycle into the original series to give a second estimate of the SI ratios.

(11) Apply a weighted 7-term moving average (3 X 5 m.a.) to each month separately, to obtain a second estimate of the seasonal component. Compute estimates of seasonal factors one year ahead by the formula

$$S_{j,t+1}=S_{j,t}+\tfrac{1}{2}(S_{j,t}-S_{j,t-1})$$

where j = 1, 2, ..., 12 denotes the month and t, the year.

(12) Repeat step (3).

(13) Divide these final seasonal factors into the original series to obtain the seasonally adjusted series.

Allan Young [8] using a linear approximation of the Census Method II arrives at the conclusion that a 145-term moving average is needed to estimate one seasonal factor with central weights if the trend-cycle component is adjusted with a 13-term Henderson moving average. The first and last 72 seasonal factors (six years) are estimated using sets of asymmetrical end weights. It is important to point out, however, that the weights given to the more distant observations are very small and therefore, the moving average can be very well approximated by taking ½ of the total number of terms plus one. So, if a 145-term moving average is used to estimate the seasonal factor of the central observation, a good approximation is obtained with only 73 terms i.e., six years of observations. This means that the seasonal factor estimates from unadjusted series whose observations end at least three years later can be considered "final", in the sense that they will not change significantly when new observations are added to the raw data.

The forecasting function specified in step (11) for each monthly seasonal factor forecasts perfectly if the seasonal factors are relatively constant through the years (stable seasonal pattern). However, if the seasonal pattern is evolving through time with a trend that is linear within the span of the moving average used to estimate the seasonal pattern, a bias is easily introduced. Months for which the seasonal factors tend to decrease will have a forecasted seasonal factor larger than expected. The opposite will happen for those months in which the seasonal factors tend to increase. Moreover, the size of the bias will be larger, the larger the slope of the line followed by the seasonal factors. It is evident then, that in the case of a linearly evolving seasonality, the seasonal factor forecasts for some months will have larger biases than others.

In the X-11-ARIMA, the unadjusted series is enlarged with one more year of data (forecasted values). Consequently, the X-11 program estimates the components with better filters.

The trend-cycle is no longer estimated with the end weights of the centered 12-term and Henderson moving averages but with their central weights. This means that the Henderson's filters will not miss a turning point at the end of series since the central weights of these filters minimize the sum of the squares of the third difference of the trend-cycle curve.

The end weights applied to the seasonal factors are closer to the central weights and can reproduce a local linearly moving seasonality with less error. In effect, for the 3 x 3 term moving averages, its forecasting filter is now .185, .407, .407 instead of –.056, .148, .426, .481. Similarly, the weights of the 3 x 3 term m.a. for the current seasonal factors are now only one step ahead of its central weights.

In the case of the 3 x 5 term moving average, its forecasting filter is .150, .283, .283, .283 instead of – .034, .134, .300, .300, .300. Observe that the new forecasting filters are those that the X-11 program apply for current seasonal factors and none of their weights is negative.

Because of their longer filtering intervals for given cutoff frequencies, smothing filters, having negative weights beyond the positive central values, tend to stretch too far the implicit assumption in filtering that the periodicities present at the time for which the filtered variable is estimated are unchanged in amplitude and phase during the filtering interval.

The replacement of the extreme values for the last year of observed data is also improved. In effect, by adding one more year of values with no extremes, since they are forecasts, a better estimate of the residuals is obtained.

4. Design of the Experiment and Conclusions

The X-11-ARIMA has been tested with Canadian and U.S. economic time series. Two statistics were chosen to determine which of the two methods, X-11-ARIMA or X-11, generates better current seasonal factors and forecasts, namely,

(1) The mean algebraic error of the current and forecasted seasonal factors for each month;

(2) the mean absolute error of the current and forecasted seasonal factors for each month.

The method giving the lowest statistics is considered the best. The Wilcoxon Signed Rank test was applied to matched pairs of statistics (1) and (2) obtained from current and forecasted seasonal factors given by both procedures to determine whether the differences are due to chance variations or whether they are really significant. Since an improvement would mean low statistics (1) and (2), a one-sided test of the null hypothesis H_0 (zero difference) versus the alternative H_1 (positive difference) was applied at a 5% level of significance.

To obtain statistics (1) and (2) corresponding to the seasonal factor forecasts given by each method for the series, we proceed as follows (the same was done for the current seasonal factors):

(1) Estimate the one-step seasonal factor forecast S_{ij}^{i-1} for each month $j=1, 2, \ldots, 12$ and year $i=1963, 1964, \ldots 1975$. (The superscript denotes the last year available of an unadjusted series with a minimum of 7 years of monthly data. In our case, we used data from 1953.)

(2) Estimate the seasonal factor S_{ij}^{i+3} for each month j=1, 2, ..., 12 and year i=1963, 1964, ..., 1972 from an unadjusted series ending in year i+3. (According to the type of filter used by Census Method II-X-11 variant this seasonal factor can be considered final in the sense that it will not change significantly when more observations are added to the original series).

(3) Define $e_{ij}=S_{ij}^{i+3} - S_{ij}^{i-1}$ as the total error in the seasonal factor forecasts.

(4) For each series build a double entry table of the e's defined in (3).

(5) For each double entry table of the e's calculate;

(5.1) The mean algebraic error for each month, that is, $1/n\sum_{i=1}^{n} e_{ij}$, j=1, 2,..., 12.

(5.2) The mean absolute error for each month, that is, $1/n\sum_{i=1}^{n} |e_{ij}|$, j=1, 2,..., 12.

The results from the Wilcoxon Signed Rank test indicated that the seasonal factor forecasts obtained from X-11-ARIMA were superior to those produced by Census Method II-X-11 variant (Dagum, E.B.[4]). The same conclusions applied to the current seasonal factors S_{ij}^{i}. Statistics Canada X-11-ARIMA was officially adopted by Statistics Canada in January 1975 for the seasonal adjustment of the main Labour Force series. In its present version, this new method is not fully mechanized. The user should be able to identify, for each series, an ARIMA model that fits the data well and produces reasonable forecasts according to the general guidelines mentioned in section (2). The identified model is only checked once a year and usually not changed for at least three years.

Since there is a class of simple ARIMA models that fit and forecast well a large number of series, we are presently working on the selections of a limited number of ARIMA models to be able to fully automate this procedure.

Decomposition of Seasonal Time Series: A Model for the Census X-11 Program

W. P. CLEVELAND and G. C. TIAO*

This paper shows that the linear filter version of the Census X-11 program for time-series decomposition can be approximately justified in terms of an additive model with stochastic trend, seasonal and noise components. Optimal estimates of the trend and seasonal components are obtained from the model and found to be in close agreement with the corresponding estimates for the Census procedure. This approach makes it possible to assess the appropriateness of the Census method. Two examples are given, one showing that the use of the X-11 procedure is largely appropriate and the other much less so.

1. INTRODUCTION

One of the goals people often have in approaching a seasonal time series is removal of the seasonal component. This may be undertaken for several reasons. It leaves a series with a simpler pattern to be studied for its implications. It is particularly helpful in revealing the latest trends, since averaging by eye is not as easy in this nonsymmetric situation. Another common reason for wanting a deseasonalized series is to make comparisons of series with different seasonal patterns.

One way of effecting the decomposition of a series into seasonal and trend components is to fit a model which has deterministic seasonal and nonseasonal parts to the data. Another is to apply moving-average filters to the data to separate out these components. The filters are often designed on the basis of allocating various parts of the power spectrum of the series to the seasonal and trend components.

The Bureau of the Census has developed a computer program for seasonal adjustment which has been widely used in government and industry. The basic feature of the program is that it uses a sequence of moving average filters to decompose a series into a seasonal component, a trend component, and a noise component. The strength and weakness of the program lies in its use of roughly the same filters for most series. Such a decomposition has the superficial advantage of uniform interpretation of the seasonal and trend components of most series. On the other hand, if one believes that observed phenomena are generated according to the physical circumstances of the problem, one could certainly be misled as to their nature by the results of the census program if no checks on its adequacy are made. Such checks would, however, be difficult to make unless one had some ideas as to the kinds of *underlying mechanisms* for which the census program would be appropriate.

This paper proposes a stochastic model for which the census procedure is nearly optimal. Specifically, we suppose that an observed time series y_t consists of three additive random components: a seasonal component s_t, a trend component p_t, and a noise component e_t, such that if particular autoregressive integrated moving average models are given for these components, the optimum estimators for s_t and p_t turn out to be very close to those obtained from the basic set of moving average filters employed in the census program. The results shed considerable light on both the merits and the demerits of the census program. In particular, one can see why the census method is generally satisfactory for a variety of series, why the residuals of the census decomposition will have smaller variance than the residuals of an autoregressive integrated moving average model fit to the same series, and why the census residuals may be correlated.

2. THE BUREAU OF THE CENSUS PROGRAM

The census procedure is summarized in Shiskin and Eisenpress [12] and described fully in Shiskin, Young, and Musgrave [13]. The program assumes the additive decomposition[1]

$$y_t = p_t + s_t + e_t \ , \qquad (2.1)$$

where y_t is the observed series, p_t is the trend component, s_t is the seasonal component, and e_t is the noise. For each series the estimates of p_t and s_t, denoted, respectively, by $\hat{p}_t$ and $\hat{s}_t$, may be obtained. The basic device used to estimate the seasonal and trend components is the symmetric moving-average operator (filter) with weights summing to one. General properties of moving average operators are discussed in Kendall and Stuart [9, p. 366] and in Whittaker and Robinson [16, p. 288]. A symmetric moving average operator $S(\delta, k)$ generates a series

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[1] There is also a multiplicative version of the program. By taking logarithms, the multiplicative version is essentially the same as the version discussed.

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x_t from a series y_t according to the relation

$$x_t = S(\delta, k)y_t = \sum_{j=-k}^{k} \delta_j y_{t-j} \qquad (2.2)$$

where k is a non-negative integer, and δ_j are the weights (constants) used in averaging y_t such that $\delta_{-j} = \delta_j$.

For a series $y_1, \ldots, y_T$ of length T, x_t cannot be computed according to (2.2) for $t = 1, \ldots, k$ and $t = T - k + 1, \ldots, T$. The operator $S(\delta, k)$ must, therefore, be modified at both ends of the series according to some extrapolation principle, as is done in the census procedure. In the analysis here, we shall be concerned mainly with version (2.2). However, the results developed can be logically extended to cover the end effects of the series.

Wallis [15] describes the sequence of moving average operators used in the census program. Although it includes procedures for adjusting outliers and for making trading day adjustments, the basic linear filter feature of the program is of the form (2.2). Also, the census program has a built-in procedure to select the 9-term, 13-term, or 23-term Henderson trend filters. In what follows we have assumed the 13-term filter because it seems to be the one applicable to the majority of series, and also because it would be rather difficult to model the procedure for selecting this one from the other two alternatives. The model in Section 3 applies to this linear filter version of the program.

3. A MODEL FOR THE BUREAU OF THE CENSUS PROGRAM

Without a specific underlying model the Bureau of the Census program lends itself to all the criticisms of moving-averages in general. If the observed series were a polynomial in time plus a trigonometric series plus noise, one should do a regression. If the preceding with time changing coefficients were the model, one could parameterize the pattern of changes in the coefficients and still do a regression. Models which fit a polynomial over a finite number of points to estimate the center point are not consistent, in the sense that different polynomials represent the same points at different times.

For a discussion of the properties of moving average filters and criticisms of the Census procedure in the context of spectral theory, the reader is referred to [3–8, 10, 11].

The best justification of moving-average filters seems to be in terms of stochastic models. For the additive decomposition in (2.1), we employ the autoregressive integrated moving average (ARIMA) processes [1] for the components. Specifically, we suppose that

$$y_t = p_t + s_t + e_t \qquad (3.1)$$

with

$$\phi_{r_1}(B)(p_t - \mu) = \psi_{q_1}(B)b_{1t}$$

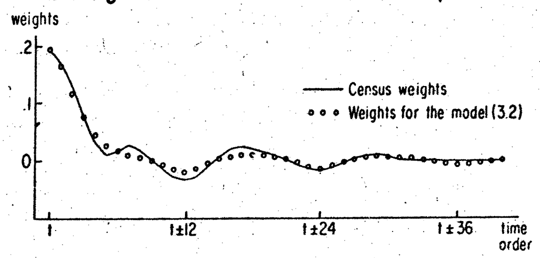
$$\phi_{r_2}(B)s_t = \psi_{q_2}(B)b_{2t}\ ,$$

where μ is a constant, B is the backward shift operator such that $Bx_t = x_{t-1}$, and $\psi_{q_1}(B)$, $\psi_{q_2}(B)$, $\phi_{r_1}(B)$ and $\phi_{r_2}(B)$ are real polynomials in B of degrees q_1, q_2, r_1 and r_2, respectively. We shall require that the zeroes of $\psi_{q_1}(B)$ and $\psi_{q_2}(B)$ lie outside the unit circle and those of $\phi_{r_1}(B)$ and $\phi_{r_2}(B)$ lie on or outside the unit circle. In (3.1), e_t, b_{1t} and b_{2t} are three independent white noise processes, normally distributed with zero means and variances σ_e^2, $\sigma_{b_1}^2$ and $\sigma_{b_2}^2$, respectively. In this framework, the minimum mean square error estimators of p_t and s_t are, respectively, the conditional expectations $E(p_t|\mathbf{y})$ and $E(s_t|\mathbf{y})$ where $\mathbf{y} = (y_1, \ldots, y_T)'$ is the vector of observations. It is shown in the appendix that, for t not close to either end of the series, $E(p_t|\mathbf{y})$ and $E(s_t|\mathbf{y})$ are, to a close approximation, symmetric moving averages of the observations $\mathbf{y}$. Thus, if one can find a model for which the conditional expectations give the same weights as those of particular symmetric moving average filters, it may then be argued that this model represents an underlying stochastic mechanism for those filters.

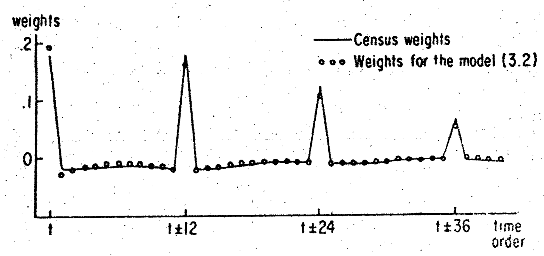
In searching for a model of the form (3.1) which would give weight functions similar to those of the census program, we kept several things in mind. When $\phi_{r_1}(B)$ contains the factor $(1 - B)^r$ to a degree at least one higher than $\phi_{r_2}(B)$, the weights in $E(p_t|\mathbf{y})$ and $E(s_t|\mathbf{y})$ sum to one and zero, respectively, as do the Census weights for $\hat{p}_t$ and $\hat{s}_t$. The use of a moving average filter for $\hat{p}_t$ rather than a polynomial suggests that one should allow for changing trend slopes. Similarly, the seasonal estimator should allow for changing amplitude and phase. Otherwise, a trigonometric series could be used.

The model stated in (3.2) is consistent with these principles and gives good correspondence to the census program using a minimal number of parameters. The weight functions for $E(p_t|\mathbf{y})$ and $\hat{p}_t$ are exceedingly close, as are the weight functions for $E(s_t|\mathbf{y})$ and $\hat{s}_t$. These are illustrated in Figures A and B.

A. Weight Functions for the Trend Component



B. Weight Functions for the Seasonal Component



$$
\begin{aligned}
y_t &= p_t + s_t + e_t \ , \\
(1 - B)^2 p_t &= (1 - \psi_{11} B - \psi_{12} B^2) b_{1t} \ , \\
(1 - B^{12}) s_t &= (1 - \psi_{21} B^{12} - \psi_{22} B^{24}) b_{2t} \ , \qquad (3.2)
\end{aligned}
$$

$$\psi_{11} = -.49 \ , \ \psi_{12} = .49 \ , \ \psi_{21} = -.64 \ , \ \psi_{22} = -.83 \ ,$$

$$\sigma_{b_2}{}^2/\sigma_{b_1}{}^2 = 1.3 \quad \text{and} \quad \sigma_e{}^2/\sigma_{b_1}{}^2 = 14.4 \ .$$

Models differing from (3.2) in the order of differencing and in the numbers of autoregressive and moving average parameters in the two component models were tried. This model was found to be as good as models with more parameters and superior to others, particularly in matching $E(s_t|\mathbf{y})$ to the corresponding census weights.

A general purpose nonlinear least-squares program was used to estimate the parameter values for each model. For a given set of parameter values, the generating functions (A.20) and (A.21) in the appendix were expanded for the first 42 terms on either side of the center. The program used these as predictions of the corresponding weights in the linear filter version of the census program, and it adjusted the parameter values iteratively until the sum of squared deviations over the seasonal and trend weights was minimized.

While we used the sum of squares as one measure in choosing among candidate models, we also visually inspected the nature of the discrepancies in the weight functions, as displayed in Figures A and B, when evaluating the adequacy of a model. Occasionally tails of the census and model weights did not appear to be converging or only the trend weights and not the seasonal were being well approximated.

3.1 The Overall Model for y_t

From (3.2), we may write

$$
\begin{aligned}
(1 - B)&(1 - B^{12}) y_t \\
&= ((1 - B^{12})/(1 - B))(1 - \psi_{11} B - \psi_{12} B^2) b_{1t} \\
&\quad + (1 - B)(1 - \psi_{21} B^{12} - \psi_{22} B^{24}) b_{2t} \\
&\quad + (1 - B)(1 - B^{12}) e_t \ , \qquad (3.3)
\end{aligned}
$$

so that the autocorrelations of $w_t = (1 - B)(1 - B^{12}) y_t$ are

k	ρ_k									
1–10	−.25	.13	.12	.11	.09	.08	.07	.05	.04	.03
11–20	.1[illegible]	−.35	.16	.00	.00	.00	.00	.00	.00	.00
21–25	.00	.00	−.01	.03	−.01					

with $\rho_k = 0$ for $k > 25$. The explicit overall model for y_t's found to be,

$$
\begin{aligned}
(1 - B)&(1 - B^{12}) y_t \\
&= (1 - .337B + .144B^2 + .141B^3 + .139B^4 \\
&\quad + .136B^5 + .131B^6 + .125B^7 + .117B^8 + .106B^9 \\
&\quad + .09\text{[illegible]}B^{10} + .077B^{11} - .417B^{12} + .232B^{13} \\
&\quad - .001B^{20} - .003B^{21} - .004B^{22} - .006B^{23} \\
&\quad + .035B^{24} - .021B^{25}) c_t \ . \qquad (3.4)
\end{aligned}
$$

where c_t is a white noise process, normally distributed with zero mean and variance $\sigma_c{}^2$.

It is important not to confuse the basic additive model (3.2) and the overall form (3.4). In particular,

1. the overall form is a logical consequence of the basic model,
2. the conditional expectations $E(p_t|\mathbf{y})$ and $E(s_t|\mathbf{y})$ are determined not by (3.4) but by (3.2),
3. different additive models can lead to the same overall model,
4. the overall form is, however, important in that it can be identified from the data and, hence, provides a means to partially assess the appropriateness of the census procedure for a given set of data.

The model in (3.2) and its consequent overall form (3.4) helps explain why the census program fits series as well as it often does. A nonstationary seasonal pattern is presumed, and the model for p_t allows for a trend of changing slope. The autoregressive part of the overall model for y_t is $(1 - B)(1 - B^{12})$. This suggests that a series obeying (3.4) or at least $(1 - B)(1 - B^{12}) y_t = \theta(B) c_t$ for some polynomial $\theta(B)$ might be fairly accurately analyzed by the census program. The overall model is broadly similar to the multiplicative model discussed by Box and Jenkins [1, Ch. 9], though not the same.

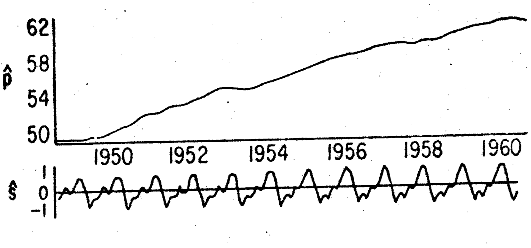
4. A COMPARISON OF THE CENSUS PROGRAM WITH OVERALL ARIMA SEASONAL MODELS FOR TWO SERIES

Two series are presented which have been fit with ARIMA models according to the procedures suggested in Box and Jenkins [1]. In doing so, the models sought were the ones that fit best in the sense of matching the correlation structure of the observed series and having uncorrelated residuals. The goal in each case was a model for forecasting rather than seasonal decomposition. It is, however, of interest to compare the results of this procedure with those of the census program.

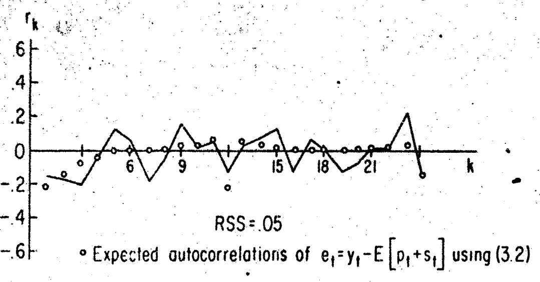
4.1 The Airline Data

These data may be found in [2, p. 429], or in [1, p. 304]. They represent logarithms of monthly passenger totals. Figures C and D show the census estimates of its components, $\hat{\mathbf{p}}$ and $\hat{\mathbf{s}}$, and the autocorrelation pattern of the residuals, $\mathbf{y} - \hat{\mathbf{p}} - \hat{\mathbf{s}}$. Also shown in Figure D are the expected residual autocorrelations, which will be explained later. The immediate impression of the series is that it exhibits a regular seasonal pattern with almost

C. Estimates of the Seasonal and Trend Components of the Airline Data from the Bureau of the Census Program



Residual Autocorrelations of the Airline Data from the Bureau of the Census Program



linear slope. The plots of the component estimates bear this out to a certain extent, but some evolution of the seasonal pattern and deviation from linearity are indicated. Some of the sample autocorrelations of the residuals appear to be rather large in magnitude, but the autocorrelation function as a whole exhibits no discernible pattern. The residual sum of squares, RSS, is .05.

The model for the airline data obtained by Box and Jenkins is

$$(1 - B)(1 - B^{12})y_t = (1 - \theta_1 B)(1 - \theta_{12} B^{12})c_t$$

$$\hat{\theta}_1 = .4 \;, \quad \hat{\theta}_{12} = .6 \;, \quad \hat{\sigma}_c^2 = .00134 \;, \quad \text{RSS} = .175 \;. \qquad (4.1)$$

The autoregressive part of the model in (4.1) is the same as that of (3.4). Also, the estimates of θ_1 and θ_{12} in (4.1) are reasonably close to the coefficients associated with B and B^{12} on the right side of (3.4). One is, however, immediately struck by the fact that the RSS in (4.1), .175, is much larger than the corresponding number, .05, for the census program.

To see why, consider a simpler case where $y_t = z_t + e_t$ and the model for z_t is $(1 - B)z_t = (1 - \psi B)b_t$ where $|\psi| < 1$. This gives

$$(1 - B)y_t = (1 - \psi B)b_t + (1 - B)e_t$$
$$= (1 - \theta B)c_t \;, \quad \text{where } |\theta| < 1 \;. \qquad (4.2)$$

Thus,

$$c_t = [(1 - \psi B)/(1 - \theta B)]b_t$$
$$+ [(1 - B)/(1 - \theta B)]e_t \;. \qquad (4.3)$$

By expanding the right side of (4.3) we find

$$\sigma_c^2 = \left\{1 + \frac{(\theta - \psi)^2}{1 - \theta^2}\right\}\sigma_b^2 + \frac{2}{1 + \theta}\sigma_e^2 \qquad (4.4)$$

so that

$$\sigma_c^2 > (2/(1 + \theta))\sigma_e^2 > \sigma_e^2 \;.$$

From (A.16) in the appendix, the variance of the residuals $\hat{e}_t = y_t - E(z_t | \mathbf{y})$ is

$$\text{Var}(\hat{e}_t) = \sigma_e^2 \cdot \frac{\sigma_e^2 (2/(1 + \theta))}{\sigma_c^2} . \qquad (4.5)$$

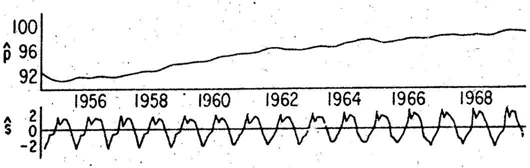
Hence, $\sigma_c^2 > \text{Var}(\hat{e}_t)$. Now the residuals from Box and Jenkins's method of model fitting are $\hat{c}_t$ which are different from the residuals $\hat{e}_t$. Thus the difference in RSS is not in this situation evidence of one method being better than the other. Two different kinds of residuals are involved. As we have said, the model for y_t in (4.1) has the same differencing structure as (3.4), indicating that the census program is a nearly appropriate decomposition procedure for the airline series.

Further evidence that the census procedure works reasonably well for the airline data is provided by the following analysis. The theoretical autocorrelations of the residuals $y_t - E(p_t + s_t | \mathbf{y})$ corresponding to the underlying model (3.2) may be computed using the general result (A.16) in the appendix. These correlations are given by the circles in Figure D. The pattern of the expected correlations is fairly close to that computed from the data using the census procedure, and the agreement in sign of the correlations is nearly perfect.

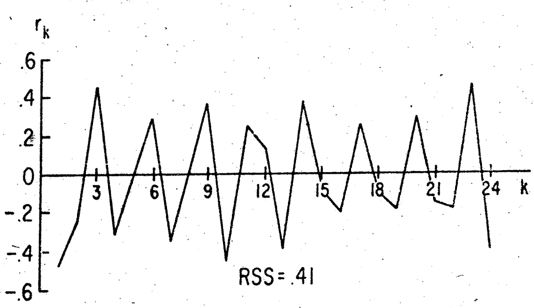
4.2 The Outward Telephone Data

The data in [14] are logarithms of monthly telephone disconnections, which the telephone companies call outward movement. Figures E and F show the census estimates of its components, and the residual autocorrelations. Note that the autocorrelations in this case exhibit a strong cyclical pattern. It seems clear that not all of the seasonality in the data has been accounted for by the program.

E. Estimates of the Seasonal and Trend Components of the Telephone Data from the Bureau of the Census Program



F. Residual Autocorrelations of the Telephone Data from the Bureau of the Census Program



The telephone data were analyzed by Thompson and Tiao and found to obey the model

$$(1 - \phi_3 B^3)(1 - \phi_{12} B^{12})y_t$$
$$= (1 - \theta_9 B^9 - \theta_{12} B^{12} - \theta_{13} B^{13})c_t$$

$$\hat{\phi}_3 = .49 \;, \quad \hat{\phi}_{12} = 1.005 \;, \quad \hat{\theta}_9 = .23 \;, \qquad (4.6)$$
$$\hat{\theta}_{12} = .334 \;, \quad \hat{\theta}_{13} = .17 \;, \quad \hat{\sigma}_c^2 = .0035 \;, \quad \text{RSS} = .59 \;.$$

The autoregressive part of (4.6) is now quite different from that in the model (3.4) underlying the census program. Again the residual sum of squares, RSS, for this model is larger than that of the census program (.59 to .4). However, the census residuals exhibit a strong quarterly correlation pattern, while the residuals $\hat{e}_t$ do not. This is probably due to the differences between the two models (3.2) and (4.6), suggesting that using the census program on this series is not justified.

5. CONCLUSIONS

In the preceding sections, it was shown that the census procedure can be approximately justified in terms of an additive model which consists of a stochastic trend and seasonal components. The specific models for the components are given in (3.2), and the resulting overall model for the observations in (3.4).

If this model could be *assumed* for a given series, the census program could be used to estimate the unobserved trend and seasonal components during the observational period. In addition, the components could be forecast by calculating the conditional distributions of their future values given the observations. However, this result also raises a number of serious questions. First, should this model or minor deviations from it, corresponding to the use of the alternative 9-term or 23-term trend filters, be assumed for all series as implied by the census program? Second, is the census procedure robust to departures from this model? The answer to the first question is clearly no. Some clues to the answer for the second are provided by the analysis of the airline and telephone data.

The model (4.1) fitted by Box and Jenkins to the airline data has the same autoregressive part as the model (3.4), but the moving average parts are somewhat different. If we take the model fitted by Box and Jenkins to be correct, the appropriateness of the decomposition (3.2) is thrown in doubt. Yet the census program appears to perform quite well on this set of data, suggesting a certain degree of robustness. On the other hand, the census residuals from the telephone data show a strong quarterly correlation pattern. Since both the autoregressive and moving-average parts of (3.4) are quite different from those of the model (4.6) fitted to the telphone data, one suspects that this robustness does not extend to models which differ markedly from (3.4).

It might be argued that the robustness of the census procedure would be enhanced if the use of the alternative 9- or 23-term trend filters were taken into account. While this is undoubtedly true to a certain extent, one ought to remember that these alternatives were designed mainly for somewhat different evolutionary trend patterns. Thus, one would expect that the models corresponding to these alternative filters would be similar to (3.2) but with different values for the parameters, so that the use of these alternative filters on the telephone data would lead to a similar pattern of the residual autocorrelations as that shown in Figure F.

APPENDIX

For the general additive model (3.1), we here derive the conditional expectations $E(p_t|\mathbf{y})$ and $E(s_t|\mathbf{y})$, as well as the variance and autocorrelations of the residuals $\hat{e}_t = y_t - E(p_t + s_t|\mathbf{y})$.

To facilitate presentation, we first consider the situation

$$y_t = z_t + e_t\ , \quad t = 1, \ldots, T\ , \tag{A.1}$$

where

$$\phi_r(B)(z_t - \mu) = \psi_q(B)b_t\ ,$$

e_t are i.i.d. $N(0, \sigma_e^2)$, b_t are i.i.d. $N(0, \sigma_b^2)$ and independent of e_t, $\psi_q(B)$ is a polynomial in B of degree q having its zeroes lying outside the unit circle, and $\phi_r(B)$ is a polynomial in B of degree r having its zeroes lying on or outside the unit circle. Now, z_t can be written as a function of r values in the past (see, e.g., [1, p. 115]).

$$z_t = c_t + \sum_{j=m}^{t} b_j \pi_{t-j}\ , \quad m \leq -\max(r, q) + 1 \tag{A.2}$$

where (i) $c_t = \sum_{l=1}^{r} A_l \alpha_l^{t-m} + \mu$, α_l^{-1} are the zeroes of $\phi_r(B)$ assumed distinct, and A_l depend on the starting values of the series at time m; (ii) π_j are obtained by equating coefficients of B^j from the relation

$$\phi_r(B)(1 + \pi_1 B + \pi_2 B^2 + \ldots) = \psi_q(B)\ . \tag{A.2a}$$

Thus the vector $\mathbf{z}$, where $\mathbf{z} = (z_1, \ldots, z_T)'$, is distributed as normal with mean vector $\mathbf{n} = (c_1, \ldots, c_T)'$ and covariance matrix $\sigma_b^2 \Sigma_z$ whose elements can be readily obtained from (A.2).

A.1 The Conditional Distribution of z Given y

Given the observation vector $\mathbf{y} = (y_1, \ldots, y_T)'$, $\mathbf{z}$ is distributed as normal with

$$E(\mathbf{z}|\mathbf{y}) = (\mathbf{I} + v\Sigma_z^{-1})^{-1}(\mathbf{y} + v\Sigma_z^{-1}\mathbf{n})$$

and

$$\operatorname{cov}(\mathbf{z}|\mathbf{y}) = \sigma_e^2(\mathbf{I} + v\Sigma_z^{-1})^{-1} \tag{A.3}$$

where $v = \sigma_e^2\sigma_b^{-2}$ and $\mathbf{I}$ is the identity matrix. Note that in $E(\mathbf{z}|\mathbf{y})$, $(\mathbf{I} + v\Sigma_z^{-1})^{-1}(\mathbf{h} + v\Sigma_z^{-1}\mathbf{h}) = \mathbf{h}$, where $\mathbf{h}$ is a vector of ones, so that for each z_t, $E(z_t|\mathbf{y})$ is a weighted average of $\mathbf{y}$ and $\mathbf{n}$ with weights summing to one.

In practice, it is often appropriate to suppose that the series began at some remote past point of time. That is, m is a large negative integer. We now distinguish between two situations.

(I) *All Zeroes of $\phi_r(B)$ Lie Outside the Unit Circle.* In this case, $\mathbf{z}$ can be thought of as a vector of observations from a stationary Gaussian process. That is, for large $-m$, $\mathbf{n}$ approaches $\mathcal{H}\mu$, and Σ_z tends to its stationary value $\dot{\Sigma}_z$ the (i, j)th element of which is $\sigma_{ij} = \sum_{l=0}^{\infty} \pi_l \pi_{l-(i-j)}$ where $\pi_0 = 1$ and $\pi_l = \pi_{-l}$. Thus, the expressions in (A.3) become

$$E(\mathbf{z}|\mathbf{y}) = (\mathbf{I} + v\dot{\Sigma}_z^{-1})^{-1}(\mathbf{y} + v\dot{\Sigma}_z^{-1}\mathbf{h}\mu)$$

and

$$\operatorname{cov}(\mathbf{z}|\mathbf{y}) = (\mathbf{I} + v\dot{\Sigma}_z^{-1})^{-1}\sigma_e^2\ . \tag{A.4}$$

(II) *Zeroes of $\phi_r(B)$ Lie on or Outside the Unit Circle.* To see the implications here, consider first the simplest case $\phi_r(B) = \phi_{r^*}(B)(1 - B)$ where $r^* = r - 1$ and the zeroes of $\phi_{r^*}(B)$ lie outside the unit circle. Then c_t in (A.2) is of the form

$$c_t = A_1 + \sum_{l=2}^{r} A_l \alpha_l^{t-m}\ . \tag{A.5}$$

Consider now the transformation

$$\begin{bmatrix} z_1 \\ \mathbf{w} \end{bmatrix} = J\mathbf{z}\ , \quad J = \begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ & \ddots & \ddots & \\ & & -1 & 1 \end{bmatrix}\ ,$$

i.e., $\mathbf{w} = (w_2, \ldots, w_T)'$ where $w_t = z_t - z_{t-1}$ follow the model

$$\phi_{r^*}(B)w_t = \psi_q(B)b_t\ . \tag{A.6}$$

Making use of the partitioned inverse of a matrix, we obtain

$$\boldsymbol{\Sigma}_z^{-1} = \mathbf{J}'\mathbf{Q}\mathbf{J} \ , \quad \mathbf{Q} = \begin{bmatrix} \sigma_{z_1}^{-2} - \sigma_{z_1}^{-4}\sigma_{z_1w}'\mathbf{G}\sigma_{z_1w} & -\sigma_{z_1}^{-2}\sigma_{z_1w}'\mathbf{G} \\ -\sigma_{z_1}^{-2}\mathbf{G}\sigma_{z_1w} & G \end{bmatrix}$$

where $\sigma_{z_1}^{-2} = \sigma_b^2/\mathrm{Var}\ (z_1)$, $\sigma_{z_1w}' = \sigma_b^{-2}\ \mathrm{cov}\ (z_1\mathbf{w}')$,

$$G = [\boldsymbol{\Sigma}_w - \sigma_{z_1}^{-2}\sigma_{z_1w}\sigma_{z_1w}']^{-1}$$

and $\boldsymbol{\Sigma}_w$ is the covariance matrix of **w**. Letting

$$\phi_{r^*}(B)(1 + \bar{\pi}_1 B + \bar{\pi}_2 B^2 + \ldots) = \psi_q(B) \ ,$$

we have from (A.2a) that $\pi_j - \pi_{j-1} = \bar{\pi}_j$, $j \geq 1$. It readily follows from (A.2) that

$$\mathrm{Var}\ (z_1) = \sigma_b^2 \sum_{j=0}^{-m} (\sum_{l=0}^{j} \bar{\pi}_l)^2 \ , \quad \bar{\pi}_0 = 1$$

and

$$\mathrm{cov}\ (z_1 w_t) = \sigma_b^2 \sum_{j=t-1}^{t-1-m} \bar{\pi}_j \pi_{j-(t-1)} \ .$$

Note that since the zeroes of $\phi_{r^*}(B)$ lie outside the unit circle, $\sum_{j=0}^{\infty} |\bar{\pi}_j| < \infty$. Thus, for large $-m$, $\sigma_{z_1}^{-2}$ and $\sigma_{z_1}^{-2}$ cov $(z_1 w_t)$ are of order m^{-1}; also, $\mathbf{G} = \dot{\boldsymbol{\Sigma}}_w^{-1} + \mathbf{R}$, where $\mathbf{R}$ is a matrix whose elements are of order m^{-1} and $\dot{\boldsymbol{\Sigma}}_w^{-1}$ is the stationary value of $\boldsymbol{\Sigma}_w$, the (ij)th element of which is $\sum_{l=0}^{\infty} \bar{\pi}_l \bar{\pi}_{l-(i-j)}$ with $\bar{\pi}_{-l} = \bar{\pi}_l$. Hence, to order $O(1)$,

$$\boldsymbol{\Sigma}_z^{-1} = \dot{\boldsymbol{\Sigma}}_z^{-1} = J' \begin{bmatrix} 0 & 0 \\ 0 & \dot{\boldsymbol{\Sigma}}_w^{-1} \end{bmatrix} \mathbf{J} \qquad \text{(A.7)}$$

To this order of approximation, then, $\dot{\boldsymbol{\Sigma}}_z^{-1}$ is finite, nonnegative definite and its rank is $T - 1$. It follows from (A.5) and (A.7) that, for large $-m$, the expressions in (A.3) becomes

$$E(\mathbf{z}|\mathbf{y}) = (I + v\dot{\boldsymbol{\Sigma}}_z^{-1})^{-1}y \ ,$$

and

$$\mathrm{cov}\ (\mathbf{z}|\mathbf{y}) = \sigma_e^2(I + v\dot{\boldsymbol{\Sigma}}_z^{-1})^{-1} \ . \qquad \text{(A.8)}$$

The preceding result can be readily extended to the situation in which $\phi_r(B)$ in (A.1) takes the form

$$\phi_r(B) = \phi_{r^*}(B)(1 - B)^{d_1}(1 - B^c)^{d_2} \qquad \text{(A.9)}$$

where d_1, d_2, c are positive integers and $r^* + d_1 + cd_2 = r$. In this case, for large $-m$, the expressions in (A.8) still hold except that the rank of $\dot{\boldsymbol{\Sigma}}_z^{-1}$ will be reduced to $T - (d_1 + cd_2)$.

A.2 The Asymptotic Form of $\dot{\boldsymbol{\Sigma}}_z^{-1}$

When the number of observations T is large, the asymptotic elements of $\dot{\boldsymbol{\Sigma}}_z^{-1}$ can be obtained from the generating function

$$\frac{\phi_r(B)\phi_r(F)}{\psi_q(B)\psi_q(F)} = X_0 + \sum_{k=1}^{\infty} X_k(B^k + F^k) \qquad \text{(A.10)}$$

where $F = B^{-1}$. Specifically, let σ^{ij} be the (i, j)th element of $\dot{\boldsymbol{\Sigma}}_z^{-1}$. Then, for i not close to 1 or T, $\lim_{T\to\infty} \sigma^{ij} = X_{|i-j|}$. This result was obtained by Wise [18] for when the model of z_t in (A.1) is stationary and invertible. It also holds when $\phi_r(B)$ takes the form (A.9). This can be seen by considering the simplest nonstationary case $\phi_r(B) = \phi_{r^*}(B)(1 - B)$. Since w_t follows the stationary model in (A.6), the generating function of the asymptotic elements of $\dot{\boldsymbol{\Sigma}}_w^{-1}$ can be written

$$\frac{\phi_{r^*}(B)\phi_{r^*}(F)}{\psi_q(B)\psi_q(F)} = \tilde{x}_0 + \sum_{k=1}^{\infty} \tilde{x}_k(B^k + F^k) \ , \quad \text{say} \ . \qquad \text{(A.11)}$$

From (A.7), the elements σ^{ij} of $\dot{\boldsymbol{\Sigma}}_z^{-1}$ and the elements $\tilde{\sigma}^{ij}$ of $\dot{\boldsymbol{\Sigma}}_w^{-1}$ are related by

$$\sigma^{ij} = \tilde{\sigma}^{ij} - \tilde{\sigma}^{i+1,j} - \tilde{\sigma}^{i,j+1} + \tilde{\sigma}^{i+1,j+1} \ .$$

Whence, from (A.11), for i not close to 1 or T

$$\lim_{T\to\infty} \sigma^{ij} = 2\tilde{x}_{|j-i|} - \tilde{x}_{|j-i-1|} - \tilde{x}_{|j-i+1|} \ .$$

Thus, the generating function of σ^{ij} is

$$2(\tilde{x}_0 - \tilde{x}_1) + \sum_{k=1}^{\infty} (2\tilde{x}_k - \tilde{x}_{k-1} - \tilde{x}_{k+1})(B^k + F^k)$$

$$= \frac{(1 - B)\phi_{r^*}(B)\phi_{r^*}(F)(1 - F)}{\psi_q(B)\psi_q(F)} = \frac{\phi_r(B)\phi_r(F)}{\psi_q(B)\psi_q(F)}$$

which gives (A.10). In a similar way, one can show that the generating function holds for any $\phi_r(B)$ of the form (A.9).

A.3 The Asymptotic Form of $E(z_t|\mathbf{y})$ and the Covariance Generating Function of $\hat{e}_t = y_t - E(z_t|\mathbf{y})$

From (A.1), the overall model of y_t can be written

$$\phi_r(B)(y_t - \mu) = \theta_u(B)c_t \qquad \text{(A.12)}$$

where the c_t are i.i.d. $N(0, \sigma_c^2)$ and $\theta_u(B)$ is a polynomial in B of degree u, $u \leq \max\ (p, q)$, having its zeroes lying outside the unit circle. The quantities $\theta_u(B)$, σ_c^2, $\psi_q(B)$, $\phi_r(B)$, σ_b^2 and σ_e^2 are related by the covariance generating function of $\phi_r(B)(y_t - \mu)$, namely,

$$\psi_q(B)\psi_q(F)\sigma_b^2 + \phi_r(B)\phi_r(F)\sigma_e^2 = \theta_u(B)\theta_u(F)\sigma_c^2 \ . \qquad \text{(A.13)}$$

Making use of (A.10), it follows from (A.4) and (A.8) that, for large T the asymptotic form of the conditional expectation of z_t given **y** is

$$E(z_t|\mathbf{y}) = [\theta_u(B)\theta_u(F)\sigma_c^2]^{-1}[\sigma_b^2\psi_q(B)\psi_q(F)y_t + \sigma_e^2\phi_r(B)\phi_r(F)\mu] \ . \qquad \text{(A.14)}$$

In particular, if $\phi_r(B)$ contains the factor $(1 - B)$, then $\phi_r(B)\mu = 0$ and

$$E(z_t|\mathbf{y}) = [\theta_u(B)\theta_u(F)\sigma_c^2]^{-1}\sigma_b^2\psi_q(B)\psi_q(F)y_t \ , \qquad \text{(A.15)}$$

which is a symmetric moving average of y_t. Further, by setting $B = 1$ in (A.13) we see that $[\psi_q(1)]^2\sigma_b^2 = [\theta_u(1)]^2\sigma_c^2$ so that the weights of (A.15) sum to one.

For the autocorrelations of $\hat{e}_t = y_t - E(z_t|y)$ it is readily verified from (A.13) and (A.14) that the autocovariance generating function of $\hat{e}_t$ is

$$\mathrm{C.G.F.}(\hat{e}_t) = \sigma_e^2 \frac{\sigma_e^2\phi_r(B)\phi_r(F)}{\sigma_c^2\theta_u(B)\theta_u(F)} \qquad \text{(A.16)}$$

from which the variance and autocorrelations can be readily calculated.

If z_t is stationary, expressions (A.14) and (A.16) are given in [17, p. 58] for $\mu = 0$. We have shown that these expressions are equally applicable when z_t is nonstationary with $\phi_r(B)$ assuming the form (A.9).

A.4 The Conditional Expectations $E(p_t|\mathbf{y})$ and $E(s_t|\mathbf{y})$

In (3.1), let $z_t = p_t + s_t$ and let $\phi^*(B)$ be the factor common to $\phi_{r_1}(B)$ and $\phi_{r_2}(B)$ and write

$$\phi_{r_1}(B) = \phi_{r^*_1}(B)\phi^*(B) \ , \quad \phi_{r_2}(B) = \phi_{r^*_2}(B)\phi^*(B) \qquad \text{(A.17)}$$

where

$$r_1^* \leq r_1 \quad \text{and} \quad r_2^* \leq r_2 \ .$$

It follows that the model for z_t in (A.1) is related to the models for p_t and s_t by

$$\phi_r(B) = \phi_{r^*_1}(B)\phi_{r^*_2}(B)\phi^*(B) \qquad \text{(A.18)}$$

and

$$\sigma_b^2\psi_q(B)\psi_q(F) = \sigma_{b_2}^2\phi_{r^*_1}(B)\phi_{r^*_1}(F)\psi_{q_2}(B)\psi_{q_2}(F) + \sigma_{b_1}^2\phi_{r^*_2}(B)\phi_{r^*_2}(F)\psi_{q_1}(B)\psi_{q_1}(F) \ ,$$

where $q \leq \max\ (r_1^* + q_2, r_2^* + q_1)$.

We now obtain the conditional expectation $E(s_t|\mathbf{y})$, by first finding the expectation $E(s_t|\mathbf{z})$. From (3.1) and adopting an approach analogous to the derivation of (A.14), the asymptotic form is

$$E(s_t|z) = [\psi_q(B)\psi_q(F)\sigma_v^2]^{-1}\sigma_{b_2}^2\psi_{q_2}(B)\psi_{q_2}(F)\phi_{r^*_1}(B)\phi_{r^*_1}(F)(z_t - \mu) \tag{A.19}$$

Since p_t, s_t and e_t are assumed independent, we see that

$$E(s_t|y) = E[E(s_t|z)|y] .$$

On substituting $E(z_t|y)$ of (A.14) for z_t on the right side of (A.19), we obtain

$$E(s_t|y) = \frac{\sigma_{b_2}^2\psi_{q_2}(B)\psi_{q_2}(F)\phi_{r^*_1}(B)\phi_{r^*_1}(F)}{\sigma_e^2\theta_u(B)\theta_u(F)}(y_t - \mu) , \tag{A.20}$$

where $\sigma_e^2\theta_u(B)\theta_u(F)$ can be readily obtained from (A.13) and (A.18).

Finally, the conditional expectation $E(p_t|y)$ is simply the difference,

$$E(p_t|y) = E(z_t|y) - E(s_t|y) . \tag{A.21}$$

Note that when $\phi_{r^*_1}(B)$ contains the factor $(1 - B)$, the constant in (A.14) and (A.20) vanishes. In this case, (i) $E(s_t|y)$ is a symmetric moving average with weights summing to zero, (ii) $E(z_t|y)$ and therefore $E(p_t|y)$ are symmetric moving averages with weights summing to one. This situation occurs for the model (3.2) because $\phi_{r_1}(B)$ and $\phi_{r_2}(B)$ have a common factor $(1 - B)$ so that $\phi_{r^*_1}(B) = (1 - B)$ and it corresponds precisely to the nature of the moving average operators for $\hat{s}$ and $\hat{p}$ in the census procedure.

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貝爾實驗室季節調整方法之介紹

貝爾實驗室季節整方法之介紹

一趙民德

— (Seasonal Adjustment Bell Laboratoies 의 소개)

1. 개설

본문은 貝爾 실험소에서 발전시킨 계절적 조정방법을 소개하고자 한다. 이 정식을 간략히 SABL 이라고 하자. 최근의 통계학 개념 및 발전하는 電腦技術을 채택했기 때문에 현재 가장 광범위하게 사용하는 것은 X-11 형이다. 원고의 대상은 그 계층의 사용자로 하였다. 이것은 필자가 직접 선택한 것이 아니므로 잘못된 곳은 책임을 지지 않겠읍니다. 본문은 특히 貝爾 실험소 회사가 미국전보전화회사에 지원한 것을 근거로 해서 개조한 것임을 밝혀둔다.

2. 時間序列의 분해

1개의 시간 서열을 T(trend), 계절(S-SEASON), 일력(C-calender 불규칙(I-Irregular) 4부분으로 나눌수 있고 일반가정을 덧셈법으로 하면 $D = T + S + C + I$, 곱셈법으로는 $D = T \times S \times C \times I$ 두가지로 된다.

圖 1은 하나의 시간 순서가 분해된 후의 상태를 뜻한다.

X-11 혹은 SABL의 주요목적은 S 및 C를 구하는데 있고, S와 C의 영향을 걸러내어 하나의 해석가능한 시간 서열을 얻는데 있다. 또한 조정된 후 시간 서열을 구하는데 있다.

3. 추향(Trend) 혹은 계절적 방법을 변경시키려면 差分法을 이용한다.

$$Y_t - Y_{t-4}$$

때로는 S 에 의해 만들어진 변화를 변경 할 수 있다

$$T_t - T_{t-12}$$

때로는 月份에 의해 생긴 변화를 변경할 수 있다.

4. 그 밖의 방법은 Smoothing operator를 거쳐서 나오는 것도 있다. 가령 平均法을 이용한다. 図二 와 図 3은 이 방법이 만든 변화를 가리킨다. 그러나 평균이 지나치지 않도록 주의하라 오히려 해롭다.

X-11은 1950년경 미국의 조사국의 발전시킨 방법인데 당시는 매우 놀라운 방법이었다. 지금도 가장 많이 쓰이는 방법으로 이것은 덧셈, 곱셈의 두가지 모형을 처리할 수 있다. S(계절) 혹은 月을 단위로 단 자료를 처리할 수 있다.

5. SASL은 근래 貝爾 실험실이 발전시킨 안정적인 결과를 채택한 것이다. X-11의 기본 방법을 이용했으며 迴歸 분석에 적합하다. 요컨데, SABL와 X-11 이 다른 것은 새것과 낡은 것일 경우이고 SABL이 쓰는 電腦技術은 비교적 신기술이다.

6. SABL 이 X-11 보다 우수한 점

1) 周期는 계절이나 月을 꼭 단위로 하지 않는다. 가령 時를 時間의 단위로 한다.

2) 곱셈법, 덧셈법의 두 모형에 한정한다.

3) 日曆(C)의 영향을 고려한다.

4) 과실의 처리에 대해서는 근대적 정신이 있다.

5) 圖示法을 사용한다.

7. 원시 數據는 SABL식에서는 우선 POWER TRANSFORMATION을 거친뒤 진일보한 분석을 한다. 図4를 보라. 이 변환 (TRANSFORMATION)은, 변환후의 원시 數據가 비교적 加法모형에 가깝게끔 한다. 이전형태를 근거로 계절 혹은 일력조정을 한다.
図4의, P 値는 -1에서 1 사이의 임의수가 될수 있다.

8. X-11 이 나타내는 P=0 및 P=1 의 2개 형태, SABL을 사용하는 사람은 더 많이 선택 할 수 있다. 비교적 좋은 P 値를 선택할 수 있다.

9. 아래의 해석은 P値선택에 대한 이유를 이해하는데 도움을 주고 먼저 日曆(C) 부분을 생략하고 加法의 모형은 : $D^{(P)} = S + T + I$ 이 된다. 만약 変方 분석의 관점에서 볼때 위의 형식은 관측치가 되며, 표준적인 変方分析은 충분한 自由度가 부족하기 때문에 S와 T의 상호영향을 관찰하기 어렵다. 유일한 방법은 TUKEY 특수정형이다. (형식)
$D^{(P)} = S + T + \theta S \cdot T + I$

그리고 SABL 정식에서 P=-1 → 1 에서 θ=0의 T의 통계량으로 상관의 법칙실험을 제출하기도 하였다.

10. SABL에서 관계있는 日曆의 영향은 매달마다 요일의 수가 다른 것과 관련이 있다. 누적된 달을 단위로 한 시간서열(序列)은 달(月份)의 장단과 특별휴가로 인한 영향등을 고려한다. SABL은 일력(日曆)을 기억해야 하며 휴가도 알고 있어야 한다.
현재의 휴일은 미국을 기준[표준]으로 하고 있다. 사용자가 자의로 조정할수 없다. 이것과 관계된 日曆의 계산은 月을 단위로 한 서열에 적용될 뿐이다.

11. SABL의 공작순서는 圖6을 보시오 계절과 일력의 조정은 그림 8을 보라.

12. SABL의 특징은 SABL 도표를 보면 알수 있다. 図9 ~図11을 보라

13. X-11과 비교해서 SABL은 특징이 몇가지 있다. 하나는 power transformation (乘法과 加法 모형은 영속적으로 적용될 수 있는것이 아니다) 둘째는 SABL은 밑부분(아래지역)은 자연적 성장이 연장되며 X-11은 下灣(대만 아래지역)에 미치는 영향은 표(도표) 17을 보라. 세째은 SABL의 뛰어난 점은 근대의 처리와 비교하여 X-11은 비록 있긴하지만 만족스럽지는 못하다 도표 18과 19를 보라.

結語

SABL이나 X-11은 모두 사용목적이 계절의 조정이라는데 있다. 이것은 비록 견실한 기초이론은 없지만 처리문제에서 시급한 해결을 도모하려고 수십년동안 폭넓은 사용과 [illegible] 계통의 원조를 받아 시간序列 분석의 중요한 도구가 된다고 할 수 있다. 필자는 몇가지 느낌을 적고자 한다.

① 이 방법을 발전시킬때 매우 조심스럽게 주의해야 함을 느낀다. 이 정식은 중국학자나 기관에서 만든 것이 아니다. 이론상의 문제 때문이 아니라 모험을 하고자 하는 정신이 없기 때문일 것이다. 새도구를 채택하는 속도가 느리고 발전시킨 程式이 대부분 사용자의 수중으로 확산되지 못한 것이 현실이다.

② X-11이 T-形의 구식이라면 SABL은 1982년 신형정식이라 하겠다. 기본상으로 두가지가 동일한 구조를 갖는다. 「계절조정」은 광범위하게 응용된 것이므로 모든 이론들이 다 세워진후 적용시키기

곤란하다. 「季節調整」은 유사 문제의 주요한 부분을 정의할 수 있기 때문에 뜻이 매우 명확하다. 하나의 程式系統을 만들기 전에 우선 목적과 원하는 결과를 상세히 해 두지 않으면 안된다.

③ 하나의 유용한 정식을 만드는 것은 비교적 쉽지만 모두가 이런 程式을 채용한다는 것은 어렵다고 생각한다. 충분한 홍보와 설득력 있는 설명이 요구되기도 한다. 또한 더욱 긴요한 일은 하나의 유효한 계통을 만드는 것이라 하겠다. 이미 발전시킨 정식은 우리에게 도움이 되고 직접 활용할 수 있도록 손쉬운 것이라야 하겠다.

이와 같은 원리는 버스가 매표소를 필요로 하는 것과 같은 것이다. 그것은 대북의 버스 품질은 당연히 뉴욕의 버스와 달아 우리나라 나름대로의 교통부가 있어 服務 편리함을 제공하는 것이다. 이원리는 X-11 이나 SABL을 이용하면 물론 맞는다. 그것은 BMDP, SAS, SPSS 등을 이용할 일반적인 程式庫로 보면 같은 것이다. 이미 발전된 정식고(程式庫)은 이미 애기 되어지지 않은 것은 연구되어 지지 않았지에 마땅이 근무을 주로 해야 한다. 사용자에게 말씀드리거나 고기앞에 가서 충분히 먹은 것이 진귀품앞에서서 보는 것보다 더욱 의의가 있는 것이 아니겠는가.

図 1

INTERSTATE TOLL MESSAGES (IN THOUSANDS)

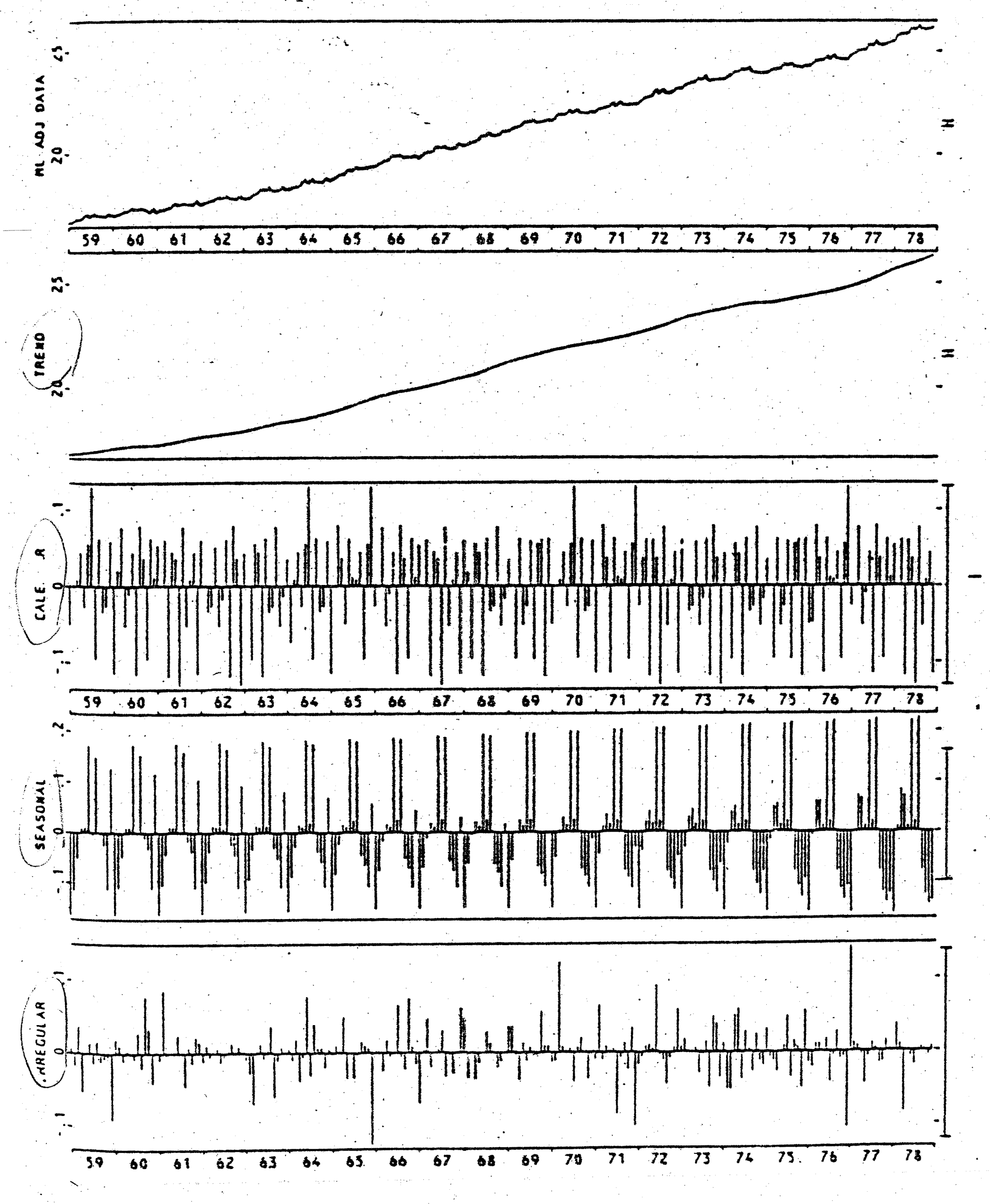


圖 2

SMOOTHING

SMOOTHING

DATA	MOVING AVERAGE OF 3	MOVING AVERAGE OF 5
-3		
-3	-2	
0	0	0
3	2	0
3	1	0
-3	-1	0
-3	-2	0
0	0	0
3	2	0
3	1	0
-3	-1	0
-3	-2	0
0	0	0
3	2	
3		

INCREASING THE WINDOW LENGTH OF A SMOOTHER INCREASES THE SMOOTHNESS OF THE SMOOTHED VALUES.

図3

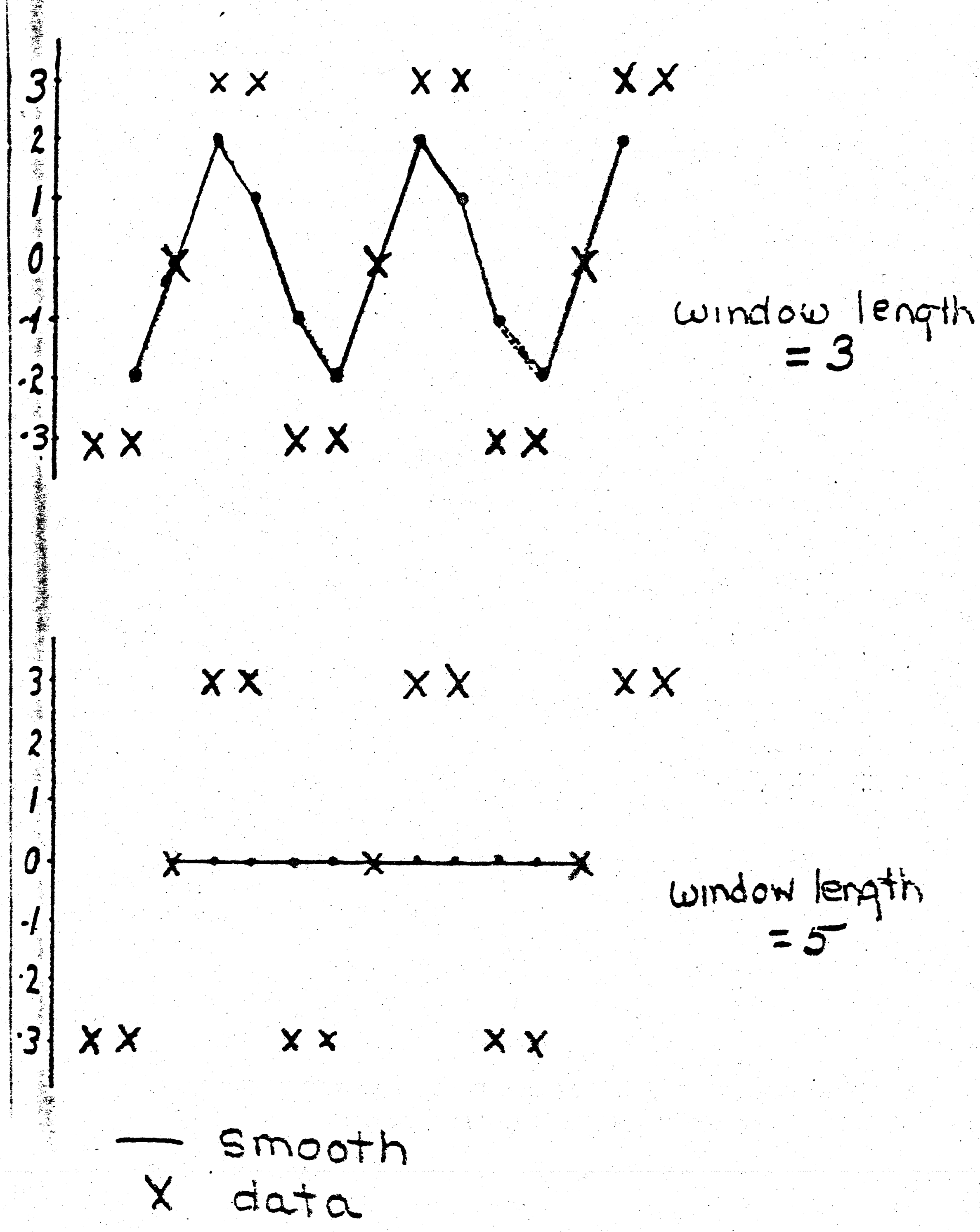


圖 4

POWER TRANSFORMATION

POWER TRANSFORMATION FAMILY DEFINED AS:

$$D^{(P)} = \begin{cases} D^{P} & P > 0 \\ \text{Log}_E D & P = 0 \\ -D^{P} & P < 0 \end{cases}$$

DATA ARE TRANSFORMED TO ALLOW AN ADDITIVE DECOMPOSITION TO BE USED

IF VALUE OF P IS INCORRECT, SEASONAL AMPLITUDES WILL DEPEND ON LEVEL OF TREND

CHOOSING P

SABL PROVIDES AN AUTOMATED PROCEDURE OF CHOOSING P FROM AMONG THE VALUES 0, ± 1/4, ± 1/2, ±1 BY MINIMIZING THE ABSOLUTE VALUE OF A CERTAIN T-STATISTIC.* FOR UNEMPLOYED MALES OVER 20 THE VALUES ARE

P	T-STATISTIC
-1	-8.3
-1/2	-4.9
-1/4	-3.1
0	-1.5
1/4	.1
1/2	1.5
1	4.2

IF 0 OR 1 IS CLOSE TO "OPTIMAL" WE MAY WANT TO USE IT FOR SIMPLICITY.

* MINIMIZES DEPENDENCE OF SEASONAL AMPLITUDE ON TREND.

圖 6

SABL PROCEDURES

1. MONTH-LENGTH ADJUSTMENT IF CALENDAR IS PRESENT

$$\text{DATA} = \frac{(\text{TOTAL FOR MONTH})\ (\text{AVERAGE MONTH LENGTH})}{\text{LENGTH OF MONTH}}$$

AVERAGE MONTH LENGTH = 30.4375

2. CHOOSE A POWER TRANSFORMATION FOR THE DATA, D,

$$(D)^{(P)} = \begin{cases} D^{P} & P > 0 \\ \text{LOG } D & P = 0 \\ -D^{P} & P < 0 \end{cases}$$

3. DECOMPOSE $D^{(P)}$ BY SMOOTHING TECHNIQUES

$$D^{(P)} = S^{*} + T^{*} + I^{*}$$

■7

4. REGRESS I^* ON SEVEN VARIABLES TO GET ESTIMATES

$\hat{A}_1,\ldots,\hat{A}_7$ COMPUTE CALENDAR.

$C(M) = \hat{A}_1\,\overline{D}_1(M)+\ldots+\hat{A}_7\,\overline{D}_7(M).$

$\overline{D}_1(M)$ = PROPORTION OF MONDAYS IN THE MONTH

$\overline{D}_2(M)$ = PROPORTION OF TUESDAYS, ETC.

MUST DO MATCHED PROCESSING OF $\overline{D}_I(M)$.

X-11 DOES NOT.

5. DECOMPOSE $D^{(P)} - C$

$D^{(P)} - C = S + T + I$

$D^{(P)} = S + T + I + C$

THE DECOMPOSITION IS ROBUST IN THE SENSE THAT OUTLIERS GO INTO THE IRREGULAR AND DO NOT DISTORT THE SEASONAL OR THE TREND. EACH DATA POINT GETS A WEIGHT BETWEEN 0 AND 1.

6. GRAPHICS AND TABLES

圖 8

ADJUSTMENT WHEN TRANSFORMATION IS USED

WITHOUT CALENDAR

$D^{(P)} = T + S + I$

$D^{(P)} - S = T + I$

SEASONALLY ADJUSTED DATA $= (T + I)^{[P]}$

WITH CALENDAR

$D^{*(P)} = T + C + S + I$

$D^{*(P)} - C - S = T + I$

CALENDAR AND SEASONALLY ADJUSTED DATA $= (T + I)^{[}$

WHERE INVERSE TRANSFORMATION IS DEFINED AS:

$$(T + I)^{[P]} = \begin{cases} (T + I)^{\frac{1}{P}} & \text{FOR } P > 0 \\ E^{(T + I)} & \text{FOR } P = 0 \\ (-T - I)^{\frac{1}{P}} & \text{FOR } P < 0 \end{cases}$$

D^{*} = MONTH LENGTH CORRECTED DATA

図9

Data and three components plot. Telephone installations. SABL with NSEASONAL = 11, NTREND = 15, and p = 0.

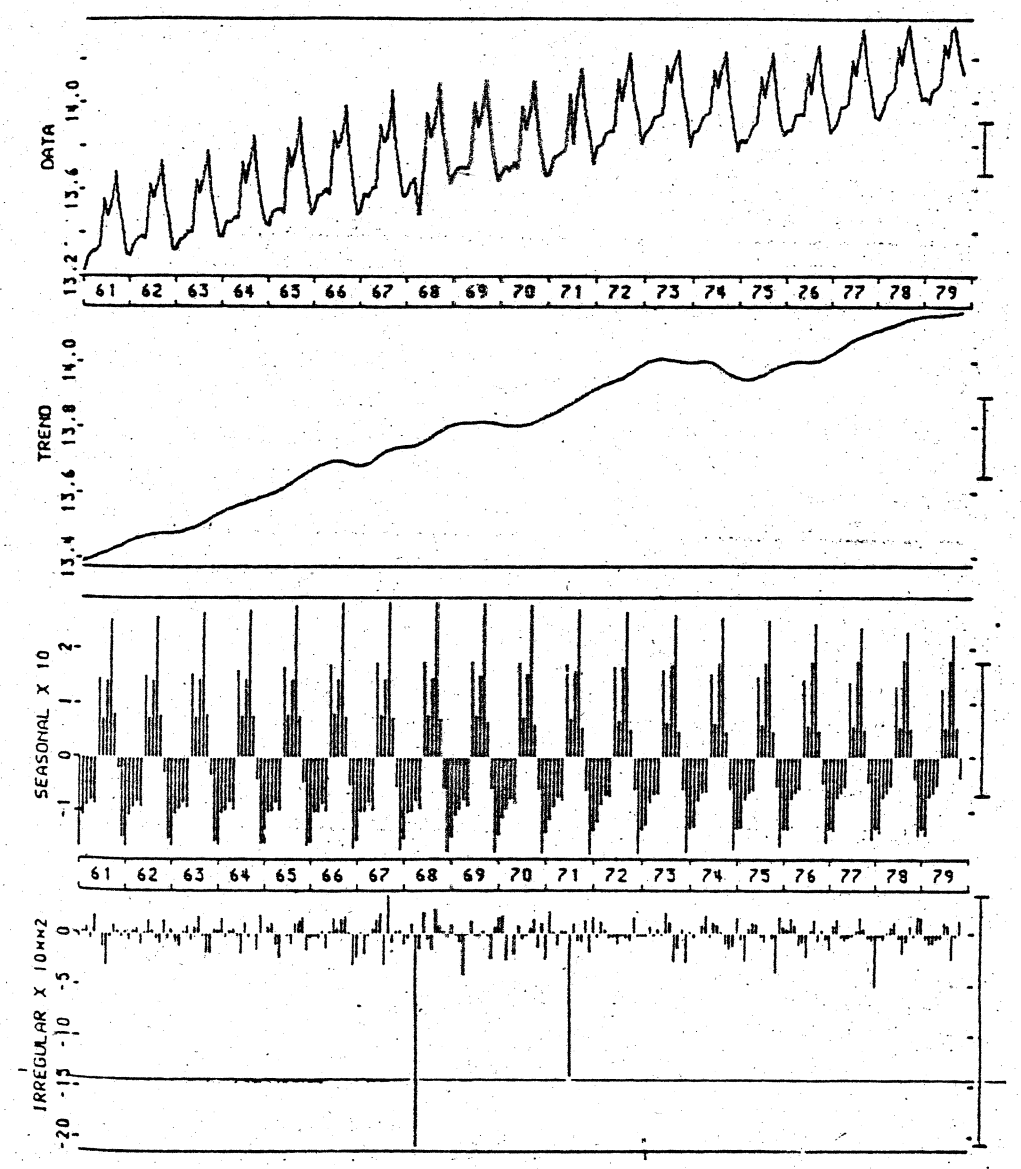


圖 10

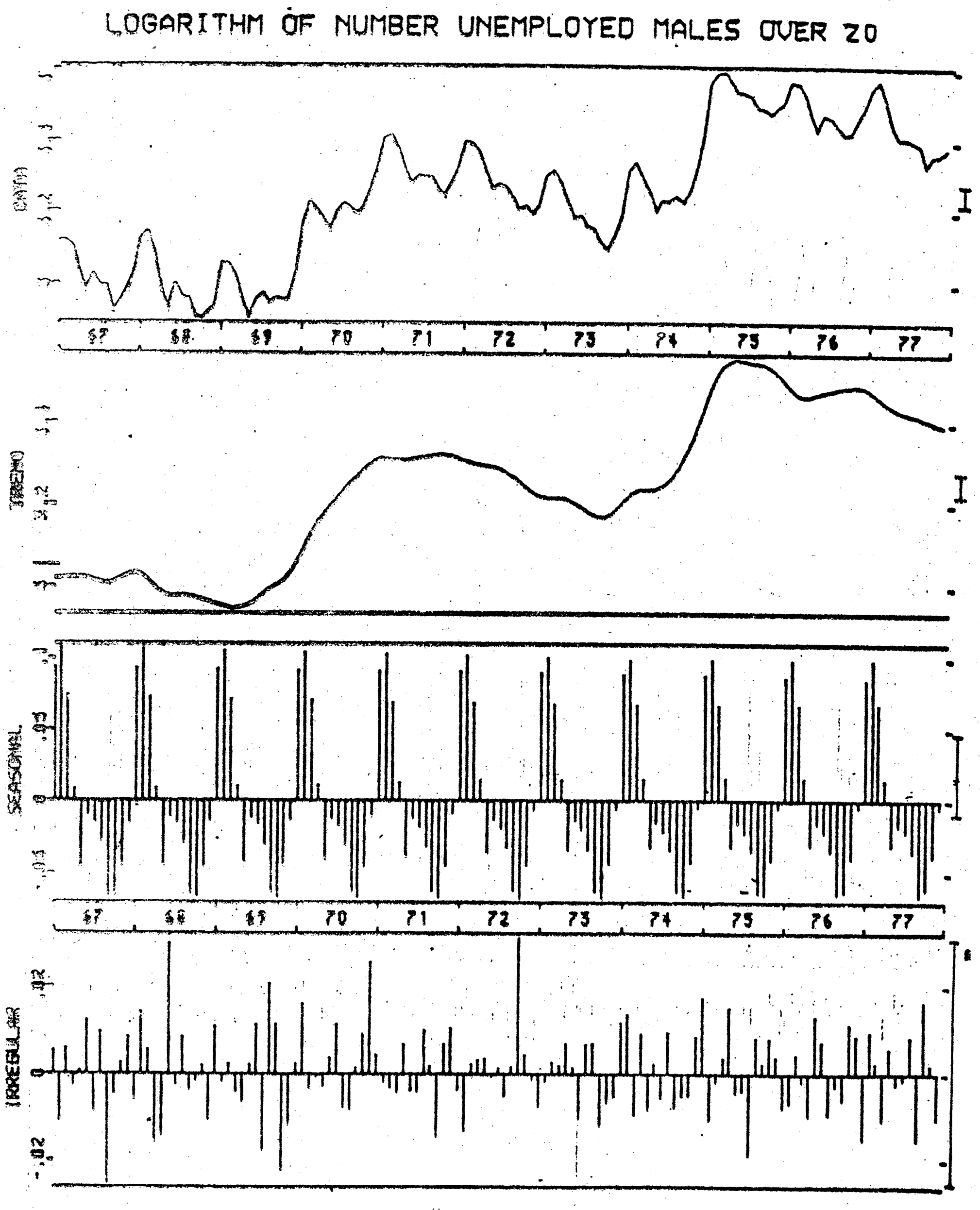

LOGARITHM OF NUMBER UNEMPLOYED MALES OVER 20
DATA
TREND
SEASONAL
IRREGULAR
67
68
69
70
71
72
73
74
75
76
77
YEARS

図 11

Seasonal-by-month plot. Telephone installations.
SABL with NSEASONAL = 11 and NTREND = 15 and p = 0.

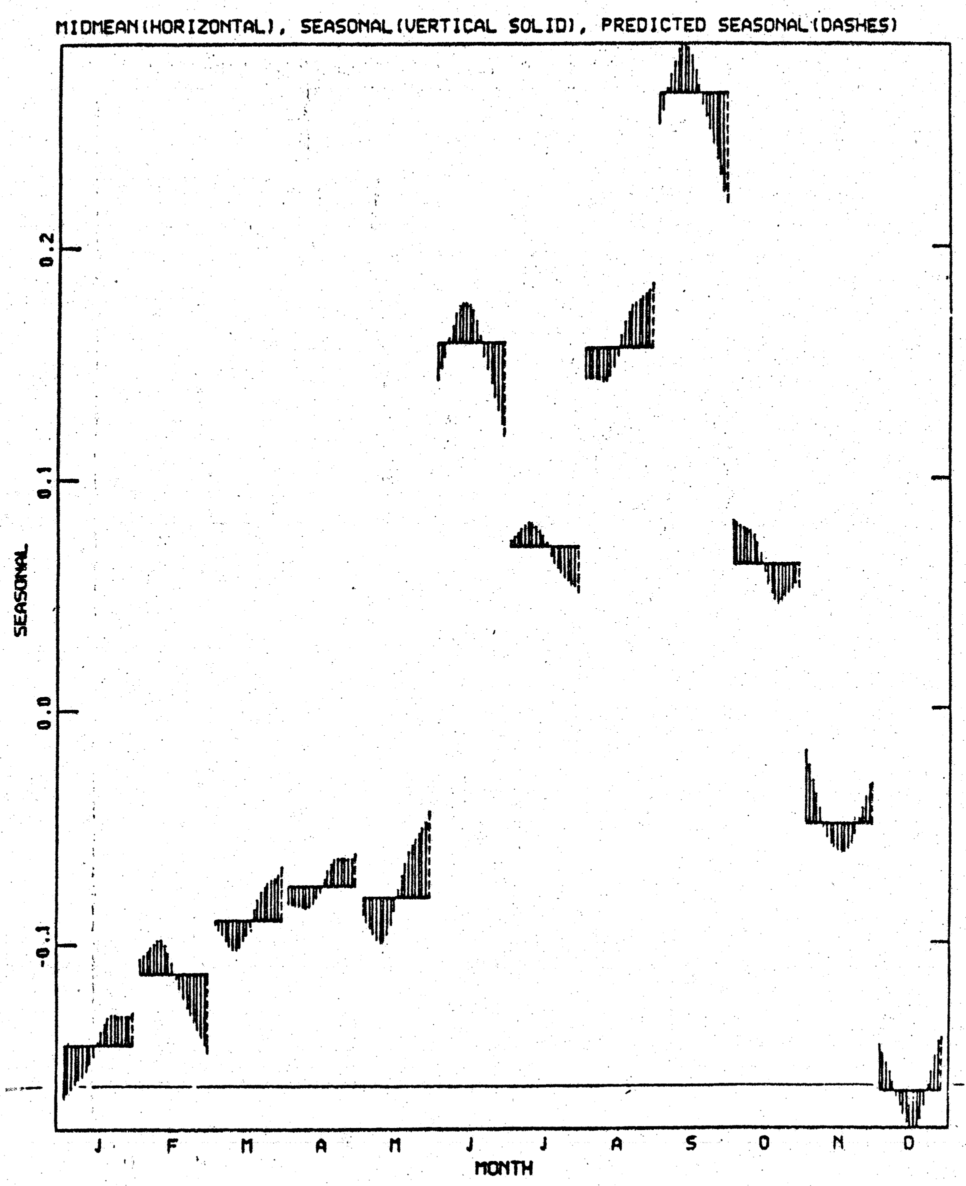


図 12

Seasonal-by-month plot. Inverse fourth roots of industrial production. SABL with NSEASONAL = NTREND = 15 and p = .25.

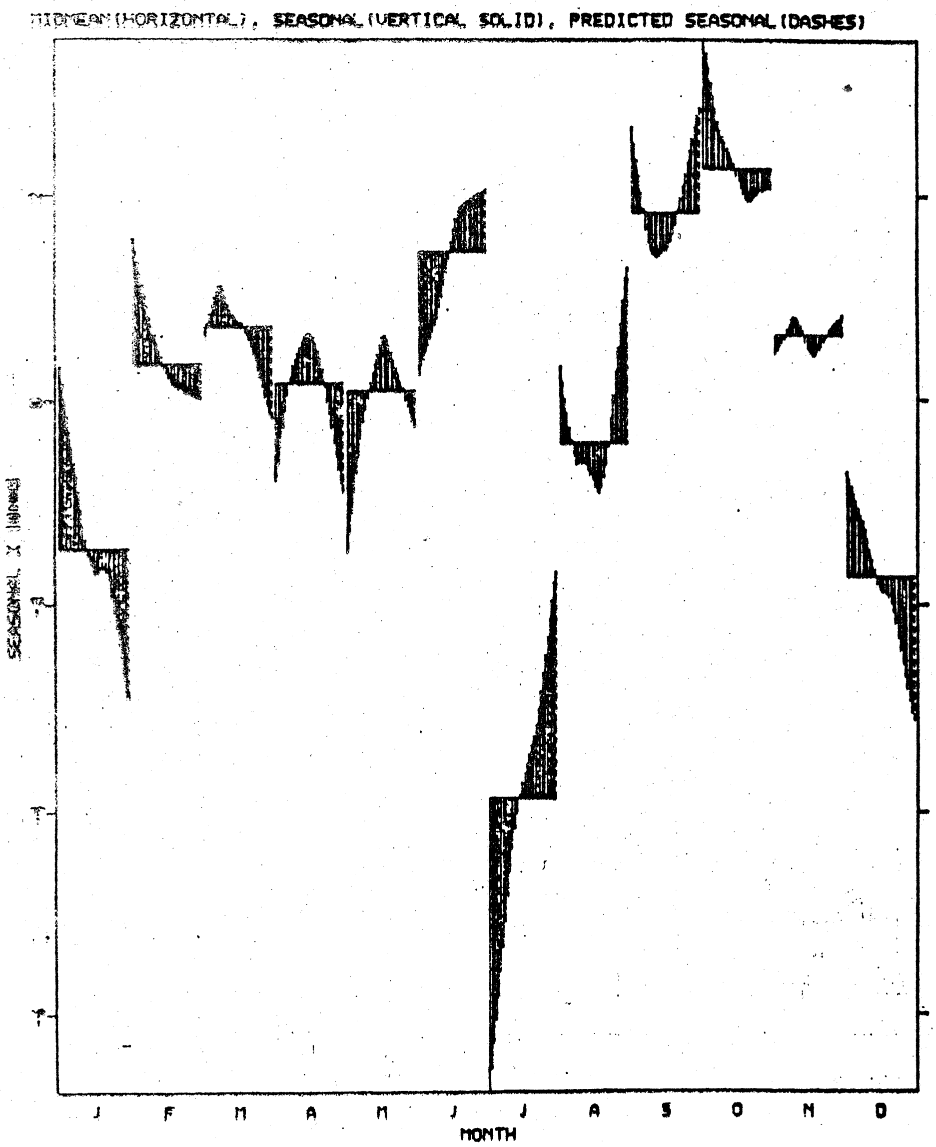


図 13

Seasonal amplitude plot. Telephone installations. SABL with NSEASONAL = 11, NTREND = 15, and p = 1.

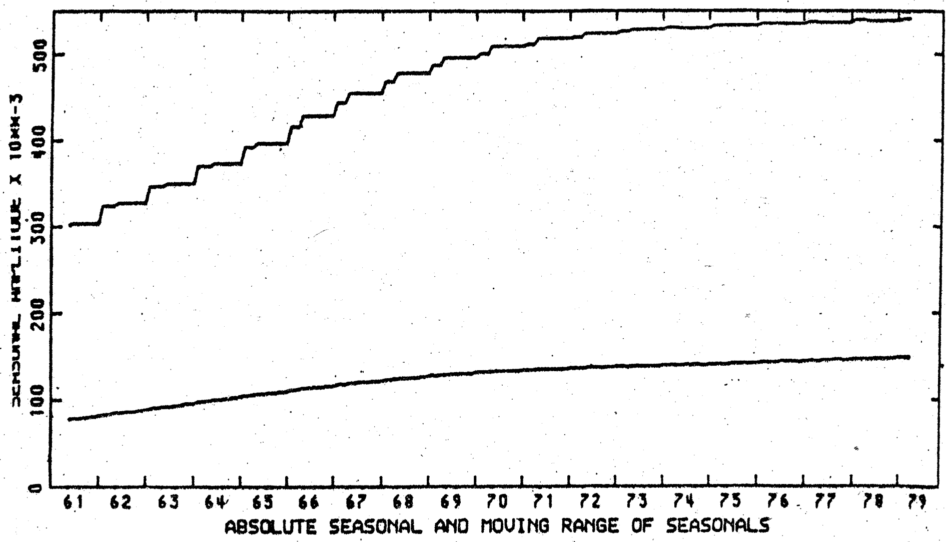


図 14

Seasonal amplitude plot. Telephone installations. SABL with #SEASONAL = 11 and #TREND = 15 and p = 1/4.

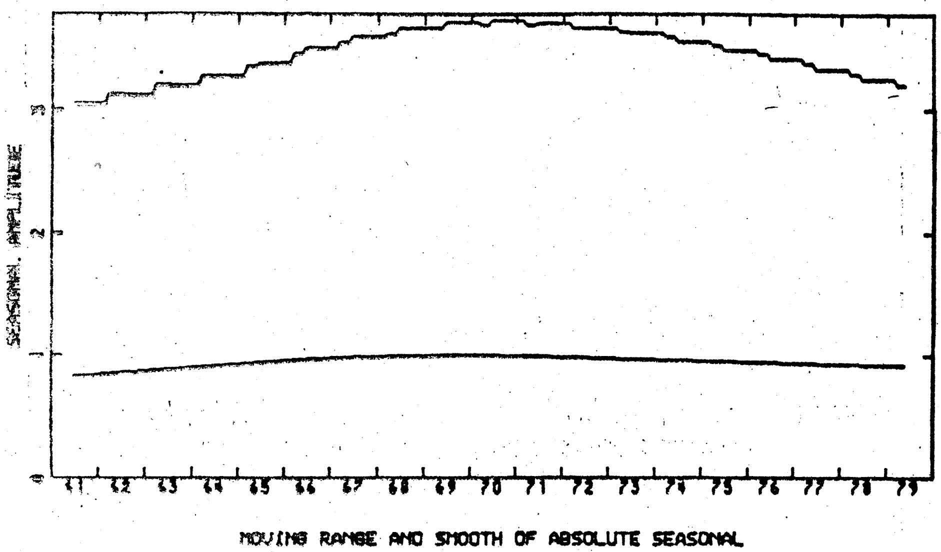


圖 15

Seasonal and data-minus-trend plot. Telephone installations. SABL with NSEASONAL = 11, NTREND = 15, and p = 0.

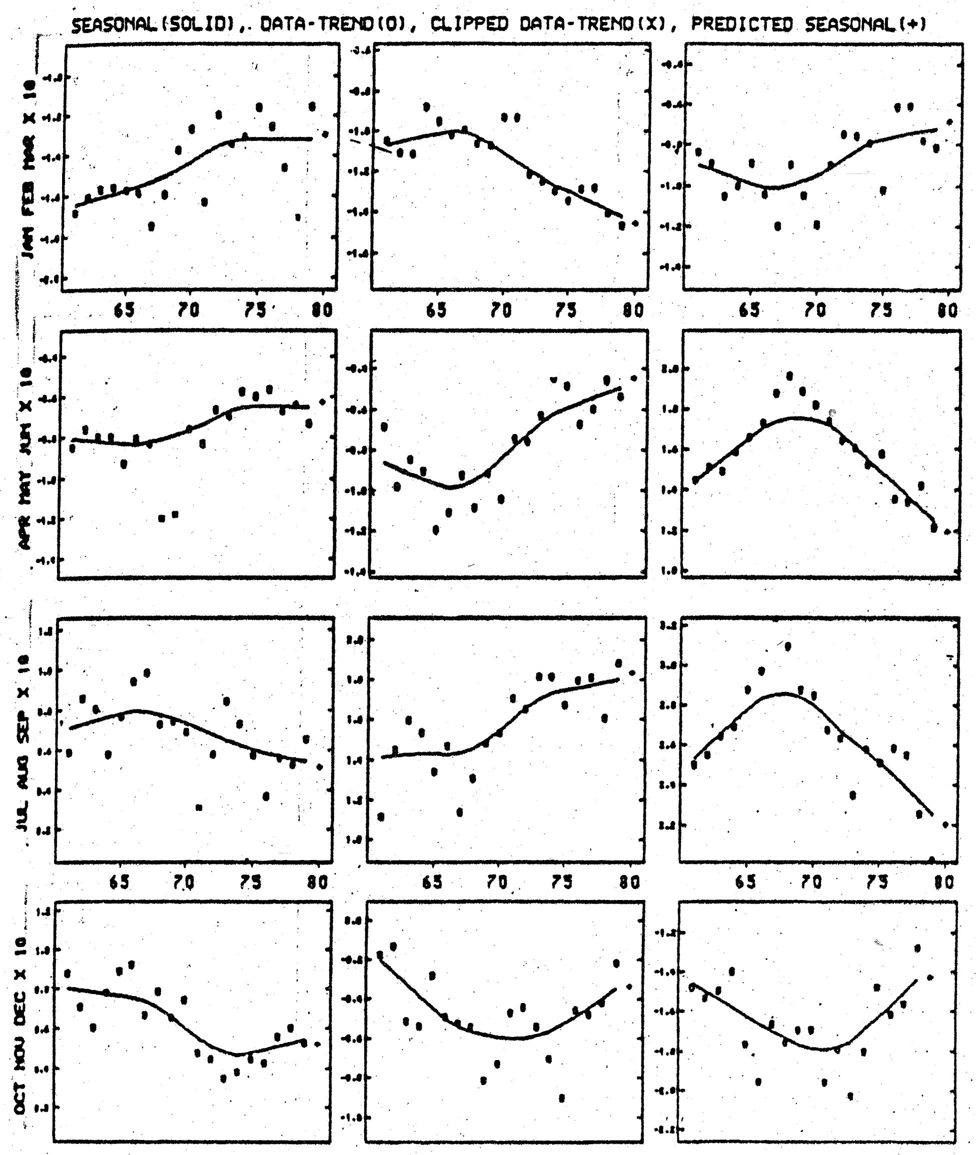


図 16

Seasonal and data-minus-trend plot. Agriculturally employed males over 19. SABL with NSEASONAL = 7, NTREND = 15, and p = 1/3.

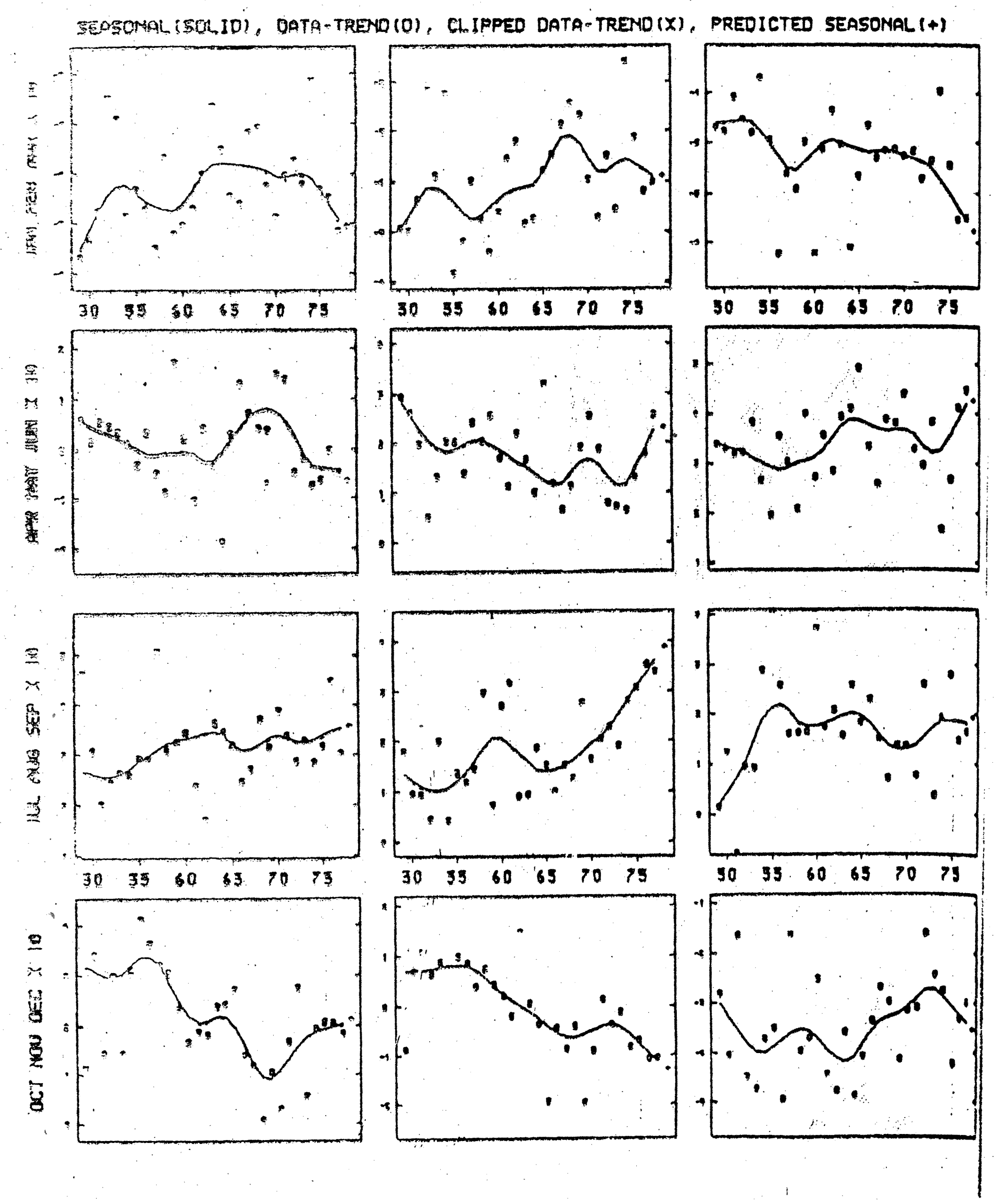
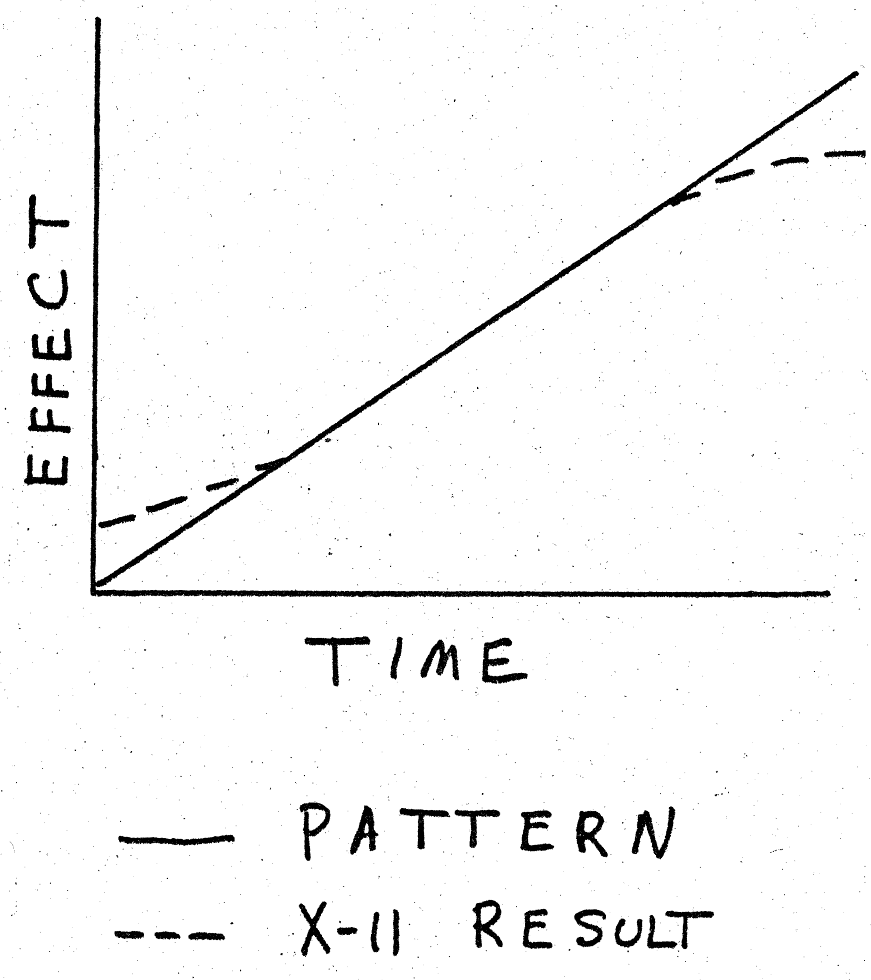
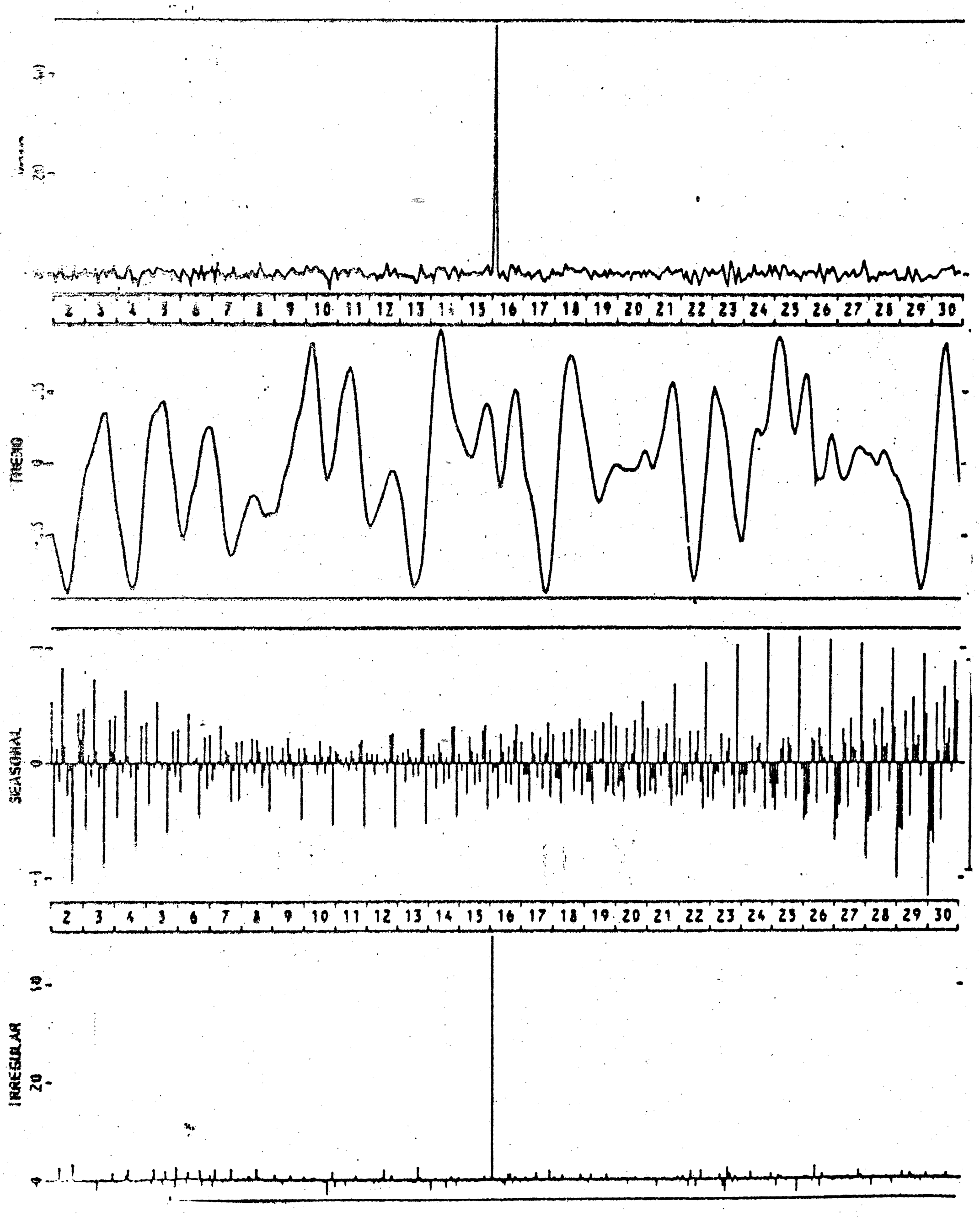


圖 17

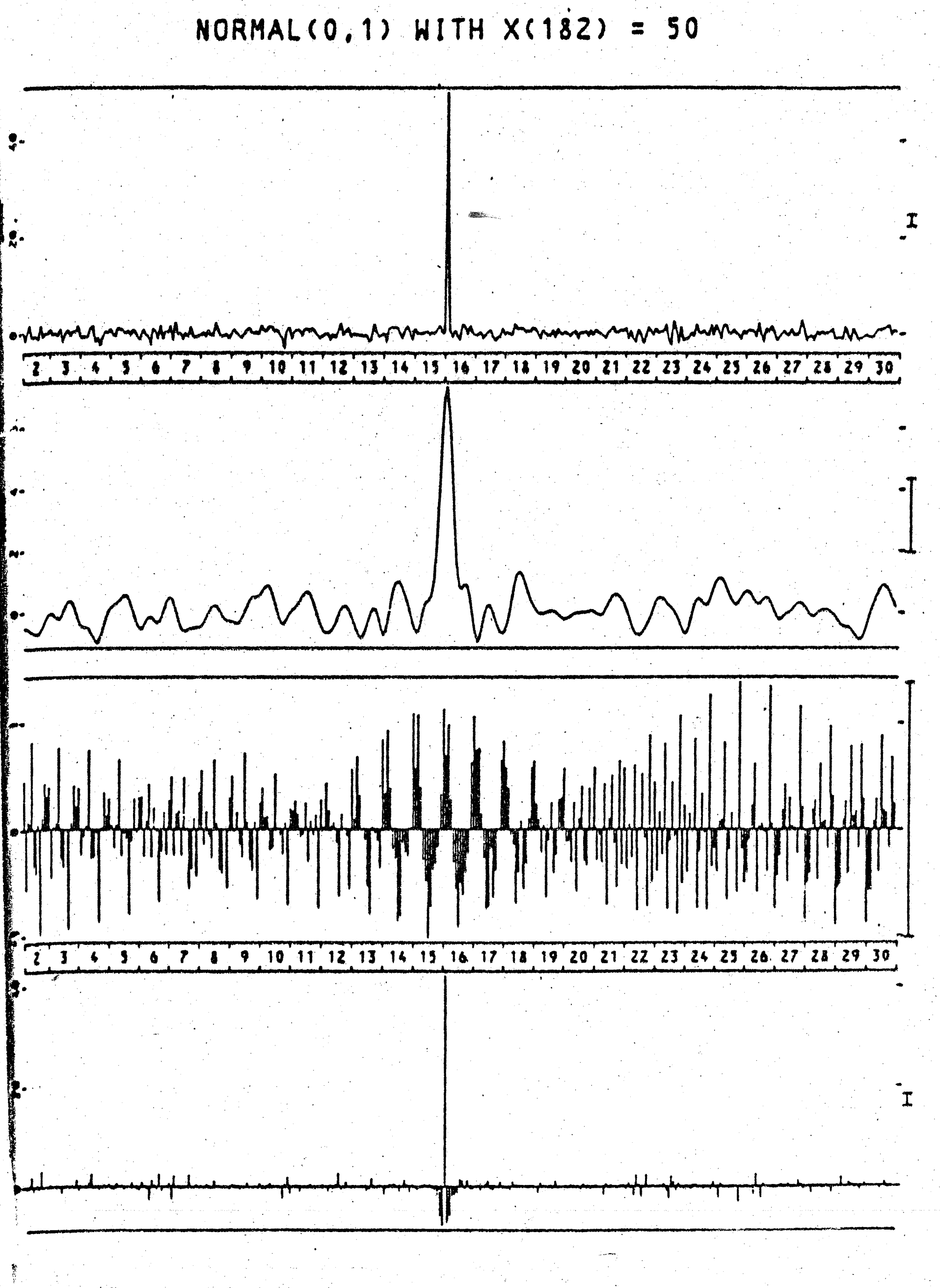


SABL

NORMAL(0,1) WITH X(182) = 50



X-11


NORMAL(0,1) WITH X(182) = 50
2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30
2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30

LINEAR APPROXIMATIONS TO THE CENSUS AND BLS SEASONAL ADJUSTMENT METHODS*

ALLAN H. YOUNG
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The Census and BLS ratio-to-moving-average seasonal adjustment methods are usually specified by procedural steps. They can be written mathematically as combinations of linear and non-linear operations on the logarithms of the original series. For some purposes the non-linear operations can be ignored. The Census and BLS historical seasonal factors are almost identical to those from a simpler model using ratios to a centered 12-term moving average. They also closely resemble those from regression models using seven years of data. In three test series, revisions between current seasonal factors and historical seasonal factors obtained when the adjustment is recomputed with additional data are smaller with the Census model than with the 12-term or regression models. By recomputing the adjustment twice a year instead of once, the same size revisions are obtained with the 12-term as with the Census model. The Census trend-cycle specifications differ from the 12-term and regression models.

1. INTRODUCTION AND SUMMARY

THE ratio-to-moving-average method of seasonal adjustment is usually specified as a sequence of steps. The sequence includes (a) calculating a moving average of the original data arranged in chronological order to estimate the trend-cycle; (b) dividing the trend-cycle estimates into the original data to obtain a series of "seasonal-irregular" ratios; and (c) for each month calculating a moving average of the "seasonal-irregular" ratios to estimate the seasonal factors for that month.

Typically, the following procedures, which complicate the mathematical statement of the method, are added: (a) The sequence of steps is iterated, starting with an improved trend-cycle estimate computed from a preliminary seasonally adjusted series. (b) Each time a moving average is computed an appropriate set of "end weights" is used to extend the moving average so that no terms are lost. (c) Extreme values are identified and modified. (d) The method, while based on the multiplicative assumptions, $Y = C \times S \times I$, equates the 12-month sums of the seasonally adjusted and the unadjusted data rather than their products.

The third and fourth procedures introduce non-linearities, but since the treatment of extremes can be considered a prior operation, the equating of sums is the major obstacle preventing an explicit statement of the method as a linear model. If the multiplicative condition were dropped, the seasonal factors could be expressed as weighted sums of the original series. If the equality of sums condition were dropped, the logarithms of the seasonal factors could be ex-

* The author is under heavy obligation to many persons at the Bureau of the Census where [illegible] of this paper was prepared. Especially helpful were Julius Shiskin and Harry Rosenblatt. Allan Zelenitz, [illegible] Marshall, Edward Melnick and David Bateman also contributed significantly. Kenneth Brewer of the [illegible] Dominion Bureau of Statistics offered several helpful comments on a draft version. I am also indebted to the [illegible] for several suggestions.

pressed as weighted sums of the logarithms of the original series. Traditionally, economists have not wanted to give up either condition just to obtain a linear model. They have observed that S and C are usually related multiplicatively and that by taking ratios they can reduce the variability of the seasonal factor curves. Also, they desire to present seasonally adjusted series in which annual totals rather than products are unchanged.

Despite the non-linearities, the ratio-to-moving-average seasonal adjustment procedure can be written in explicit mathematical statements. The logarithms of the seasonal, trend-cycle and irregular are each represented as a combination of non-linear and linear operations on the logarithms of the original series.

The linear operations are expressed as moving weighted averages of the logarithms of the original series; the weights are obtained from the various moving averages used in Method II, the Census ratio-to-moving-average seasonal adjustment procedure. The logarithms of the seasonal factors are estimated by a 145-term moving average, and those of the trend-cycle and irregular components by 157-term moving averages. The first and last 72 seasonal factors and the first and last 78 trend-cycle and irregular factors are estimated by asymmetrical sets of end weights.

The non-linearities arise from the conjunction of additive operations on the series and its logarithms. For some purposes the effect of the non-linear operations on the seasonal factors can be considered negligible and the linear operations can be viewed as an approximation to the ratio-to-moving-average methods.[1]

The linear statements permit comparison of the Census ratio-to-moving-average model with the BLS model and comparison of these with the ratio to centered 12-term moving average model. These comparisons provide fresh insights into the Census and BLS seasonal adjustment methods. They suggest several avenues for simplification or modification, although the high speed of electronic computers may make such changes unnecessary as far as computing costs.

The linear statements show that the Census and BLS seasonal factors are almost identical. Except for the six terms at the beginning and end of the series and the six corresponding year-ahead factors, the Census Method II seasonal factors are also almost identical to those computed from the ratios to a centered 12-term moving average. At the ends, the Method II factors are superior to those obtained in the 12-term moving average model by simply repeating the factor for the year before. The revisions which occur in the end terms when the adjustment is recomputed with several additional years of data are smaller for Method II than for the 12-term moving average model. However, by recalculating the adjustment every six months instead of once a year the 12-term

[1] Several investigators have indicated that a linear statement approximating the ratio-to-moving-average method could be derived. Hyman B. Kaitz [13] derived measures similar to some of those presented here for the BLS method. J. Durbin [5] presented the case of a stable, additive seasonal based on a 12-month moving average. J. P. Burman [2] and George R. Hext [10] presented transfer functions for the central estimates of the Census and BLS methods. Also, Dale W. Jorgenson's discussion of the general linear model for seasonal adjustment [12] considers many of the points discussed here. An alternative to the methods considered here are non-linear "amplitude adjustments" recently considered by Mi[illegible] Godfrey [7], and David Fairbarns [6].

moving average model could provide current factors of the same quality as Method II. While the use of the 12-term moving average model in this manner might be a worthwhile simplification, the problem of identifying extremes and the desire in some instances for a better trend-cycle estimate than that provided by a 12-term moving average should also be considered.

The linear statements also permit comparisons with multiple regression models which have been considered as alternatives to the ratio-to-moving-average models. Two differences are noted which although never presented in this manner before have contributed to the preference for the ratio-to-moving-average method. (a) The central trend-cycle expressions in the regression models are less flexible. (b) The statements for the current seasonal factors are quite different. The latter difference is evaluated empirically by comparing the revisions between the current and historical seasonal factors obtained by recomputing the adjustment with additional data. The current ratio-to-moving-average factors are judged superior to regression factors derived from seven years of data.

A simplified computer program is possible using the linear statements. The irregular component could be computed directly from the original series and extremes identified. Then seasonal factors could be computed directly from the original series. Also, while not developed in the article, the linear statements may provide a basis for significance tests and for improving the end weights used to estimate the current seasonal factors.

2. THE LINEAR OPERATIONS

A linear expression of the seasonal factor estimate of the X-11 variant of Method II can be derived as follows:

(a) The logarithm of the 12-term moving average of the original series is represented by

$$\widehat{\log C_t^{(12)}} = \sum_{m=-6}^{+6} c_{1,m} \log Y_{t+m} \qquad (1)$$

where the c_1's are the weights for the centered 12-term moving average and Y is the original series.

(b) The logarithm of the preliminary seasonal factor is represented by

$$\widehat{\log S_t^{(P)}} = \sum_{l=-2}^{+2} s_{1,l}\left(\log Y_{t+12l} - \sum_{m=-6}^{+6} c_{1,m} \log Y_{t+12l+m}\right) \qquad (2)$$

where the s_1's are the weights for the five-term (3×3) moving average.

(c) The logarithm of the preliminary trend-cycle is represented by

$$\widehat{\log C_t^{(P)}} = \sum_{k=-6}^{+6} c_{2,k}\left(\log Y_{t+k} - \sum_{l=-2}^{+2} s_{1,l}\left(\log Y_{t+k+12l} - \sum_{m=-6}^{+6} c_{1,m} \log Y_{t+k+12l+m}\right)\right) \qquad (3)$$

where the c_2's are the weights for the 13-term Henderson moving average. (In earlier variants the 15-term Spencer moving average was used.)[2]

(d) The seasonal factor estimate, $\hat{S}_t$, is represented by

$$\widehat{\log S_t} = \sum_{j=-3}^{+3} s_{2,j}\left(\log Y_{t+12j} - \sum_{k=-6}^{+6} c_{2,k}\left(\log Y_{t+12j+k} - \sum_{l=-2}^{+2} s_{1,l} \cdot \left(\log Y_{t+12j+k+12l} - \sum_{m=-6}^{+6} c_{1,m} \log Y_{t+12j+k+12l+m}\right)\right)\right) \tag{4}$$

where the s_2's are the weights for the seven-term (3×5) moving average. If N is the number of months in the original series, then $73 \leq t \leq N-72$. Combining the s_1, s_2, c_1 and c_2 weights into a single set of 145 weights, the expression can be written as follows:

$$\widehat{\log S_t} = \sum_{h=-72}^{+72} a_h \log Y_{t+h}, \qquad 73 \leq t \leq N - 72. \tag{5}$$

The seasonally adjusted series estimate is represented by

$$\widehat{\log (CI)_t} = \log Y_t - \widehat{\log S_t}, \qquad 73 \leq t \leq N - 72. \tag{6}$$

The final trend-cycle estimate is represented by

$$\begin{aligned} \widehat{\log C_t} &= \sum_{h=-6}^{+6} c_{2,h} \widehat{\log (CI)_{t+h}} \\ &= \sum_{h=-78}^{+78} b_h \log Y_{t+h}, \qquad 79 \leq t \leq N - 78, \end{aligned} \tag{7}$$

and the irregular component estimate by

$$\begin{aligned} \widehat{\log I_t} &= \widehat{\log (CI)_t} - \widehat{\log C_t} \\ &= \sum_{h=-78}^{+78} d_h \cdot \log Y_{t+h}, \qquad 79 \leq t \leq N - 78. \end{aligned} \tag{8}$$

The a_h and d_h sum to zero, the b_h to 1.0.

Each of the first and last 72 seasonal factors requires an additional set of weights based on the end weights used for the s_1, s_2, c_1 and c_2 moving averages. Thus there are 73 sets of a's, —one set of central weights and 72 end weight sets. The expressions for the first and last 72 seasonal factor estimates can be written in a manner similar to that used in equation (4). Due to their length they are presented here only in terms of the combined weights. For example,

$$\widehat{\log S_1} = \sum_{h=1}^{73} a_h^{(1)} \log Y_h. \tag{9}$$

At the end of the series the weights are applied in reverse order. For example,

[2] The Spencer and Henderson moving averages are described in Macaulay [16]. Henderson's general formula provides moving averages of any length which will reproduce a third degree parabola and in which the sum of squares of the third differences of the weights are a minimum. Minimizing the sum of squares of the third differences was used as a criterion for achieving smoothness in the fitted curve. Spencer's 15-term moving average approximates the same properties, but was designed for computational convenience.

$$\widehat{\log S_N} = \sum_{\lambda=1}^{73} a_\lambda^{(N)} \log Y_{N-73+\lambda} \tag{10}$$

where

$$a_\lambda^{(N)} = a_{74-\lambda}^{(1)}.$$

Similarly, there are 78 sets of end weights for the trend-cycle and 78 for the irregular.

The central weight sets for the seasonal, trend-cycle and irregular components are presented in chart 1, parts A to C. For each component three sets are shown—the set derived above using the 13-term Henderson moving average for the c_2 weights and sets based on 9-term and 23-term Henderson moving averages (see [28]). In part D, end weight sets based on the 13-term Henderson moving average are shown for the seventh from end and end seasonal factor, the two corresponding year-ahead factors,[3] and the seventh from end and end trend-cycle factors.

The derivations presented here for Method II can be extended directly to the BLS method [29]. The BLS central seasonal factors (1966 version) are approximated by a 165-term moving average of the logarithms of the original series.

The only difference of note is that the BLS method assigns zero weight to the last 3 terms of the series in the last 3 seasonal weight patterns, while *X*-11 assigns substantial weight to the end terms of the series (see chart 2).

3. THE NON-LINEAR OPERATIONS

Two procedures cause the ratio-to-moving-average factor to differ from the pure multiplicative factors which were derived in the preceding section. They are: (a) the use of arithmetic moving averages, and (b) the forcing of each 12-month period as close as possible to 12.

The use of arithmetic moving averages can be represented by introducing a correction, P, each time a geometric average is taken as in the pure multiplicative case, rather than an arithmetic average as in the ratio-to-moving-average case. Let S^* represent the final "unforced" ratio-to-moving-average factor, then

$$\begin{aligned}\widehat{\log S_t^*} = \log P_t^{(j)} + \sum_j s_{2,j}\Bigg(&\log Y_{t+12j} - \log P_{t+12j}^{(k)} \\ &- \sum_k c_{2,k}\Bigg(\log Y_{t+12j+k} - \log P_{t+12j+k}^{(l)} \\ &- \sum_l s_{1,l}\Bigg(\log Y_{t+12j+k+12l} - \log P_{t+12j+k+12l}^{(m)} \\ &- \sum_m c_{1,m} \log Y_{t+12j+k+12l+m}\Bigg)\Bigg)\Bigg).\end{aligned}$$

[3] Year-ahead seasonal factors are estimates for the 12 months following the last month of data included in the computer run. They are used to adjust data as it becomes available in the interval between annual computer runs.

CHART 1. CENSUS X-II COMBINED WEIGHTS

Part A. Central Seasonal Factor Estimates

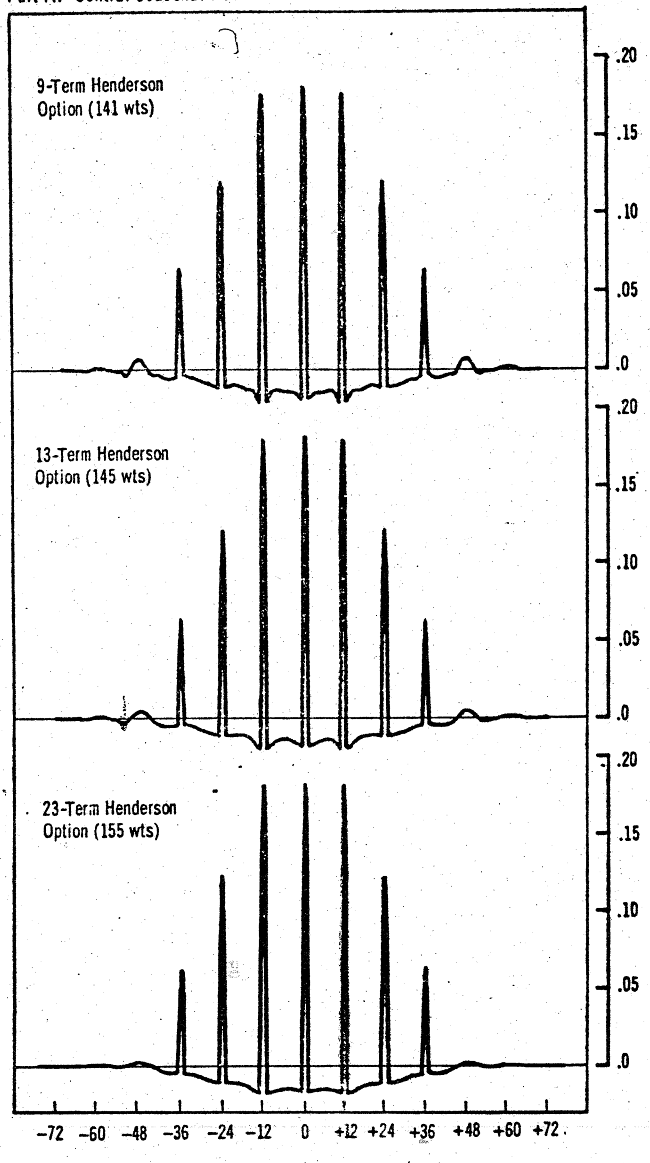


CHART 1. CENSUS X-II COMBINED WEIGHTS—Continued

Part B. Central Trend-Cycle Estimates

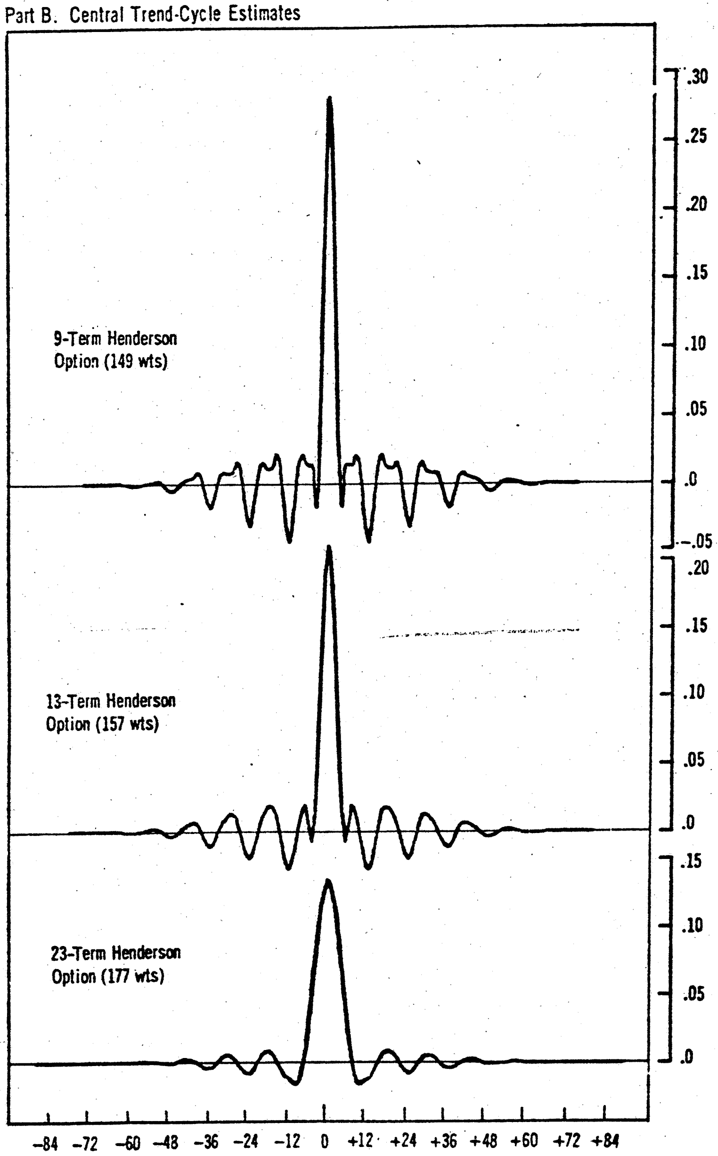


CHART 1. CENSUS X-11 COMBINED WEIGHTS—Continued

Part C. Central Irregular Estimates

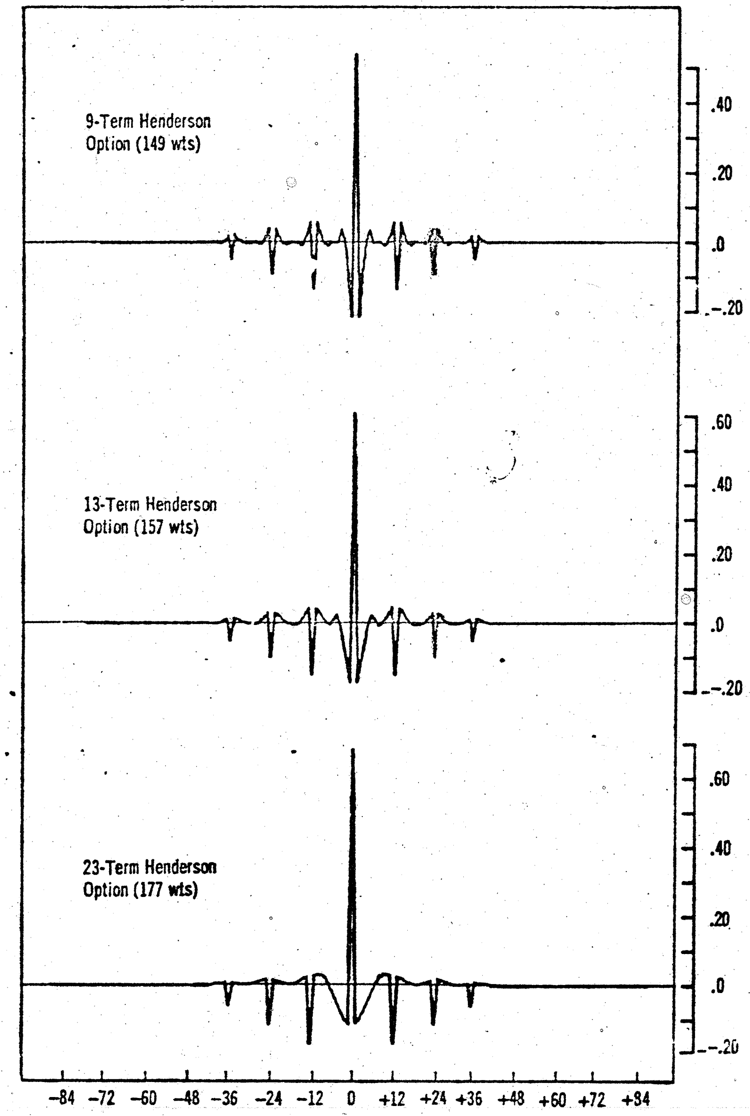
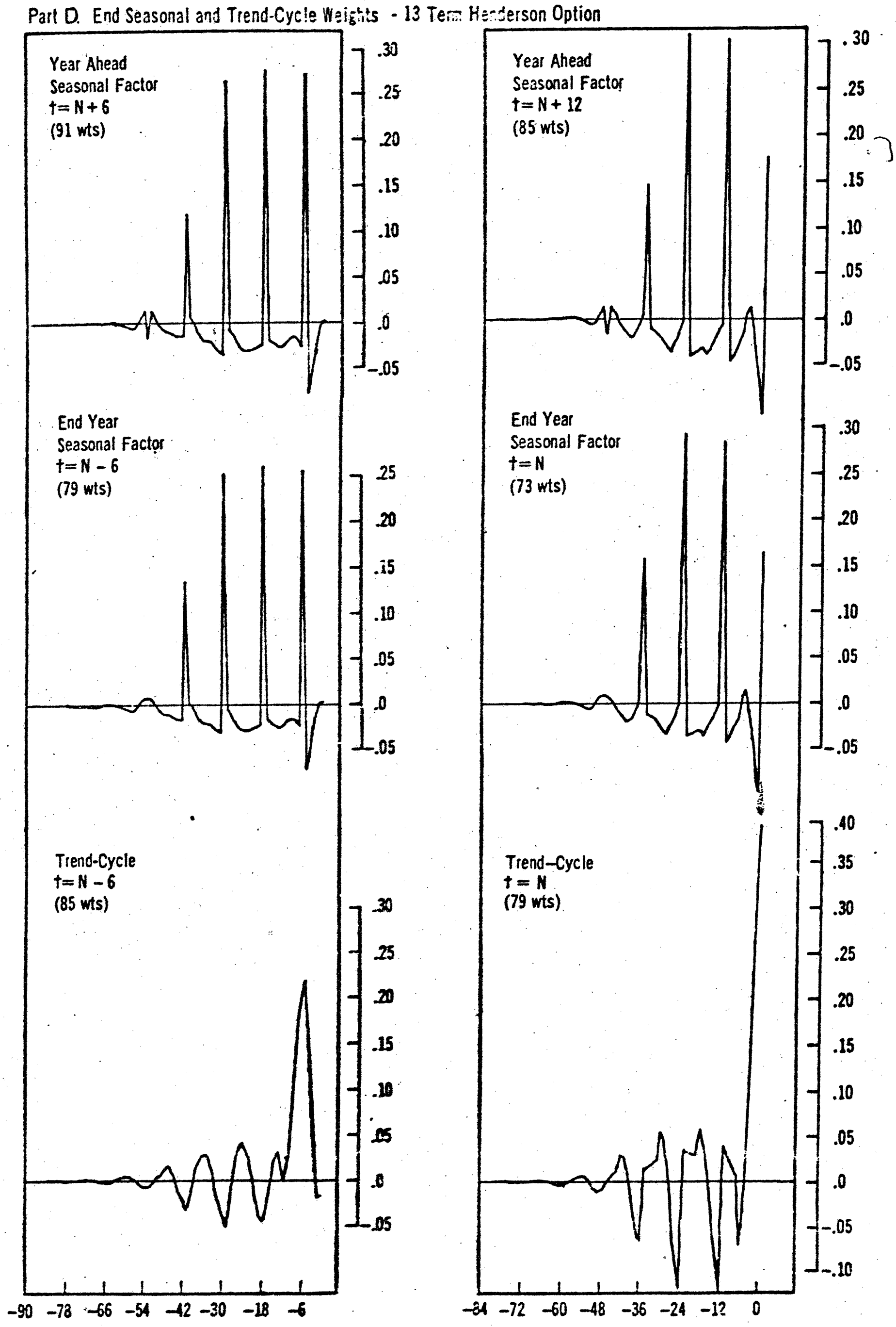


CHART 1. CENSUS X-11 COMBINED WEIGHTS—Continued

Part D. End Seasonal and Trend-Cycle Weights - 13 Term Henderson Option



$$\widehat{\log S_t^*} = \log P_t^{(j)} - \sum_j s_{2,j} \log P_{t+12j}^{(k)} + \sum_j \sum_k s_{2,j} c_{2,k} \log P_{t+12j+k}^{(l)} - \sum_j \sum_k \sum_l s_{2,j} c_{2,k} s_{1,l} \log P_{t+12j+k+12l}^{(m)} + \sum_{h=-72}^{+72} a_h \log Y_{t+h} \tag{11}$$

where $P^{(j)}$ is the ratio of a weighted arithmetic mean to a weighted geometric mean of the seasonal-irregular ratios for a given month over a span of seven years; $P^{(l)}$ is a similar measure computed over a span of five years; $P^{(m)}$ is the ratio of a weighted arithmetic to geometric mean of the original data computed over 13 months; and $P^{(k)}$ is the ratio for the seasonally adjusted series computed over 13 months.

$P^{(l)}$ and $P^{(j)}$ depend on the variance of the irregular component and on the amount of moving seasonality; $P^{(m)}$ depends on the variance of the original series computed over 13-month spans; and $P^{(k)}$ depends on the variance of the seasonally adjusted series computed over 13-month spans (particularly on the middle 5 months where most weight is concentrated in the c_2 pattern).[4]

Whether $\hat{S}$ lies above or below $\hat{S}^*$ depends upon the nature of the series. In general $\hat{S}$ is greater than $\hat{S}^*$ since $P^{(k)}$ depends on the variation in the irregular and cyclical and $P^{(m)}$ depends on these and also on the seasonal, while $P^{(j)}$ and $P^{(l)}$ depend only on the variation in the irregular with a small additional effect from moving seasonality. An extremely high or low value in a series, however,

---

[4] The P's are defined as:

$$P^{(m)} = \sum_{m=-6}^{+6} c_{1m} Y_m \Big/ \prod_{m=-6}^{+6} Y_m^{c_{1m}}$$

and

$$P^{(l)} = \frac{\sum_{l=-2}^{+2} s_{1,l} Z_{12l}}{\prod_{l=-2}^{+2} Z_{12l}^{s_{1,l}}}$$

where Z represents the SI ratio,

$$Z_{12l} = \frac{Y_{12l}}{P_{12l}^{(m)} \prod Y_{12l+m}^{c_{1,m}}} = \frac{Y_{12l}}{\sum_m c_{1,m} Y_{12l+m}}.$$

$P^{(j)}$ and $P^{(k)}$ are similarly defined.

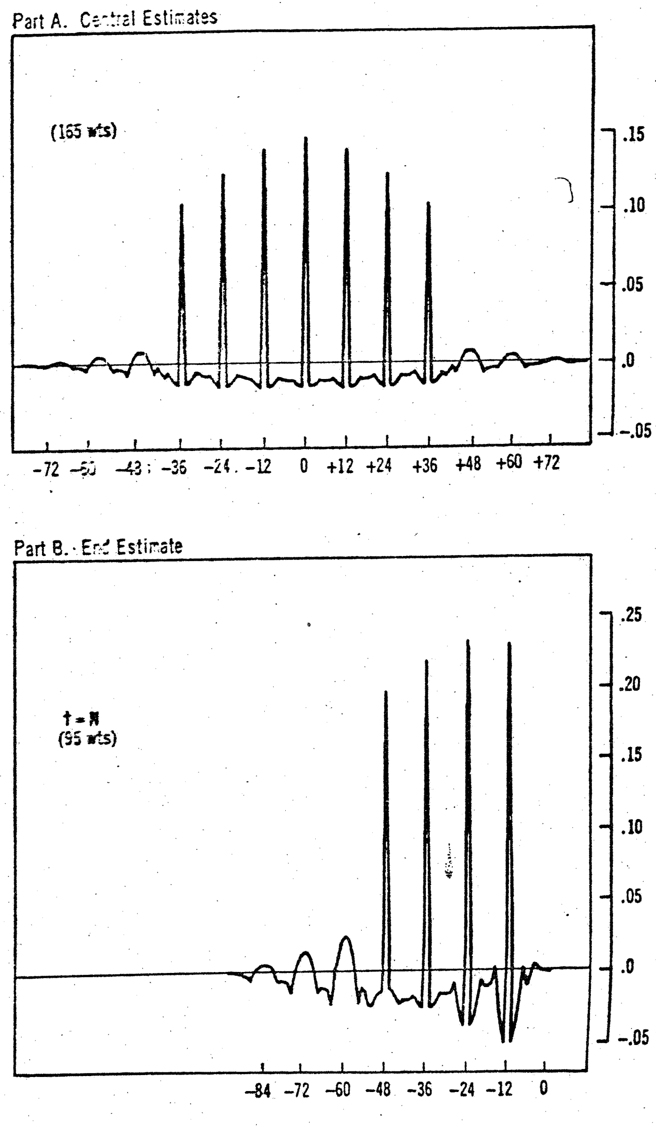
Since, for $x > 0$, $\Pi X_i^{1/n} + \sigma^2_{\sqrt{X_i}} \leq \frac{X_i}{n} \leq \Pi X_i^{1/n} + \sigma^2_{X_i}$, $i = 1, \cdots, n$.

each P (disregarding the weights) is bounded by an expression containing the variance and one containing the variance of the square roots of the numbers:

$$1 + \frac{\sigma^2_{\sqrt{X_i}}}{\Pi X_i^{1/n}} \leq P \leq 1 + \frac{\sigma^2_{X_i}}{\Pi X_i^{1/n}} \quad i = 1, \cdots, n.$$

I am indebted to Max Bershad for deriving the left half of the approximation from reviews of articles by Dianande and Dinghas, *Mathematical Reviews*, Vol. 26, No. 5, reviews 5111 and 5112, November 1

CHART 2. BLS COMBINED SEASONAL WEIGHTS



results in some values of $\hat{S}$ being less than $\hat{S}^*$ since its effect on $P^{(i)}$ is greater than on the other P's where it is averaged out.

The forcing step can be represented as follows:[5] Let $\hat{S}'$ represent the final "forced" ratio-to-moving-average factor, then

$$\hat{S}'_t = \hat{S}^*_t \Big/ \sum_{h=-6}^{+6} c_{1,h} \hat{S}^*_{t+h}. \qquad (12)$$

[5] To keep things simple the forcing step is specified once. In Method II it occurs at the end of the first round as well as the second. The forcing step is considered by Burns and Mitchell [3], who show that it results in a seasonally adjusted series in which the expected value of the annual total equals the total of the unadjusted series. Cowden suggests that the forcing step can largely overcome "the bias due to inflexibility" of the 12-term moving average [4].

The pure multiplicative factors can also be forced to 12

$$\hat{S}_t'' = \hat{S}_t \Big/ \sum_{h=-6}^{+6} c_{1,h}\hat{S}_{t+h}. \tag{13}$$

For "well behaved" series, defined as series where the P's tend to be constant over the series, the expression for $\hat{S}'$ in equation (12) approaches that for $\hat{S}''$ in equation (13).

It is simpler to determine the importance of the non-linear operations empirically than to examine them further theoretically. The extent of variation among $\hat{S}$, $\hat{S}'$ and $\hat{S}''$ is presented in table 1 for four series and between $\hat{S}$ and $\hat{S}'$ in chart 3. The average differences between $\hat{S}'$ and $\hat{S}$ range from 0.10 per cent for imports to 1.52 per cent for unemployed men. The $\hat{S}$ tends to lie above the $\hat{S}'$ as indicated by the range of differences between them. Method II was used to modify extreme values in the imports and unemployed men series, prior to computing the seasonal estimates. In order to illustrate the effect of extremes, factors were computed with and without extreme values for carbon steel, a series greatly affected by extremes. With extremes included the average difference for carbon steel is 3.7 per cent, instead of 0.18 per cent. Forcing the pure multiplicative factors to sum to 12 over 12-month periods reduces their deviations from the ratio-to-moving-average factors. The average differences between $\hat{S}'$ and $\hat{S}''$ range from 0.03 per cent for imports to 0.45 per cent for unemployed men. Once again the effect of not removing extreme values is substantial. With extremes included the average difference is 3.14 per cent for carbon steel while it is 0.13 per cent with extremes removed.

The assessment of the non-linear operations indicates the following: (a) The effect of the non-linear operations is not very large and for some purposes can be ignored. The expression for the pure multiplicative seasonal factor can be

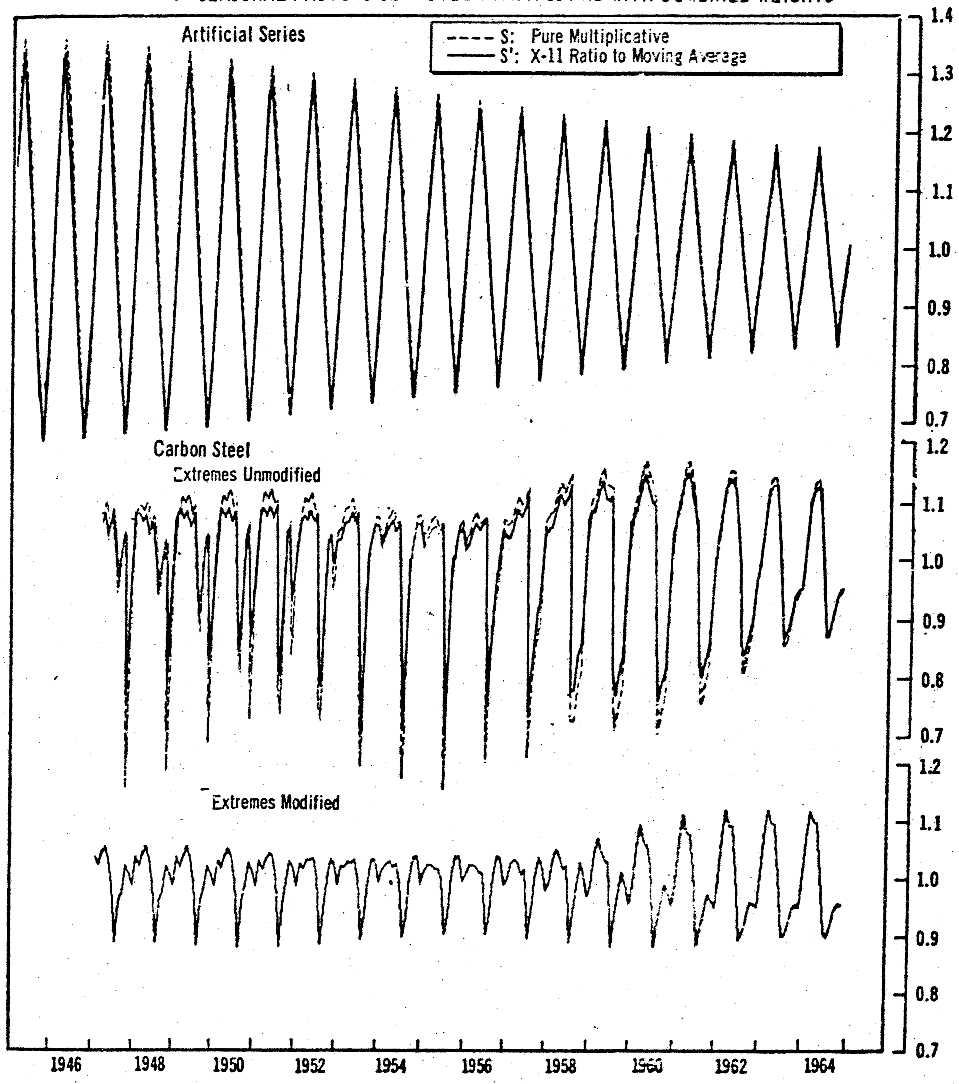
TABLE 1. PURE MULTIPLICATIVE SEASONAL FACTORS COMPARED WITH X-11 FACTORS

(Average without regard to sign and range of percentage differences between indicated factors)

Series	$100[\hat{S}'/\hat{S}''-1.0]$		$100[\hat{S}'/\hat{S}-1.0]$		$100[\hat{S}/\hat{S}''-1.0]$	
	Average	Range	Average	Range	Average	Range
Artificial Seasonal Pattern (20 yrs.)	0.11	+0.2 to −0.2	1.25	− 0.5 to −2.1	1.25	+2.0 to +0.6
Imports (1948 to 1965)	0.03	+0.1 to −0.1	0.10	+ 0.1 to −0.2	0.07	+0.2 to −0.1
Unemployed Men (1950 to 1964)	0.45	+0.8 to −0.8	1.52	− 0.8 to −2.1	1.59	+1.8 to 0.0
Carbon Steel Production (1947 to 1964)						
Including Extremes	3.14	+15.9 to −2.9	3.70	+14.9 to −4.9	1.23	+2.4 to +0.2
Extremes Modified	0.13	+0.3 to −0.3	0.18	+ 0.3 to −0.5	0.14	+0.3 to −0.1

Note: $\hat{S}$: Pure multiplicative factor.
$\hat{S}''$: Pure multiplicative factor forced to 12.
$\hat{S}'$: X-11 ratio-to-moving-average factor forced to 12.
Seasonal factors were computed with the central weight patterns and cover periods 6 years shorter at each end than periods shown on table.

CHART 3. SEASONAL FACTORS COMPUTED WITH X-11 AND WITH COMBINED WEIGHTS



used to evaluate the weight patterns used in Method II and can serve as a basis for simplifying the ratio-to-moving-average procedure. (b) The pure multiplicative seasonal factors based on the Method II weights are a close but slightly biased approximation to the ratio-to-moving-average factors. Forcing the pure multiplicative factors to sum to 12 removes the bias and reduces the differences, on average, between them and the ratio-to-moving-average factors. These results may be extended to pure multiplicative factors estimated by other methods. In general, pure multiplicative factors may be forced to 12 as a final step to achieve the annual total criterion. Whether such factors are acceptable in comparison to the ratio-to-moving-average factors depends not on the forcing step but on the particular model used to estimate the pure multiplicative factors. All this assumes that extreme values are modified when necessary. (c) If statistical tests are derived for the pure multiplicative approximation to Method II, they can be extended to the ratio-to-moving-average estimates with little sacrifice in validity. This does not depend directly on the average difference between $\hat{S}'$ and $\hat{S}$, but on whether standard errors derived for $\hat{S}$ may be considered applicable to $\hat{S}'$. Differences ranging from 0.10 to 1.52 per cent probably affect the variance of I and the standard errors of S very little.

4. SIMPLIFICATION OF METHOD II

The linear approximation of the Method II seasonal factor involves 145 terms of the original series. Is it possible to simplify this expression without significantly affecting the results? Examination of the seasonal weight pattern (chart 1, part A) reveals that most weight is concentrated in the central 73 terms, which include the seven "spikes." The weights at each end beyond the spikes are close to zero. This suggests that the pattern could be truncated to 73 or 85 terms, if the value of the discarded weights were distributed among the remaining weights in a manner which maintained the seasonal characteristics of the pattern.

Further examination reveals the seasonal characteristics which must be maintained. The center spike is the central weight of the pattern and the other spikes are separated from it by intervals of 12 terms. The seven spikes arise from the seven-term (3×5) moving average applied to the SI ratios in the second round. The sum of the central weight and the weights separated from it by multiples of 12 sum to $+11/12$. Each of the remaining 11 sets sum to $-1/12$.

These properties are an extension of the properties Durbin found for a stable seasonal. Durbin showed that, with an end point correction, the stable, additive seasonal is equivalent to the difference between the unweighted mean of the month and the unweighted mean of the series, [5].[6] The pure multiplicative, stable seasonal can be expressed as the ratio of the unweighted geometric monthly mean to the unweighted geometric mean of the series, also with an end point correction. In Method II the sum of the central weight and the weights separated from it by a multiple of 12 can be considered as the sum of $+1$ and $-1/12$. Combining the $-1/12$ with the $-11/12$'s for the other 11 months separates the weight pattern into two parts, one summing to $+1$ and the other to -1. It is apparent that the Method II moving seasonal (except for the forcing to 12) is equivalent to the ratio of a weighted geometric monthly mean to a weighted geometric mean of the series calculated over 145 terms.

To truncate the 145-term pattern, each discarded weight should be distributed among weights to be retained, which are separated from the discard by a multiple of 12. This preserves the sums of $+11/12$ for the spikes and $-1/12$ for each of the other months. Thus to obtain a 73-term pattern the values of the 36 weights at each end of the 145-term pattern should be distributed equally over the retained weights separated by multiples of 12 from the discard. This truncated pattern is similar to the pattern for a single-round seasonal adjustment based on a moving average which combines the seasonal eliminating property

[6] It is interesting to note an antecedent to this in a paper by Shiskin [24].

TABLE 2. SIMPLIFIED SEASONAL WEIGHT PATTERNS COMPARED WITH 145-TERM PATTERN

(Average without regard to sign and range of percentage differences between indicated factors)

Series	$100[\hat{S}''^{(t)}/\hat{S}''-1.0]$		$100[\hat{S}''^{(12)}/\hat{S}''-1.0]$	
	Average	Range	Average	Range
Imports (1948 to 1965)	0.09	+0.2 to −0.3	0.07	+0.1 to −0.2
Unemployed Men (1950 to 1964)	0.38	+0.6 to −0.6	0.15	+0.2 to −0.5

Note: $\hat{S}''$: Pure multiplicative factor forced to 12.
$\hat{S}''^{(t)}$: Pure multiplicative factor truncated to 73 weights and forced to 12.
$\hat{S}''^{(12)}$: Pure multiplicative factor based on centered 12-term moving average for trend cycle, 3×5 moving average for seasonal, and forced to 12.

Seasonal factors were computed with the central weight patterns and cover periods 6 years shorter at each end than periods shown on table.

of the 12-term and the flexibility of the Henderson moving average, see for example [1] and [11]. Seasonal factors computed with the truncated pattern are compared with those from the 145-term pattern in table 2. The differences are small.

Durbin showed that, except for the end point correction, the second round had no effect on the stable seasonal. The second-round stable seasonal is the same as the first-round stable seasonal based on the 12-term moving average.

Following Durbin, the Method II seasonal factor can be compared with a moving seasonal factor based on a 12-term moving average. In table 2 and chart 4 the 145-term Method II pattern is compared with the 85-term pattern obtained by combining the 3×5 moving average and the 12-term moving average. The differences are even smaller than in the first exercise.

The Method II seasonal factors are almost identical to factors computed from the ratios to a centered 12-term moving average. With the exception of the first and last six terms of the series, the conclusion holds for the end weight patterns as well as the central pattern. One might question whether this particular result carries over with full force from the linear combinations of the weights to the ratio-to-moving-average method. It does. Tests indicate that, if the 3×3 first-round moving average is replaced with the 3×5, the first and second-round seasonals of Method II are almost identical.

While it appears possible to simplify Method II, there are two additional points which should be considered: (a) The economist may desire a trend-cycle estimate which does not contain the deficiencies of the 12-term moving average. The trend-cycle estimates are provided by the weight patterns in chart 1, part B. Here the properties of the Henderson curve are important. (b) Method II identifies extreme values in a preliminary estimate of the irregular component, after C is estimated by a Henderson or Spencer curve. If the irregular estimate were based on a 12-term moving average for C, it would be affected by the deficiencies of the 12-term moving average and a poorer identification of extremes would result.

5. COMPARISON OF RATIO-TO-MOVING-AVERAGE AND REGRESSION MODELS

Multiple regression seasonal adjustments can also be expressed as linear functions of the original series and compared with the moving average method, i.e., the seasonal factor estimated by regression for month t can be expressed by

$$\widehat{\log S_t} = \sum_{i=1}^{N} a_i \log Y_i$$

and these a's compared with those for the moving average method.[7] Specifications of the types considered by Rosenblatt [22] and Henshaw [9] are compared with the X-11 variant of Method II. The purpose of these comparisons is to point out the basic similarity of the ratio-to-moving-average and regression approaches and to indicate the nature of the difference between the specifications being considered.

Model I may be written

$$\begin{aligned} \log Y_t &= \log S_t + \log C_t + \log I_t, \\ \log S_t &= \sum_{k=1}^{6}\left[\alpha_{1,k} \sin \frac{2\pi kt}{12} + \alpha_{2,k} \cos \frac{2\pi kt}{12}\right] \\ &\quad + \sum_{k=1}^{2}\left[\beta_{1,k} t^k \sin \frac{2\pi t}{12} + \beta_{2,k} t^k \cos \frac{2\pi t}{12}\right], \qquad (14) \\ \log C_t &= \gamma_0 + \sum_{k=1}^{5} \gamma_k t^k + \delta_1 \sin \frac{2\pi t}{72} + \delta_2 \cos \frac{2\pi t}{72} \cdot \end{aligned}$$

The seasonal expression contains the sine and cosine curves for the seasonal periods of 12, 6, 4, 3, 2.4 and 2 months. These completely describe any stable seasonal pattern. In addition, the last four terms allow for some long-term change in the seasonal pattern over the series, i.e., moving seasonality. The trend-cycle expression contains a fifth degree polynomial and sine and cosine curves of 72 months which increase the flexibility of the trend-cycle. Model Ia contains the same specification as Model I with the addition in the trend-cycle of a sixth degree to the polynomial and 24- and 48-month sine and cosine curves.

---

[7] From the solution to the linear regression model

$$\hat{Y} = (T \vdots S)\begin{pmatrix}\hat{\alpha} \\ \cdots \\ \hat{\gamma}\end{pmatrix} = (TA_1)(T \vdots S)'Y + (SA_2)(T \vdots S)'Y,$$

where

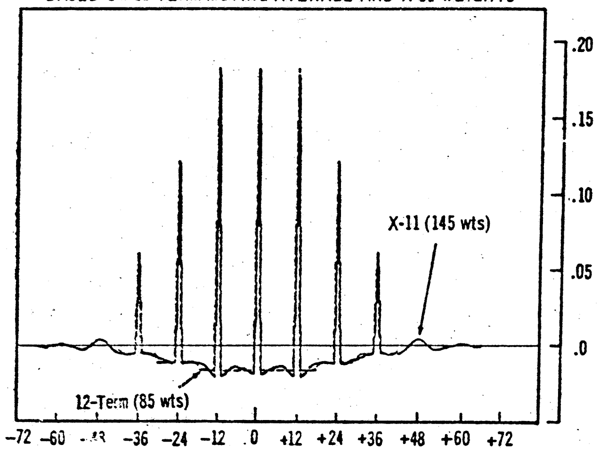
T is the matrix of trend-cycle vectors,
S is the matrix of seasonal vectors,
α and γ are the parameters associated with T and S,

$$\begin{pmatrix}\hat{\alpha} \\ \cdots \\ \hat{\gamma}\end{pmatrix} = [(T \vdots S)'(T \vdots S)]^{-1}(T \vdots S)'Y,$$

$$\begin{pmatrix}A_1 \\ \cdots \\ A_2\end{pmatrix} = [(T \vdots S)'(T \vdots S)]^{-1}, \quad \text{and}$$

$(TA_1)(T \vdots S)'$ and $(SA_2)(T \vdots S)'$ are the matrices of the trend-cycle and seasonal weights respectively.

CHART 4. COMBINED SEASONAL WEIGHTS
BASED ON 12-TERM MOVING AVERAGE AND X-11 WEIGHTS



Model II contains a fifth degree polynomial trend-cycle and a simplified seasonal expression. The moving seasonal allowance and the 3 and 2.4 month periods of the stable seasonal pattern are omitted.

Model III provides for a stable seasonal and may be written

$$\log Y_t = \log S_t + \log C_t + \log I_t,$$
$$\log S_t = \sum_{k=1}^{12} \alpha_k D_k, \qquad (15)$$
$$\log C_t = \sum_{k=1}^{10} \gamma_k t^k$$

where $D = 1$ when t is the kth month of the year and equals 0 otherwise. The seasonal expression provides for a stable seasonal pattern and the trend-cycle is a 10th degree polynomial. Henshaw found a fourth degree polynomial sufficient for his test series, additional terms were not significant, but for comparison a 10th degree is chosen here.

Model IV is written

$$\log Y_t = \log S_t + \log C_t + \log I_t,$$
$$\log S_t = \sum_{k=1}^{12} \alpha_k D_k + \sum_{k=1}^{12} \beta_k D_k t, \qquad (16)$$
$$\log C_t = \sum_{k=1}^{5} \gamma_k t^k.$$

The seasonal expression contains a first degree polynomial for each month and the trend-cycle is a fifth degree polynomial.

The number of coefficients in the models is 24 in Model I, 29 in Model Ia, 14 in II, 22 in III and 29 in IV.

In deriving the weights it was assumed that the series contained 85 months resulting in sets of 85 weights which provide interesting comparisons with the moving average expressions. Weights derived for a regression covering more than 85 months are in a sense analogous to weights obtained by substituting a longer moving average for the seven-term (3×5) average used in Method II.

Selected sets of weights for the regression models are shown in chart 5. The seasonal weights for the central term in Models I, III and IV are similar to the Method II weights with "spikes" at 12-month intervals summing to 11/12's and each of the other 11 sets summing to −1/12. Model I and III end weights show little resemblance to those of Method II, but those of Model IV do, except for the two negative spikes on the left side of the pattern.

It is interesting to note that the weight assigned in the end seasonal weight pattern to the end spike is less than the weight assigned to the two preceding spikes in both Method II and the regressions. In general, the more flexible the trend-cycle specification, the more weight assigned to the trend-cycle and the less to the seasonal at the end of the series.

The central weight pattern for the fifth degree polynomial trend-cycle is quite different from the Method II patterns. In Method II there is more bunching of weight in the center resulting in a more flexible trend-cycle. The 10th degree polynomial in Model III begins to approach the 23-term Henderson pattern of Method II. The sixth degree polynomial and 24-, 48- and 72-month sine and cosine terms in Model Ia come close to the 23-term Henderson, and begin to approach the 13-term Henderson. None of the regression specifications approach the 9-term Henderson. At the ends none of the regression specifications show much resemblance to Method II.

Model II illustrates a property of the regression method not found in the ratio-to-moving-average model. The removal of the 2.4- and 3-month sine and cosine periods substantially affects the seasonal weight pattern. The spikes do not sum to +11/12 nor do the other sets sum to −1/12. If a series does not have significant variation associated with particular seasonal harmonics, terms may be omitted from the specification, thereby reducing the number of parameters. The moving average model is not easily modified in this respect. This is a practical advantage of the regression method to the extent that there are economic series which may be described with less than the full seasonal specification implied in the ratio-to-moving-average method.

6. COMPARISON OF CURRENT YEAR REVISIONS

Several studies have considered the quality of the historical Census and BLS seasonal factors—those generated by the central weight patterns and patterns not too far removed from the center. Few studies have considered the quality of the current seasonal factors—those generated by the year-ahead weight patterns. The current factors are important, however, since they have the greatest effect on economic policy. In this part of the paper the current factors provided by the Census method are compared with the current factors from the ratio to 12-term moving average model and three regression models. The criterion selected to examine the current factors is the amount of revision between the current and the historical seasonal factors. Since the criterion involves both the

Chart 5. Regression Weights Compared with X-11

Part A. Seasonal Estimates

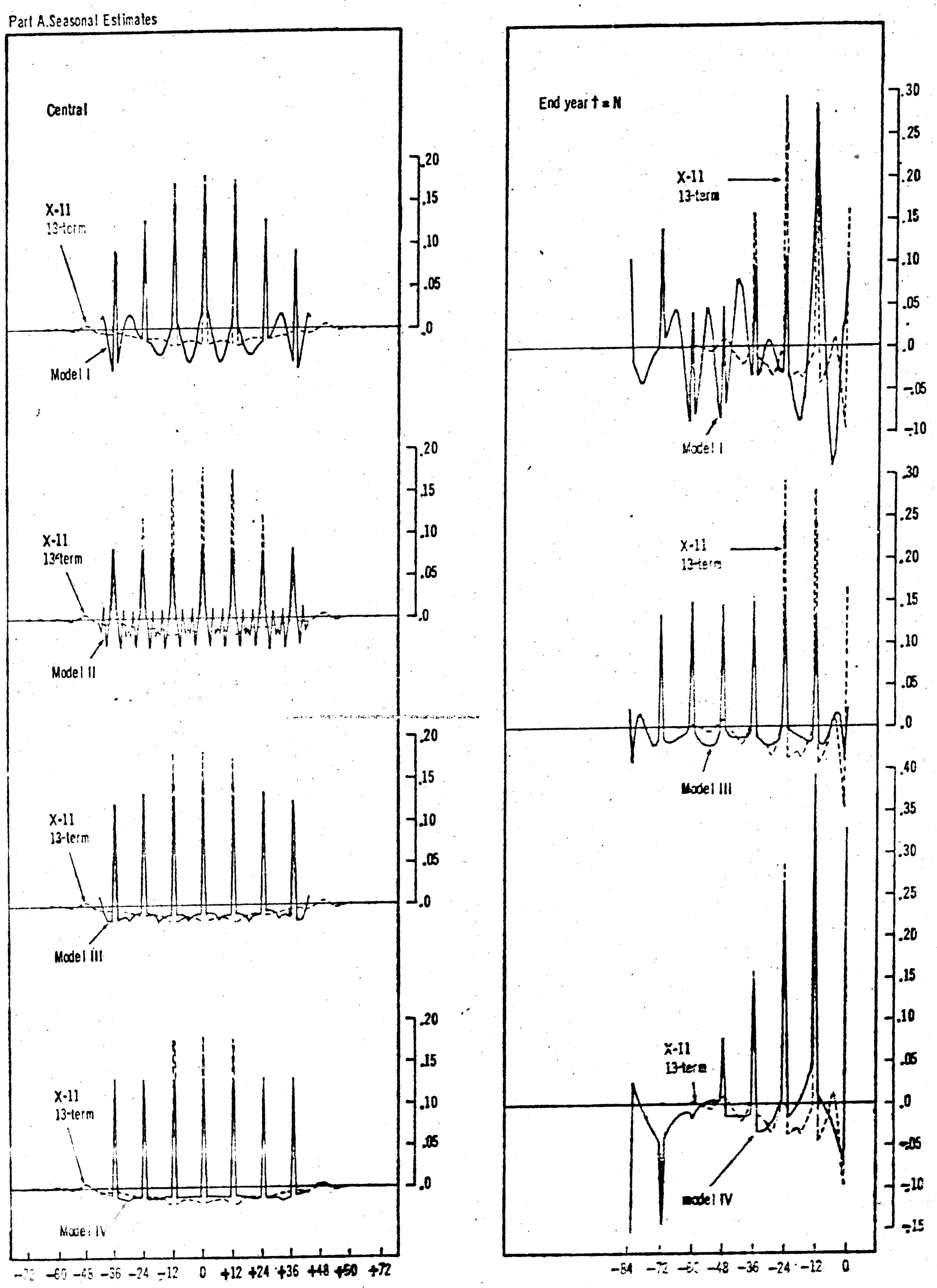


Chart 5. Regression Weights Compared with X-11 --Continued

Part B. Trend-Cycle Estimates

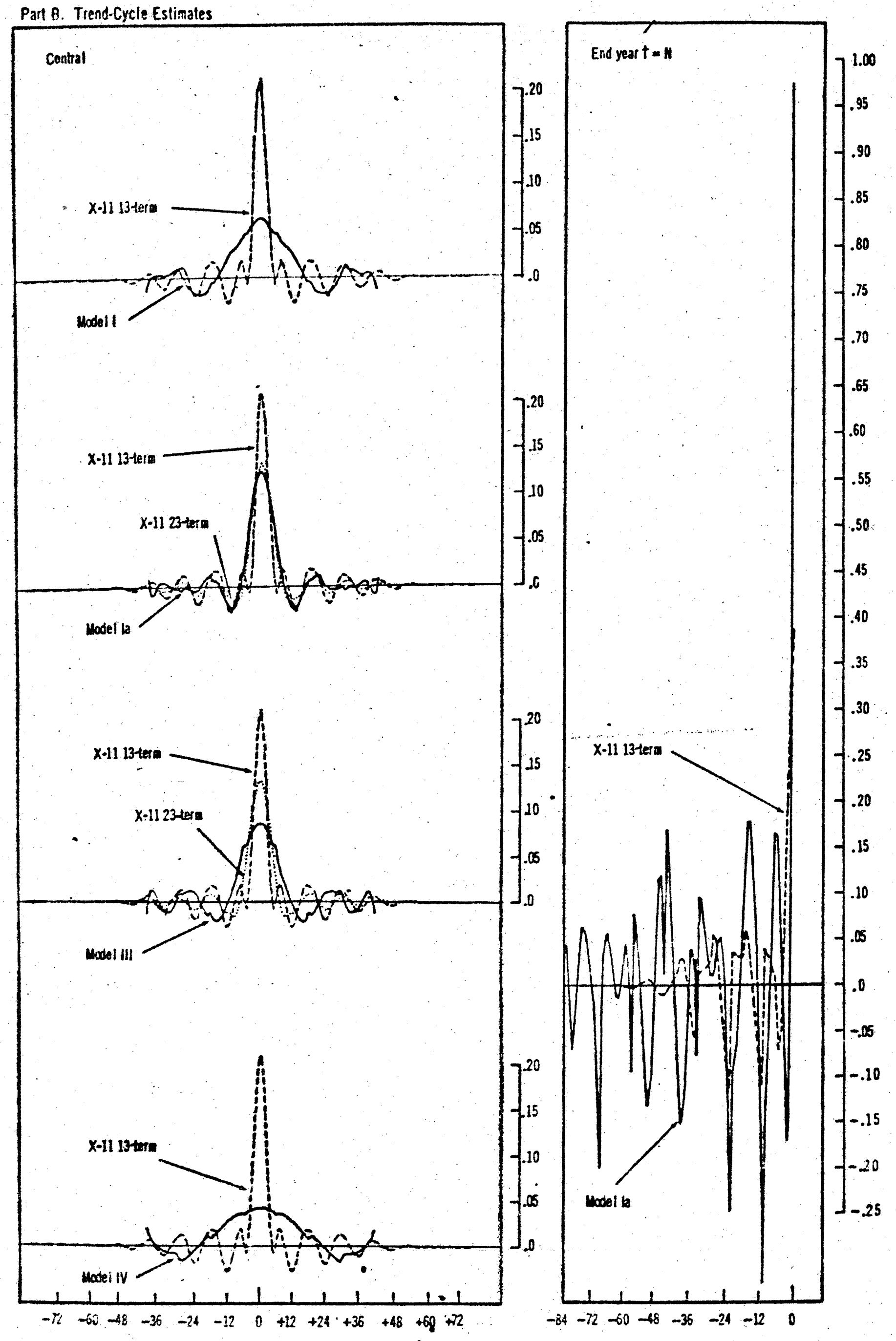
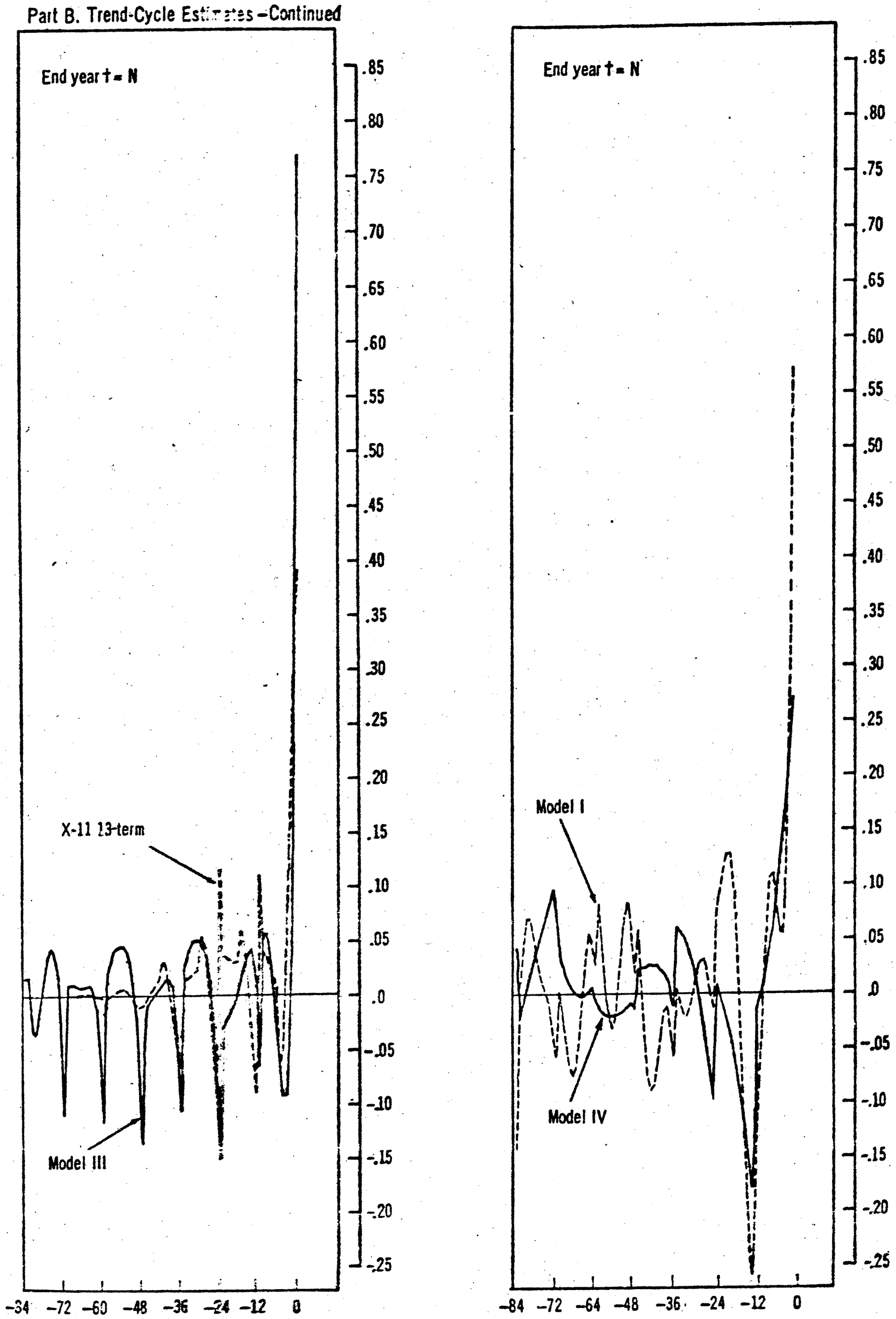


Chart 5. Regression Weights Compared with X-11 —Continued

Part B. Trend-Cycle Estimates —Continued



historical and current estimates, it is necessary first to consider briefly the quality of the historical factors.

The Census and BLS historical seasonal factors have been compared with alternatives and evaluated with various non-spectral criteria advanced for judging seasonal adjustments, for example, Shiskin [25] and [27]. In general, the Census and BLS historical seasonal factors have closely resembled factors provided by other methods. These empirical comparisons are consistent with the derivations in the previous sections which show that the Census and BLS central seasonal weight patterns are almost identical to the pattern of the 12-term moving average model and closely resemble those of the regression models.

Beginning with the report of the President's Committee to Appraise Unemployment Statistics in 1962 the historical factors have been evaluated in terms of desired spectral properties, [21] and also, for example, Nerlove [17], [18], Netheim [20] and Rosenblatt [22]. Rosenblatt in a recent paper re-examines the Census and BLS models using spectral criteria [23]. He shows that changes made in the procedures have improved their spectral properties, specifically their loss of spectral power at seasonal and trend-cycle frequencies. But he questions the significance for the economist of departures from desired spectral properties. He argues that the effect of any departures from desired properties should be examined in the time domain in which the figures are used.

The author's evaluation of the historical seasonal factors provided by the ratio-to-moving-average method is that they stand up well to various criteria, both spectral and non-spectral. While the assessment of the historical factors is favorable there are qualifications which should be noted.[8]

Unlike the central weight patterns the X-11 end weight patterns do not closely resemble the regression weight patterns. Which end weight patterns are best? A recent study considered this question. Nerlove [19], using spectral criteria, compared current seasonal factors provided by an early variant of the BLS procedure with those from a modification of a model suggested by Hannan. He found little basis for choosing one method as superior to the other. In view of the differences in the weight patterns and the results of Nerlove's study it is appropriate to examine the question further.

The amount of revision between the current and historical factors follows naturally as a criterion for evaluating the end weight patterns once a model is judged to provide satisfactory historical estimates. (It also is a criterion on which great emphasis is placed by both the user and publisher of seasonally adjusted data.) In general, the end weight patterns giving the smallest revisions are considered best. An overly sensitive model, and in some instances a too stable model, will have large revisions. On the other hand, a too stable model may have small revisions consistently biased in one direction. A check for excessive stability, therefore, is necessary as a secondary criterion. The check used here is to examine the current seasonally-adjusted data for residual seasonality

[8] All spectral evidence may not be in. Also, the present variants perform better than those to which spectral analysis was first applied. Recent work by Fairbarns [6] and Godfrey [7] has raised anew the question of whether a model which allows for rapid year-to-year changes in the amplitude of the historical seasonal factors might be preferable to the standard ratio-to-moving-average procedure for a significant proportion of economic series. In an econometric model it may be preferable to include a seasonal allowance rather than to use data adjusted by the ratio-to-moving-average method, see for example, Ladd [14] and Lovell [15]. Also it would be desirable to have estimates which are amenable to statistical significance tests.

using the X-11 test for stable seasonality. This test computes the ratio of the mean square between-month variation to the mean square residual variation in the seasonal-irregular ratios. In X-11 the computed ratio is compared with the tabled F ratio at the 1% level (2.4 for a 10-year series). The computed ratios, however, especially as presented here, are difficult to assess in probability terms since the data have had extreme values modified and because the test is only approximately correct.

The revisions are summarized in table 3 and presented in detail in table 4. The details of the computations and some limitations of the comparisons are provided in a technical note at the end of this section.

The results show that X-11 provides the smallest revisions in the three series compared. The 12-term moving average model provides revisions of the same magnitude as X-11 in the first half of the year and slightly larger revisions in the second half. Overall the 12-term moving average model is second. Regression Model III does as well as X-11 in the second half of the year for two series and is third overall. Models IV and I are fourth and fifth respectively. Ranking the models by the size of the revisions yields similar results with Models I and IV changing places and Model III replacing the 12-term moving average model in second place for the second half of the year.

The computed F ratios are quite high for the year-ahead estimates of unemployed men provided by the two moving average models and regression Model III. The ratios for the final estimates are also high. These ratios would be lower if computed from data not modified for extremes. For example, the ratio for the X-11 year-ahead estimates of unemployed men drops from over 20 to about 7, and a similar reduction occurs in the ratio for the final unemployment estimates when extremes are not removed. Nevertheless, these ratios are probably evidence of a small amount of residual seasonality in the year-ahead estimates. In other words, the moving average models and regression Model III which do best in the revision test may provide year-ahead estimates for this particular series which are too stable.

One might question the basic premise that the historical factors provided by all the methods are quite similar. Possibly the X-11 revisions are smallest not because the X-11 year-ahead weights are better than the other end weights, but because the central weights are not as good as the other central weights. One way to examine this possibility is to compare the differences between the current X-11 factors and the historical regression factors with the revisions between the current and historical regression factors. The differences between the X-11 current factors and the historical regression factors (table 5) are smaller than the revisions between the current and historical regression factors (table 3). The larger revisions in the regression models appear to be due to the poor quality of the current regression factors, not to the quality of the X-11 historical factors.

Although the performance of the regressions might be improved by more closely tailoring the regression specifications to the particular series, it probably is not an easy task to derive regression estimates which, from the standpoint of revisions, are better than the moving average estimates.

Given certain assumptions about the series, it may be possible to derive improved Method II end weights from the central weight pattern which minimize

TABLE 3. SUMMARY OF REVISIONS IN SEASONAL FACTORS

Series	X-11			12-Term			Model I			Model III			Model IV		
	1-6	7-12	1-12	1-6	7-12	1-12	1-6	7-12	1-12	1-6	7-12	1-12	1-6	7-12	1-12
Part A. Average Revision (Average of revisions in table 4.)															
Unemployed Men (1958 to 1961)	1.8	2.0	1.9	1.8	2.3	2.1	4.8	6.5	5.7	2.1	2.9	2.5	4.0	4.7	4.3
Imports (1959 to 1962)	1.1	1.9	1.5	1.1	2.2	1.7	1.6	2.4	2.0	1.7	1.9	1.8	1.3	2.5	1.9
Retail Sales (1958 to 1961)	0.3	0.6	0.5	0.3	0.7	0.5	0.6	1.0	0.8	0.4	0.6	0.5	0.5	1.3	0.9
Part B. Average Rank (Rank of 1 assigned to smallest revision for each year in table 4.)															
Unemployed Men (1958 to 1961)	2.25	1.62	1.75	2.12	2.75	2.25	3.50	4.12	3.75	2.88	3.00	3.25	4.25	3.50	4.00
Imports (1959 to 1962)	2.25	2.25	2.62	2.25	3.38	2.62	3.75	2.88	2.88	4.12	2.25	3.38	2.62	4.25	3.50
Retail Sales (1958 to 1961)	2.00	2.12	2.12	2.50	2.50	2.50	3.38	3.25	3.62	3.00	2.12	2.12	4.12	5.00	4.62
Three Series Average Rank,															
Rank of 1 Assigned Lowest	2.17	2.00	2.17	2.20	2.87	2.46	3.54	3.41	3.41	3.33	2.46	2.92	3.67	4.25	4.04
Average Rank	1	1	1	2	3	2	4	4	4	3	2	3	5	5	5

Part C. Residual Seasonality Test—F ratios

Series	X-11 Current	X-11 Historical	12-Term Current	12-Term Historical	Model I Current	Model I Historical	Model III Current	Model III Historical	Model IV Current	Model IV Historical
Unemployed Men (1958 to 1961)	22.5	5.8	28.2	5.0	2.8	1.7	43.4	7.3	2.3	.8
Imports (1959 to 1962)	1.8	0.6	1.4	0.6	1.8	0.7	3.1	0.7	1.7	0.6
Retail Sales (1958 to 1961)	2.7	1.6	3.5	1.6	2.2	0.9	4.0	1.9	7.4	1.5

TABLE 4. REVISIONS BETWEEN CURRENT AND HISTORICAL SEASONAL FACTORS

(Average without regard to sign of percentage differences between seasonal factors)

Series	X-11		12-Term		Model I		Model III		Model IV	
	1–6	7–12	1–6	7–12	1–6	7–12	1–6	7–12	1–6	7–12
Unemployed Men										
1958	1.6	1.2	1.2	1.4	8.3	11.1	1.2	3.4	3.1	2.4
1959	2.6	3.4	2.9	3.2	7.6	9.6	2.1	3.0	5.7	4.4
1960	2.1	2.6	1.9	3.1	1.7	2.6	2.6	2.9	5.2	10.9
1961	1.0	0.8	1.2	1.6	1.5	2.7	2.3	2.4	2.0	1.1
Imports										
1959	1.3	1.0	1.4	1.2	2.5	3.0	2.0	1.6	1.2	1.8
1960	0.9	2.2	0.9	1.8	1.1	1.6	1.4	1.7	1.8	2.7
1961	1.0	2.5	0.9	2.9	1.5	2.2	1.4	2.5	1.4	3.3
1962	1.4	1.6	1.4	2.8	1.3	2.8	1.0	1.6	1.0	1.8
Retail Sales										
1958	0.4	0.4	0.3	0.4	0.8	1.1	0.7	0.7	0.4	1.3
1959	0.3	0.3	0.4	0.4	1.1	1.3	0.4	0.4	0.6	1.8
1960	0.3	0.8	0.4	0.7	0.3	0.9	0.3	0.6	0.5	1.0
1961	0.3	0.9	0.3	1.1	0.3	0.7	0.3	0.8	0.5	1.2

revisions between the current and final seasonal factors. This approach would also provide end weights for the 12-term moving average model.

TECHNICAL NOTE

Revisions between the year-ahead factors and the factors based on four additional years of data were computed for four years for unemployed men, imports, and retail sales. The seasonal factors were computed by (a) applying the weight patterns to the logarithms of the original series to estimate the logarithms of the seasonal factor, (b) taking antilogs, and (c) forcing the factors to sum to 12. To describe the test the derivation of the revisions for 1960 is indicated. For X-11 (13-term Henderson option) the percent differences between the 1960 factors based on the year-ahead weight patterns applied to data ending in 1959, and those based on the third from end year weight patterns applied to data ending in 1963 were computed. The percent differences were then averaged without regard to sign; two averages were taken, one for January to June and one for July to December. The year was split in half for comparison with the 12-term moving average model, where the loss of six values may affect the quality of the factors for the second half of the year.

The year-ahead 1960 seasonal factors based on the centered 12-term and 3×5 moving averages were also computed from data through 1959. However, there are no seasonal-irregular ratios for July to December 1959. The most recent seasonal-irregular ratios for the July to December period are those for 1958. Using the year-ahead weight patterns, factors were computed for July to December 1959. These factors were repeated and used as the year-ahead

TABLE 5. REVISIONS BETWEEN CURRENT *X*-11 AND HISTORICAL REGRESSIONS SEASONAL FACTORS

(Average without regard to sign of percentage differences between seasonal factors)

Series	*X*-11/Model I			*X*-11/Model III			*X*-11/Model IV		
	1–6	7–12	1–12	1–6	7–12	1–12	1–6	7–12	1–12
Unemployed Men (1958 to 1961)	2.6	2.9	2.7	1.9	1.9	1.9	2.0	2.5	2.3
Imports (1959 to 1962)	1.2	1.8	1.5	1.1	1.6	1.4	1.0	1.6	1.2
Retail Sales (1958 to 1961)	0.4	0.6	0.4	0.3	0.6	0.5	0.3	0.7	0.5

factors for July to December 1960. As with *X*-11, the final factors were computed from data extending through 1963.

The regression year-ahead 1960 factors were computed from data for the years 1953 to 1959 and the final factors from data for 1957 to 1963. The year-ahead 1960 factors were the 1959 factors repeated.

The comparisons have several limitations: (a) The regression models were applied as seven-year moving regressions. (b) Different procedures for obtaining current estimates are possible. Instead of repeating the end-year factors for the regression models, the time variables could have been extended. Or, both the moving average and regression models could be refit each month, thereby eliminating year-ahead factors. (c) In applying a regression model one would prefer to test alternative explanatory variables in an attempt at the best specification for the particular series. (d) Extreme values were removed from the data before the seasonal factors were computed. The presence of extremes in the data would cause larger revisions than shown here, either because they were not removed or because they were not removed consistently in successive runs. (e) There may be deficiencies in the current data which are not exposed by the residual seasonality test.

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Revision of Seasonally Adjusted Labor Force Series

*Robert J. McIntire

At the beginning of each calendar year, the Bureau of Labor Statistics reestimates the seasonality of the unemployment, employment, and other labor force series derived from the Current Population Survey based on the inclusion of another full year of data in the estimation process. The initial estimates of seasonality used for current adjustment are projections based on average seasonal patterns in the recent past. Since seasonal patterns tend to change somewhat over time, it is usually possible to improve on those initial estimates by incorporating more recent data. Based on this annual reestimation, BLS issues the projected factors for the first 6 months of the new year as well as revised estimates of historical seasonally adjusted data for the last 5 years. Each year's data are generally subject to 5 revision cycles before the values are considered final. This year's revisions incorporate data through December 1983 and provide revised estimates for January 1979 through December 1983.

Revised data for many of the major seasonally adjusted labor force series were published in the news release on the December 1983 employment situation, issued January 6 (USDL 84-5). Data for recent months and quarters for many more of the revised series appear in this issue of *Employment and Earnings*. In addition, this issue provides the projected seasonal factors for the first 6 months of 1984 for the 12 component series used in the computation of the seasonally adjusted civilian labor force and unemployment rate (see table 3 at the end of this article). Projected factors for the last 6 months of 1984 will be estimated in early July, based on data through June 1984, and will be published in the July issue of this publication. Next month's issue will contain the 1979-83 revisions for a few hundred of the seasonally adjusted labor force series most in demand. These revisions replace the data published in the February 1983 edition for 1979-82 and the seasonally adjusted estimates for 1983 published during the past year. Seasonally adjusted data for 1978 and earlier years were not revised.

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Effect of the revisions

One of the criteria used to evaluate alternative methods of seasonal adjustment is how close initial estimates come to later revisions. Policymakers and analysts must make determinations based on current information, and so it is important that the initial estimates of current factors for the seasonal adjustment of major economic series produce estimates of level and change that are as close as possible to the improved estimates that will be made after more data have become available. With respect to this criterion as applied to the overall and civilian unemployment rates, the Bureau's official method for seasonal adjustment of the labor force series performed quite well in 1983. Table 1 shows these rates as first computed and as revised, as well as the change due to revision. The civilian rate to one decimal place was unchanged by the revisions in 10 of the 12 months, and the overall rate was unchanged in 6 of the 12 months. In all months in which either rate changed, the revision never exceeded 0.1 in absolute value. These are the fewest and smallest revisions since 1979, a pleasantly surprising result, since 1983 should have been much harder to adjust well given the magnitude of the cyclical movements within the year and in the preceding 1980-82 period.

Table 1. Seasonally adjusted unemployment rates in 1983 and change due to revision

Month	As first computed		As revised		Change due to revision	
	Overall	Civilian	Overall	Civilian	Overall	Civilian
January	10.2	10.4	10.3	10.4	0.1	0
February	10.2	10.4	10.2	10.4	0	0
March	10.1	10.3	10.2	10.3	.1	0
April	10.1	10.2	10.1	10.2	0	0
May	10.0	10.1	9.9	10.1	-.1	0
June	9.8	10.0	9.8	10.0	0	0
July	9.3	9.5	9.3	9.5	0	0
August	9.4	9.5	9.3	9.5	-.1	0
September	9.1	9.3	9.1	9.2	0	.1
October	8.7	8.8	8.7	8.8	0	0
November	8.2	8.4	8.3	8.4	.1	0
December	[1]8.0	[1]8.1	8.1	8.2	.1	.1

[1] These rates were never published; they reflect the use of seasonal factors projected for December 1983 given in the July 1983 issue of *Employment and Earnings* and were revised before regular publication of December data.

Adjustment methods and procedures

Through most of the 1970's, the Census Bureau's X-11 method[1] was used for seasonal adjustment of labor force series. At the end of 1979, the X-11 ARIMA procedure,[2] developed by Statistics Canada as an extension of the X-11 method, was adopted as the official seasonal adjustment procedure. The switch was made after extensive tests had shown that initial seasonally adjusted estimates were, on average, better (closer to later revisions) with the X-11 ARIMA than with the X-11 alone. Its use for seasonal adjustment of labor force data at BLS was also consistent with the recommendations in the final report of the National Commission on Employment and Unemployment Statistics.[3]

The X-11 ARIMA method improves current estimates by allowing recent observations, especially the last 6 months, to weigh more heavily in the estimates of current and recent seasonal factors than did the X-11 alone. The method provides this improvement through the use of ARIMA models to extend the data series by 12 months. The X-11 algorithm for seasonal adjustment is applied to the extended series. The addition of projected observations at the end of the series allows the actual observations from the most recent year to make a stronger contribution to estimates of current and recent seasonality than they could with the old X-11 method.

ARIMA projections are based only on the past experience observed in a series itself. ARIMA models have proved to have good properties for short-term projection or extrapolation of a large class of time series, especially in a seasonal adjustment context, since the extrapolations tend to track intrayear movements quite well. The ARIMA models in the X-11 ARIMA program used to seasonally adjust the labor force series are of the Box-Jenkins type.[4] They can generally be described with the notation

(p,d,q) (P,D,Q) TRANSFORMATION,

Where:

(1) p is the number of regular (nonseasonal) autoregressive parameters
(2) d is the number of regular differences
(3) q is the number of regular moving average parameters
(4) P is the number of seasonal autoregressive parameters
(5) D is the number of seasonal differences
(6) Q is the number of seasonal moving average parameters
(7) TRANSFORMATION may be NONE, LOG, or POWER (n).

While the lettered elements within the parentheses of the model specification can theoretically take on many values, in practice only small values are useful.

For each labor force series which has been extended based on an ARIMA model, the model has been specifically chosen as well suited to the particular series, based on a set of established criteria. The criteria essentially require a model to: (1) fit the series well, (2) have low average forecasting errors in the last 3 years prior to the projected year, and (3) produce residuals (the differences between the observed values and the values forecast by the model for the observed period) which follow a random pattern. Acceptable ARIMA models have been identified and were used for 148 of the 188 labor force series which were directly adjusted at the end of 1983, including all 12 major civilian labor force components, whose ARIMA models are shown in table 2. The 40 remaining series for which acceptable models have not been identified were simply run through the X-11 part of the program without any ARIMA extrapolations.

The procedures used for the actual seasonal adjustment of the labor force series with the X-11 part of the process were basically the same as those followed for the last 4 years. There were, however, two minor changes. The first concerned the time span of historical data used. Instead of using all available data back to January 1967, the Bureau used a 10-year time period for the adjustment of the labor force series. This is consistent with the standard practice for the establishment survey series on nonagricultural payroll employment. Also, tests have shown that additional observations before the last 10 years generally have a negligible impact on the projected factors and the revisions for the most recent 5 years. Also a 10-year span is often more appropriate than longer periods for ARIMA identification, estimation, and forecasting, since economic time series are more likely to be affected by significant structural changes over longer periods. For all directly adjusted series, the time period used was January 1974 through December 1983. The second minor change was the discontinuance of the prior adjustment which has been applied to the unemployment series for adult men for the last several years, described in previous articles on seasonal adjustment. Since that adjustment was only applied to data prior to November 1974, it would have

[1] The X-11 method is described in *The X-11 Variant of the Census Method II Seasonal Adjustment Program*, by Julius Shiskin, Alan Young, and John Musgrave (Technical Paper No. 15, Bureau of the Census, 1967).

[2] ARIMA is an acronym for Auto-Regressive Integrated Moving Average. The primary documentation for the X-11 ARIMA procedure is in *The X-11 ARIMA Seasonal Adjustment Method*, by Estela Bee Dagum (Statistics Canada Catalogue No. 12-564 E, January 1983).

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[4] For a more detailed discussion of ARIMA models, refer to previously cited Dagum (1983) and to: Box, G.E.P. and Jenkins, G.M., *Time Series Analysis Forecasting and Control* (San Francisco, Holden Day, 1970); and Granger, C.W.J. and Newbold, P., *Forecasting Economic Time Series* (New York, Academic Press, 1977).

Table 2. ARIMA models for the 12 major civilian labor force components, 1984

Series	Model	Transformation
Agricultural employment:		
Men, 20 years and over	(0,1,4)(0,1,1)	NONE
Women, 20 years and over	(0,1,2)(0,1,1)	LOG
Men, 16 to 19 years	(0,1,2)(0,1,1)	NONE
Women, 16 to 19 years	(2,0,1)(0,1,1)	NONE
Nonagricultural employment:		
Men, 20 years and over	(0,1,1)(0,1,1)	LOG
Women, 20 years and over	(0,1,1)(0,1,1)	LOG
Men, 16 to 19 years	(0,1,1)(0,1,1)	NONE
Women, 16 to 19 years	(0,1,1)(0,1,1)	NONE
Unemployment:		
Men, 20 years and over	(2,1,2)(0,1,1)	NONE
Women, 20 years and over	(2,1,2)(0,1,1)	LOG
Men, 16 to 19 years	(0,1,1)(0,1,1)	NONE
Women, 16 to 19 years	(0,1,4)(0,1,1)	NONE

had only a negligible impact on the projected factors or the 5-year revisions.

The X-11 method of seasonal adjustment contained in the X-11 ARIMA procedure assumes that the original series, including the 12 extrapolated observations if an ARIMA model has been applied, is either the product or the sum of 3 components—trend-cycle, seasonal, and irregular. The method uses either a ratio-to- or difference-from-moving-average approach to estimate the components, depending on whether the multiplicative or additive model is used. The seasonally adjusted series values are computed by dividing each month's original value by the corresponding seasonal factor if the multiplicative model is used, or by subtracting the factor if the additive model is used. Of the 12 major civilian labor force components, the 4 teenage unemployment and nonagricultural employment series were adjusted using the additive model, and the other 8 series with the multiplicative model. Of all the 188 directly adjusted series, 37 were adjusted with the additive model, primarily those involving teenage employment and unemployment, for which the seasonal component seems to be fairly independent of the trend-cycle.

Aggregation procedures

BLS maintains and publishes several hundred seasonally adjusted labor force series in addition to the 188 directly adjusted series discussed above. The additional series are produced by arithmetically combining or aggregating the directly adjusted series with each other or, in some cases, with series on population or resident Armed Forces levels, which are not seasonally adjusted because they are not considered to have any significant seasonal variation. For example, the seasonally adjusted levels of total unemployment, civilian employment, and civilian labor force, and the seasonally adjusted civilian unemployment rate are all produced by aggregation of the seasonally adjusted results for the 12 major civilian labor force components. The seasonally adjusted level of total unemployment is the sum of the seasonally adjusted levels of unemployment for the 4 age-sex groups—men and women 16 to 19, and men and women 20 years and over. Seasonally adjusted civilian employment is the sum of the seasonally adjusted levels of employment for the 8 employment components—the same 4 age-sex groups as noted above employed, respectively, in nonagricultural and agricultural industries. The seasonally adjusted civilian labor force is the sum of all 12 components. The seasonally adjusted civilian unemployment rate is calculated by taking the total seasonally adjusted unemployment level as a percent of the total seasonally adjusted civilian labor force. For the overall labor force, the resident Armed Forces level is added to civilian employment to produce total employment. The labor force is the sum of total employment and total unemployment. The seasonally adjusted overall unemployment rate is calculated by taking total seasonally adjusted unemployment as a percent of the seasonally adjusted labor force.

The principal reason for producing many of the major seasonally adjusted estimates for the labor force by aggregation rather than by direct adjustment is that this approach ensures that the major seasonally adjusted totals will be consistent with at least one major set of components. If the totals were directly adjusted along with the components, this would not generally be true since the X-11 is not a sum-preserving procedure—the sum of the result for two or more directly adjusted series will not generally be the same as the result of directly adjusting the sum of the unadjusted versions of the same series. The various components tend to have significantly different patterns of seasonal variation—for example, teenage unemployment tends to peak in June, while unemployment of adult men tends to peak in the winter months of January and February. It is necessary to directly adjust the components in order to properly estimate these varying seasonal patterns. Of course, one of the implications of producing seasonally adjusted estimates for many major series by aggregation is that exact factors cannot be projected for those series. However, implicit seasonal factors can be calculated after the fact by taking the ratio of the unadjusted aggregate to the seasonally adjusted aggregate, or, for additive implicit factors, the difference between those two aggregates.

Availability of revised series

As indicated above, much of the revised seasonally adjusted data is being published in this and next month's issues of *Employment and Earnings*. Additional data for any of the several hundred seasonally adjusted labor force series, as well as the January-June 1984 factors for any of the directly adjusted series beyond the 12 major components, can be obtained from BLS upon request. Requests for data or inquiries concerning seasonal adjustment methodology or the availability of machine-readable files of labor force data should be addressed to the Division of Data Development and Users' Services, Office of Employment and Unemployment Statistics, Bureau of Labor Statistics, Washington, D.C. 20212.

Table 3. Current seasonal adjustment factors for the 12 major civilian labor force components, January-June 1984

Procedure and series	January	February	March	April	May	June
Multiplicative Adjustment (Divide factor into original value)						
Agricultural employment:						
Men, 20 years and over	0.904	0.895	0.912	0.960	1.031	1.076
Women, 20 years and over	.797	.795	.864	.921	1.068	1.204
Men, 16 to 19 years	.608	.558	.659	.844	1.079	1.529
Women, 16 to 19 years	.667	.537	.573	.753	1.020	1.494
Nonagricultural employment:						
Men, 20 years and over	.985	.985	.990	.995	1.000	1.008
Women, 20 years and over	.997	1.001	1.003	1.003	.999	.986
Unemployment:						
Men, 20 years and over	1.149	1.149	1.113	1.021	.972	.969
Women, 20 years and over	1.048	1.018	.983	.932	.957	1.005
Additive Adjustment (Subtract factor from original value)						
Nonagricultural employment:						
Men, 16 to 19 years	-296	-346	-295	-224	-119	472
Women, 16 to 19 years	-246	-295	-261	-210	-170	283
Unemployment:						
Men, 16 to 19 years	7	13	-17	-98	-97	251
Women, 16 to 19 years	-70	-78	-99	-91	-20	302

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A COMPARISON OF VARIOUS TREND-CYCLE ESTIMATORS

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Various alternatives to the two by twelve and two by four moving averages used in the X-11-ARIMA seasonal adjustment method (Dagum, 1980) for the preliminary trend-cycle estimation are compared by means of spectral analysis. For monthly series, its is found that the Cholette (1978) estimator performs better than: (1) a variant of the Leser (1963) estimator, (2) the Bongard (1960) estimator and (3) the two by twelve moving average. For quarterly series, the Leser estimator proves preferable. The filters discussed here were recently incoporated as optional in the X-11-ARIMA programme.

INTRODUCTION

The main object of this paper is to choose substitutes for the centred twelve and centred four term filters (averages) which are used by the X-11-ARIMA seasonal adjustment method (Dagum, 1975, 1980) in order to estimate the preliminary trend-cycle. The criteria considered for the selection of the two substitutes are: first, the efficiency to reproduce movements of cyclical nature - particularly those of shorter periodicity - and, second, the elimination of seasonal and irregular fluctuations. The spectral analysis of the linear filters discussed will determine to what extent they satisfy these criteria.

The centred twelve term filter will be compared to (1) a variant of the 24-term monthly filters developed by Leser (1963); (2) to the 24-term monthly filters by Cholette (1978); and (3) to the 19-term monthly filters by Bongard (1960). In the case of quarterly series, the corresponding quarterly versions of the same filters will be compared with the centred four term filter - except for the Bongard filters which do not have quarterly equivalents.

Although the substitute filters contemplated do not "loose" any estimates like the centred twelve and centred four term filters, the properties of their six (two in the quarterly case) last and first estimates will be ignored. In other words, we are only interested in the estimates which could alternatively be obtained by the centred twelve (four) term filter.

1. CONSTRUCTION OF THE TREND-CYCLE ESTIMATORS

First, a few definitions are in order. The *estimation interval* will mean the T consecutive time points or periods of a series - that is the segment of series considered - for which estimates are to be found. To at least one *estimation period* t ($1 \leq t \leq T$) of the interval corresponds a linear filter or weighted average with weights $[w_{t,k}, k = 1,...,T]$, which applied to the observations $[z_k, k = 1,...,T]$ of the interval, provides an estimate for that period (using (3)). The *estimator* will be the set of filters available for the interval. It follows naturally that the first subscript, t, of weight $w_{t,k}$ denotes the estimation period; and the second, k, the *application period* of the weight. For some estimators, like the centred twelve, there is only one estimation period; the words estimator, average and filter are then used indistinctly.

There are several ways of building linear estimators. Those discussed here were derived either on the basis of certain properties to be fulfilled by the weight system (e.g. smoothness) or on the basis of the time behaviour to which the estimates should comply (e.g. quadraticity). To the first case belong the centred twelve and the centred four term averages, the Bongard (1960) and the Henderson (1916) estimators; and to the second, those by Leser (1961, 1963), Cholette (1978) and Jorgenson (1964).

a) The Centred Twelve and Four Term Averages— The weights of a linear filter, which are to preserve the scale of the original observations, have to sum to one. If elimination of seasonality is also required, they must sum to a same value over the same month application periods. The weights of the centred twelve and of the centred four are respectively 1/24, eleven times 1/12 and 1/24; and, 1/8, 1/4, 1/4, 1/4 and 1/8. They obviously do satisfy the two conditions just mentioned.

The use of such weights for each period clearly implies a constant behaviour of the estimates over the entire 13-term (5-term) estimation interval, which is a very unrealistic assumption. It can, however, be shown (Anderson, 1971, p. 50) that the middle seventh (third) estimate reproduces a straight line exactly. Linearity is a much more sensible assumption to make about the trend-cycle over such an interval. The rejection of the former hypothesis and the acceptance of the latter entail dropping the six (two) first and last estimates; and this leaves only one estimation period (the central) for these estimators.

b) The 19-Term Monthly Bongard Estimator— Bongard (1960) starts from the following principle to build his trend-cycle estimator: In order to reduce irregularity, the weights of a weighted sum should be all positive and as small as possible. Like for the centred twelve, he requires from his weight system scale preservation (i.e. summation to 1.0) and elimination of seasonal effects (i.e. equal sums for same months); and also, the reproduction of a third degree polynomial (i.e. of the trend-cycle). For a given estimation period t ($1 \leq t \leq 19$) of the 19-period interval, this amounts to find the weights $w_{t,k}$ which minimize the following constrained objective function:

$$\begin{aligned} f(w,\gamma,\lambda,\theta;t) \; = \; & \sum_{k=1}^{19} w_{t,k}^2 \; + \; \gamma((\sum_{k=1}^{19} w_{t,k}) - 1.0) \; + \; \lambda_1(w_{t,1} + w_{t,13} - w_{t,10}) \\ & + \lambda_2(w_{t,2} + w_{t,14} - w_{t,10}) + \ldots + \lambda_7(w_{t,7} + w_{t,19} - w_{t,10}) \\ & + \lambda_8(w_{t,8} - w_{t,10}) + \lambda_9(w_{t,9} - w_{t,10}) + \lambda_{10}(w_{t,11} - w_{t,10}) \\ & + \lambda_{11}(w_{t,12} - w_{t,10}) + \sum_{j=1}^{3} \theta_j (\sum_{k=1}^{19} w_{t,k}(k-10)^j - (t-10)^j), \quad t=1,\ldots,19. \end{aligned} \tag{1}$$

The first term means that the weights should be as small as possible. The terms in γ, λ and θ respectively impose the summation to unity, the equalities of the sums for same months and the reproduction of the three first time powers on the estimation interval. By appropriately setting A and T (of dimensions 19 by 11 and 19 by 3 respectively), by forming the normal equations and by making the necessary algebraic manipulations, one finds the following solution for the 19 estimation periods

$$\begin{bmatrix} W \\ \Gamma \\ \Lambda \\ \Theta \end{bmatrix} = \begin{bmatrix} 2I & B & A & T \\ B' & 0 & 0 & 0 \\ A' & 0 & 0 & 0 \\ T' & 0 & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ B' \\ 0 \\ T' \end{bmatrix}, \tag{2}$$

where B (of dimensions 19 by 1) is a vector ones and I (19 by 19) the unity matrix. Each row t of W (19 by 19) contains the weights of the filter required to calculate the value corresponding to the t-th estimation period of the interval

$$\hat{c}_t = \sum_{k=1}^{19} w_{t,k} z_k, \quad t=1,\ldots,19, \tag{3}$$

where $\hat{c}$ and z respectively stand for the trend-cycle estimates and the observations in the given interval.

c) The 24-Term Monthly and 8-Term Quarterly Modified Leser Estimators— Leser (1963) built his estimators based on criteria that the estimates and not necessarily the weights should fulfil. Over a 2-year interval, he assumes the additive model for the series components with constant seasonality

$$z_{(i-1)J+j} = c_{(i-1)J+j} + s_j + e_{(i-1)J+j}, \quad j=1,\ldots,J; \; i=1,\ldots,2, \tag{4}$$

where z, c, s and e respectively stand for the observations, the trend-cycle, the seasonals and the irregulars. J is the number of time periods (12 or 4) per year. The components minimize the constrained objective function

$$f(c,s,\lambda) = \sum_{t=3}^{T} (c_t - 2c_{t-1} + c_{t-2})^2 + \sum_{i=1}^{2} \sum_{j=1}^{J} (z_{(i-1)J+j} - c_{(i-1)J+j} - s_j)^2 + \lambda \sum_{j=1}^{J} s_j, \quad T=2J; \tag{5}$$

or using matrix notation

$$F(C,S,\Lambda) = C'AC + (Z - C - RS)'I(Z - C - RS) + \Lambda'B'S, \tag{5'}$$

where A (of dimension T by T), R (T by J) and B (J by 1) are appropriately defined. The first term of (5) means that the trend-cycle should behave as linearly as possible; the second, that the size of the irregulars should be as small as possible. The third term insures the cancellation of seasonality. The solution of (5') is

$$\begin{bmatrix} \hat{C} \\ \hat{S} \\ \hat{\Lambda}/2 \end{bmatrix} = \begin{bmatrix} (A+I) & R & 0 \\ R' & R'R & B \\ 0 & B' & 0 \end{bmatrix}^{-1} \begin{bmatrix} I & 0 \\ R' & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} Z \\ 0 \end{bmatrix} = \begin{bmatrix} W \\ W^* \end{bmatrix} \begin{bmatrix} Z \\ 0 \end{bmatrix} \Rightarrow \hat{C} = \underset{T\times T}{W} Z, \tag{6}$$

where again the t-th row of W contains the weights used for the estimation of the t-th value of the interval according to (3).

For monthly series, the Leser estimator was modified by inserting in (5) trend-cycle second differences taken every second period:

$$\sum_{t=5}^{T} (c_t - 2c_{t-2} + c_{t-4})^2 \tag{7}$$

It will be seen later that his modification improves significantly the efficency of the estimator. Furthermore, applying the 13-term (5-term) Henderson trend-cycle estimator (see Shiskin, Young and Musgrave, 1967, p. 63; and Dagum, 1980, p. 114) to the resulting estimates produces estimates which fulfil better the criteria given in the Introduction.

$$\hat{C}^H = [H]\,\hat{C} = [H]\,W Z = W^H Z. \tag{8}$$

Rows 7 to 18 (3 to 6 in the quarterly case) of matrix [H] contain the same central weights of the 13-term (5-term) Henderson filters; and the other rows, the non-central weights. Estimation periods 7 to 18 (3 to 6) consequently get the very same treatment from the Henderson (in the sense that they are applied the same central weights).

d) The 24-term Monthly and 8-Term Quarterly Cholette Estimators— Contrary to Leser's formulation, which specifies each of the three components, the monthly 24-term and quarterly 8-term estimators by Cholette (1978) assume that the series comprises only the trend-cycle and an aggregate of seasonal-irregular residuals

$$z_t = c_t + (s_t + e_t), \quad t = 1,\ldots,T; \quad T = 2J, \tag{9}$$

where J equals 12 or 4. The trend-cycle minimizes second differences taken on ajacent observations and on observations one and two periods apart (3 first terms of (10)); and the seasonal-irregular residuals, annual same month first differences (4th term of (10)) as well as annual sums (5th term)

$$\begin{aligned} f(c) &= \sum_{t=3}^{T} (c_t - 2c_{t-1} + c_{t-2})^2 + K \sum_{t=5}^{T} (c_t - 2c_{t-3} + c_{t-6})^2 \\ &+ \Big(\sum_{t=J+1}^{T} (z_t - c_t) - (z_{t-J} - c_{t-J})\Big)^2 + \sum_{i=1}^{2} \sum_{j=1}^{J} (z_{(i-1)J+j} - c_{(i-1)J+j})^2, \end{aligned} \tag{10}$$

where K equals one for monthly and zero for quarterly series. Once more the solution takes the form of a linear combination of the observations:

$$\dot{C} = W\,Z \Rightarrow \dot{c}_t = \sum_{k=1}^{T} w_{t,k}\, z_k, \quad t = 1,\ldots,T. \tag{11}$$

For the same reasons and in the same manner as the Leser estimators, the estimates are then smoothed by means of the 13- or 5-term Henderson trend-cycle filters.

e) Centring the Modified Leser and Cholette Estimators— The length of the estimation intervals of the Leser and Cholette estimators is even. (T + 1) "centred" weights are obtained by the following transformation

$$\begin{aligned} w_{c,k} &= (w_{J+1,k} + w_{J,k-1}) / 2, \quad k = 2,\ldots,T, \\ w_{c,1} &= w_{J+1,1} / 2, \qquad w_{c,T+1} = w_{J,T} / 2, \end{aligned} \tag{12}$$

and will advantageously be used for the (N-T) "central" periods of series containing more than T (N > T) observations.

f) Weight Tables— Tables 1 to 5 display the weights of the estimators obtained in this section for the relevant estimations periods. Because of their symmetry, only the centred weights and those associated to one half of the estimation interval are given.

Table 1 Weights $w_{t,k}$ of the Monthly 19-Term Bongard Trend-Cycle Estimator for Estimation Periods 10 to 13

Estimation periods t	Application periods (k = 1,2,...)							
10	-0.0662202	-0.0302579	0.0057044	0.0416667	0.0776290	0.1135913	0.1495536	0.0833333
	0.0833333	0.0833333	0.0833333	0.0833333	0.1495536	0.1135913	0.0776290	0.0416667
	0.0057044	-0.0302579	-0.0662202					
11	0.0040509	-0.0391865	-0.0513393	-0.0324074	0.0176091	0.0987103	0.2108962	0.0833333
	0.0833333	0.0833333	0.0833333	0.0833333	0.0792824	0.1225198	0.1346726	0.1157407
	0.0657242	-0.0153770	-0.1275628					
12	0.0733300	-0.0421627	-0.0994544	-0.0985450	-0.0394345	0.0778770	0.2533896	0.0833333
	0.0833333	0.0833333	0.0833333	0.0833333	0.0100033	0.1254960	0.1827877	0.1818783
	0.1227679	0.0054563	-0.1700562					
13	0.1316964	-0.0391865	-0.1326885	-0.1488095	-0.0875496	0.0510913	0.2671131	0.0833333
	0.0833333	0.0833333	0.0833333	0.0833333	-0.0483631	0.1225198	0.2160218	0.2321429
	0.1708829	0.0322421	-0.1837798					

Table 2 Weights $w_{t,k}$ of the Monthly 24-Term Modified Leser Trend-Cycle Estimator for Estimation Periods 13 to 18

Estimation periods t	Application periods (k = 1,2,...)							
13	-0.0161892	-0.0261407	-0.0218688	-0.0087176	0.0080503	0.0250910	0.0406147	0.0568322
	0.0749326	0.0946288	0.1121857	0.1189141	0.0995225	0.1094740	0.1052022	0.0920509
	0.0752830	0.0582423	0.0427186	0.0265011	0.0084007	-0.0112955	-0.0288524	-0.0355807
14	0.0054695	-0.0183576	-0.0283538	-0.0244657	-0.0112767	0.0056164	0.0231347	0.0404258
	0.0587401	0.0823235	0.1098276	0.1319163	0.0778639	0.1016909	0.1116872	0.1077990
	0.0946101	0.0777170	0.0601986	0.0429076	0.0245933	0.0010098	-0.0264942	-0.0485830
15	0.0244566	-0.0069044	-0.0287103	-0.0353332	-0.0286950	-0.0143286	0.0040592	0.0240770
	0.0441445	0.0695309	0.1029488	0.1364210	0.0588767	0.0902377	0.1120436	0.1186665
	0.1120283	0.0976619	0.0792741	0.0592564	0.0391889	0.0138024	-0.0196155	-0.0530877
16	0.0364047	0.0056406	-0.0224556	-0.0383072	-0.0403683	-0.0318789	-0.0153860	0.0057695
	0.0287802	0.0563213	0.0919382	0.1318749	0.0469287	0.0776928	0.1057890	0.1216406
	0.1237016	0.1152122	0.0987193	0.0775639	0.0545531	0.0270120	-0.0086048	-0.0485416
17	0.0389746	0.0161718	-0.0111427	-0.0324140	-0.0431597	-0.0432053	-0.0324726	-0.0134039
	0.0104456	0.0401814	0.0768735	0.1181514	0.0443588	0.0671616	0.0944761	0.1157473
	0.1264931	0.1265387	0.1158059	0.0967372	0.0728877	0.0431520	0.0064599	-0.0348181
18	0.0323555	0.0221563	0.0021165	-0.0191807	-0.0360473	-0.0451084	-0.0432854	-0.0308003
	-0.0097361	0.0186453	0.0542750	0.0962762	0.0509778	0.0611771	0.0812168	0.1025140
	0.1193807	0.1284417	0.1266187	0.1141336	0.0930695	0.0646880	0.0290583	-0.0129429
†	-0.0080946	-0.0308607	-0.0253606	-0.0100065	0.0082255	0.0257960	0.0416667	0.0575373
	0.0751078	0.0933399	0.1086940	0.1141940	0.0995225	0.1141940	0.1086940	0.0933399
	0.0751078	0.0575373	0.0416667	0.0257960	0.0082255	-0.0100065	-0.0253606	-0.0308607
	-0.0080946							

† centred weights

Table 3 Weights $w_{t,k}$ of the Quarterly 8-Term Modified Leser Trend-Cycle Estimator for Estimation Periods 5 and 6

Estimation periods t	Application periods (k = 1,2,...)							
5	-0.0516923	0.0012821	0.1323846	0.2930256	0.3016923	0.2487179	0.1176154	-0.0430256
6	-0.0036410	-0.0579487	0.0079487	0.1786410	0.2536410	0.3079487	0.2420513	0.0713590
†	-0.0258462	-0.0208718	0.1250000	0.2708718	0.3016923	0.2708718	0.1250000	-0.0208718
	-0.0258462							

† centred weights

Table 4 Weights $w_{t,k}$ of the Monthly 24-Term Cholette Trend-Cycle Estimator for Estimation Periods 13 to 18

Estimation periods t	Application periods (k = 1,2,...)							
13	-0.0225546	-0.0234459	-0.0160151	-0.0026792	0.0121628	0.0279138	0.0413240	0.0564321
	0.0728637	0.0913320	0.1064322	0.1145676	0.1058879	0.1067793	0.0993484	0.0860125
	0.0711706	0.0554195	0.0420093	0.0269012	0.0104697	-0.0079986	-0.0230988	-0.0312343
14	-0.0106354	-0.0195678	-0.0208123	-0.0144215	-0.0026399	0.0121064	0.0272793	0.0432593
	0.0595187	0.0811315	0.1018984	0.1178834	0.0939688	0.1029011	0.1041457	0.0977549
	0.0859732	0.0712269	0.0560541	0.0400741	0.0238147	0.0022018	-0.0185651	-0.0345500
15	0.0019024	-0.0124004	-0.0214279	-0.0229499	-0.0159235	-0.0036926	0.0116272	0.0297519
	0.0471940	0.0692546	0.0931847	0.1151461	0.0814309	0.0957337	0.1047612	0.1062832
	0.0992569	0.0870260	0.0717061	0.0535814	0.0361393	0.0140787	-0.0098514	-0.0318128
16	0.0121814	-0.0035929	-0.0177541	-0.0263876	-0.0255157	-0.0176449	-0.0039735	0.0141749
	0.0338072	0.0566876	0.0805875	0.1057635	0.0711520	0.0869263	0.1010874	0.1097209
	0.1088491	0.1009782	0.0873068	0.0691584	0.0495261	0.0266457	0.0027458	-0.0224301
17	0.0181990	0.0047570	-0.0107207	-0.0239906	-0.0292988	-0.0274358	-0.0176746	-0.0019316
	0.0172443	0.0407996	0.0654582	0.0895942	0.0651343	0.0785764	0.0940541	0.1073239
	0.1126322	0.1107692	0.1010079	0.0852650	0.0660891	0.0425338	0.0178751	-0.0062608
18	0.0192206	0.0109225	-0.0021151	-0.0163613	-0.0263849	-0.0309178	-0.0270820	-0.0170193
	-0.0009008	0.0188178	0.0438290	0.0696580	0.0641127	0.0724108	0.0854485	0.0996946
	0.1097182	0.1142511	0.1104153	0.1003527	0.0842342	0.0645156	0.0395043	0.0136754
†	-0.0112773	-0.0273401	-0.0195570	-0.0053389	0.0113162	0.0274075	0.0416667	0.0559258
	0.0720171	0.0886723	0.1028903	0.1106735	0.1058879	0.1106735	0.1028903	0.0886723
	0.0720171	0.0559258	0.0416667	0.0274075	0.0113162	-0.0053389	-0.0195570	-0.0273401
	-0.0112773							

† centred weights

Table 5 Weights $w_{t,k}$ of the Quarterly 8-Term Cholette Trend-Cycle Estimator for Estimation Periods 5 and 6

Estimation periods t	Application periods (k = 1,2,...)							
5	-0.0477000	-0.0182500	0.1250000	0.3159500	0.2977000	0.2682500	0.1250000	-0.0659500
6	0.0190500	-0.0880000	-0.0182500	0.2122000	0.2309500	0.3380000	0.2682500	0.0378000
†	-0.0238500	-0.0421000	0.1250000	0.2921000	0.2977000	0.2921000	0.1250000	-0.0421000
	-0.0238500							

† centred weights

2. SPECTRAL COMPARISON OF THE ESTIMATORS

Spectral analysis of filters refers to their effects on sinusoidal fluctuations assumed present in the series to be filtered. These effects are divided into two parts: the gain and the phase shift. The former states with which percentual amplitude, reduction or amplification, a sine wave of given periodicity is passed to the filtered series; and the latter, with what lag the passage takes place. The *periodicity* measures the length of time needed for a sine wave to complete one full cycle. The *frequency* is the inverse of the periodicity.

It has been common practise among economists to associate the *trend-cycle periodicities* to relatively long fluctuations. Generally, the trend-cycle is well approximated by sines and cosines of one and a half year and more. The seasonal movement is represented by sines and cosines of periodicity 12/1, 12/2 up to 12/6 and of periodicities lying in a small neighbourhood around the numbers quoted. These "bands" of periodicity will be referred to as the *seasonal periodicities*; and the remaining periodicities, which accrue to the irregular, as *pre-seasonal periodicities* and *inter-seasonal periodicities*.

In the figures to follow, the periodicities are frequentially expressed. For monthly series, the trend-cycle frequencies range from 0 to 20/360 (18 month periodicity): and the seasonal frequencies lie in the immediate neighbourhoods of 30/360 (12 months), 60/360 (6 months) up to 180/360. For quarterly series, the corresponding frequencies respectively become 0 to 20/120 (6 quarters); and, 30/120 (4 quarters) and 60/120 (2 quarters).

Ideally, a trend-cycle filter should preserve the trend-cycle frequencies, reduce the inter-seasonal and pre-seasonal frequencies (i.e. irregularity) as much as possible and completely eliminate the seasonal frequencies.

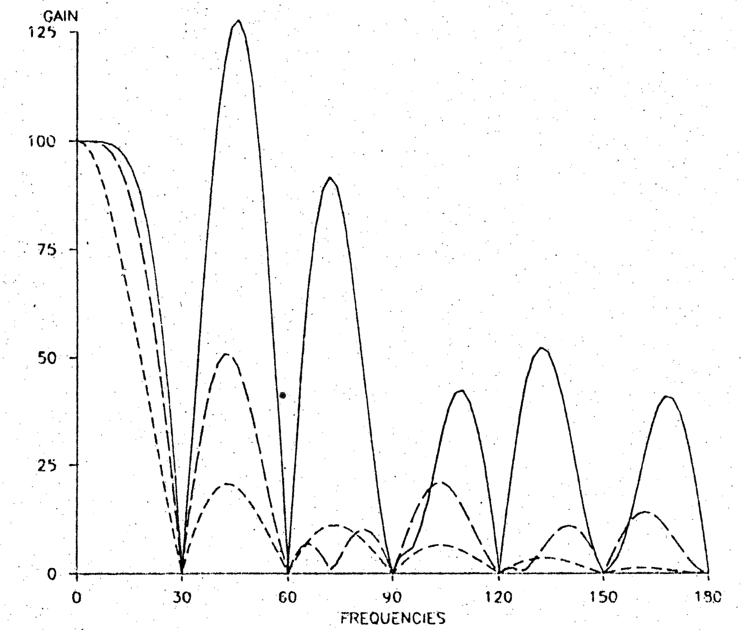

Figure 1
Gains of the Centred Twelve (-------) and of the Filters Associated to the 10th (— — —) and 13th (————) Estimation Periods of the Bongard 19-Term Monthly Estimator

a) Monthly Estimators— Figure 1 shows the gains of the filters associated to the central (10th of 19) estimation period of the Bongard estimator and to its 13th estimation period together with the gain of the centred twelve. The 13th period of the former estimator corresponds to the last period for which it would be possible to apply the centred twelve (without extending beyond the 19-period interval). The Bongard estimator preserves more of the trend-cycle frequencies than the centred twelve; but also, more of the inter-seasonal irregular frequencies, especially for the 13th estimation period. For these reasons, it is here considered a poor substitute of the centred 12-term filter. (As for the phase shifts, they are in all cases negligible at the relevant trend-cycle frequencies.)

Figure 2 displays the gains of the centred filters associated to the monthly 24-term Cholette and modified Leser estimators with that of the centred twelve. The latter retains more irregularity and less of the cycle frequencies than the two other filters, which in turn give very similar gains to one another. However, as the estimation period departs from the centre of the estimation interval, the Cholette estimator better fulfils the criteria required from a good substitute of the centred 12-term filter. In fact, as shown in Figure 3 for the 18th estimation period, the Cholette filter passes the trend-cycle frequencies better while still reducing the irregular more than the centred twelve; on the other hand, the modified Leser filter keeps more of the irregular frequencies than the centred twelve. In other words, only the Cholette estimator performs better than the centred twelve in all respects.

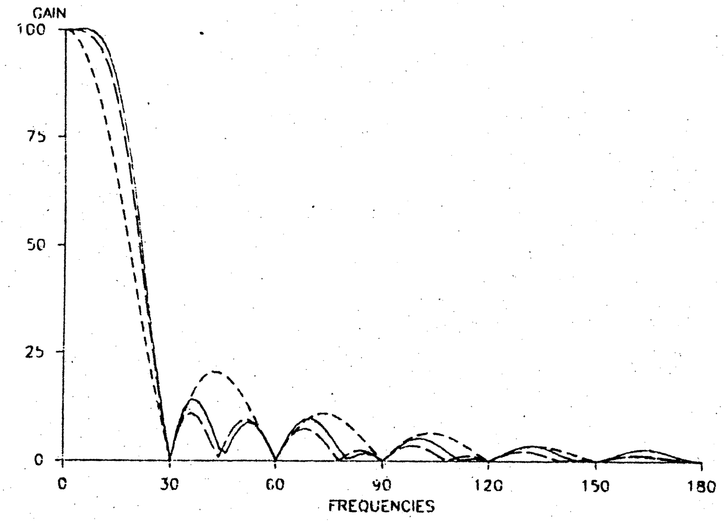

Figure 2
Gains of the Centred Twelve (--------) and of the Centred Filters Associated to the 24-Term Monthly Modified Leser (———) and Cholette (————) Estimators

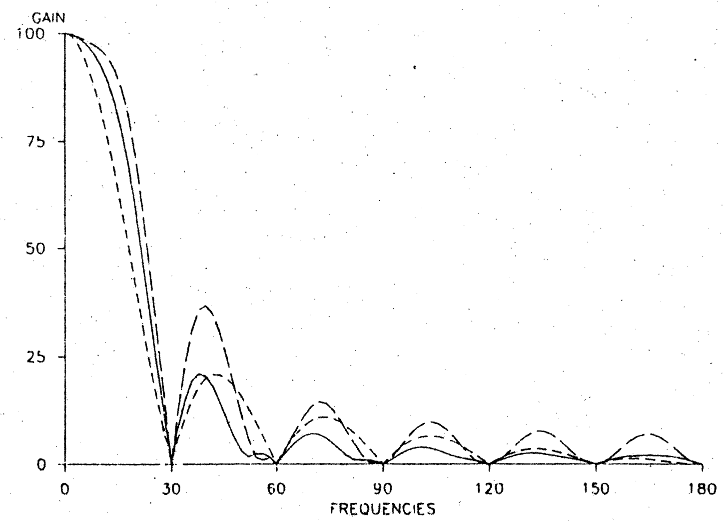

Figure 3
Gains of the Centred Twelve (--------) and of the Filters Associated to the 18th Estimation Periods of the Monthly 24-Term Modified Leser (———) and Cholette (————) Estimators

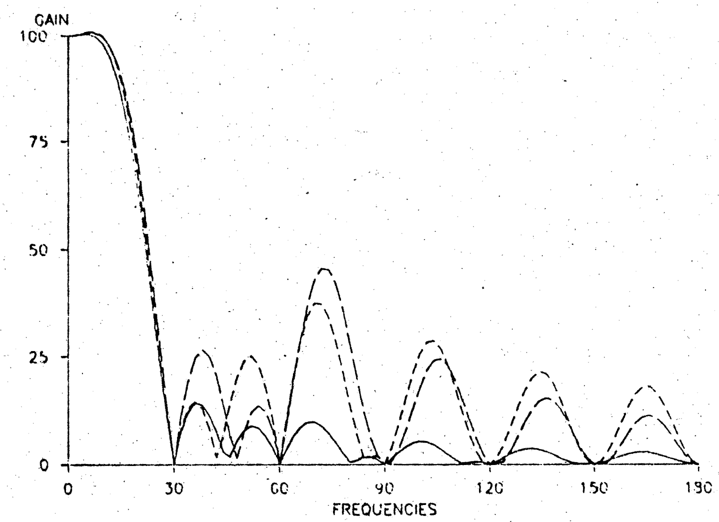

Figure 4
Gains of the Centred Filters Associated to the 24-Term Original (--------), to the Henderson Modified (— — —) and to the Henderson and Extra Second Differences Modified (————) Leser Monthly Estimators

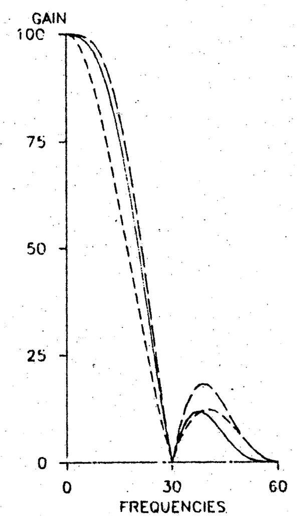

Figure 5
Gains of the Centred Four (--------) and of the Centred Filters Corresponding to the Quarterly 8-Term Cholette (— — —) and Leser (————) Estimators

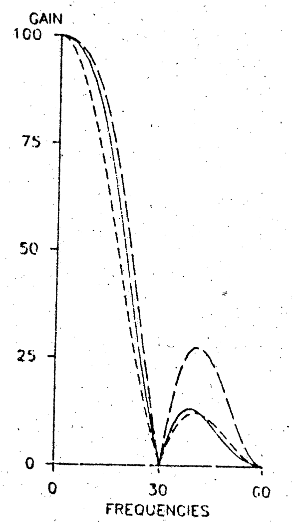

Figure 6
Gains of the Centred Four (--------) and of the Filters Associated to the 6th Estimation Periods of the Quarterly 8-Term Cholette (— — —) and Modified Leser (————) Estimators

Figure 4 illustrates the effect of the modifications made to the original monthly 24-term Leser estimator. The gains of the centred filters associated to the estimator as proposed by Leser (1963), to the estimator modified by the Henderson trend-cycle estimator and to that modified both by the extra trend-cycle second differences of (7) and by the Henderson estimator are shown. Applying the Henderson alleviates the intensity of the irregular, especially at the higher inter-seasonal frequencies; whereas the insertion of extra second differences further reduces it very perceptibly at all the inter-seasonal and pre-seasonal frequencies.

b) Quarterly Estimators— In Figure 5, one finds the gains of the centred filters corresponding to the quarterly 8-term Cholette and modified Leser estimators with that of the centred four; and in Figure 6, the gains of the filters associated to the sixth estimation periods of the same estimators. The reverse situation from that observed in the monthly case now prevails: The Leser estimator preserves less irregularity and slightly less of the cycle frequencies than its contender while still performing better than the centred four in every respects. For quarterly series, the modified Leser estimator then better fulfils the criteria chosen for a substitute to the centred four.

CONCLUSION

This study analysed the spectral properties of trend-cycle estimators that can be used instead of the centred 12-term and centred 4-term filters. The substitutes must fulfil the double condition of better preserving the frequencies associated to the trend-cycle without increasing those pertaining to the irregular component as compared to the filters to be replaced.

Three trend-cycle estimators were considered: that of Bongard (1960), that of Cholette (1978) and a variant of that of Leser (1963) (see Section 1). The findings showed that (1) the Cholette estimator performs better than the centred twelve; and (2), that the modified Leser variant is superior to the centred four.

Series which either contain fast busines cycles (i.e. movements between 18 and 30 months or 6 to 10 quarters) would benefit the most from the substitutions; whereas other series would remain rather unaffected.

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APPENDIX: CALCULATING THE GAINS AND PHASE SHIFTS OF FILTERS

Let the interval [1,...,T] contain the application time periods of a linear filter. Let t $(1 \leq t \leq T)$ be the estimation period of the filter; $[w_k, k = 1,...,T]$, the weights of the filter for which the gain $G(\omega)$ and the phase shift $\phi(\omega)$ are desired; and $A(\omega)$ and $B(\omega)$, the following quantities defined for each frequency ω, $0 \leq \omega \leq 1/2$:

$$A(\omega) = \sum_{k=1}^{T} w_k \cos(2\pi\omega(k-t)), \qquad B(\omega) = \sum_{k=1}^{T} w_k \sin(2\pi\omega(k-t)).$$

The gain of the filter and its phase shift respectively write:

$$G(\omega) = (A(\omega)^2 + B(\omega)^2)^{1/2}, \quad G(\omega) \geq 0.$$

$$\phi(\omega) = \tan^{-1}(B(\omega) \ / \ A(\omega)), \quad -\pi/2 \leq \phi \leq \pi/2.$$

The phase shift can advantageously be expressed in terms of absolute time periods by the following transformation:

$$\phi_A(\omega) = \phi(\omega) \ / \ 2\pi\omega.$$

The absolute phase shift gives the number of lag periods the estimates have over reality at a given frequency.

In Figures 1 to 6, the frequencies ω take the values n/360, n = 0,2,4,...,180, in the monthly case; and n/120, n = 0,2,4,...,60, in the quarterly case.

A Set of Quality Control Statistics
for the X-11-ARIMA
Seasonal Adjustment Method

by
J. Lothian and M. Morry

Seasonal Adjustment and Time Series
Analysis Staff
Statistics Canada.

78-10-005 E

October, 1978.

A SET OF QUALITY CONTROL STATISTICS FOR THE X-11-ARIMA SEASONAL ADJUSTMENT METHOD

I. INTRODUCTION

The X-11 Variant of the Census Method II Seasonal Adjustment Method (J. Shiskin et al 1967) contained a summary measures table denoted by F.2. The purpose of the F.2 Table was to give a set of statistics pertaining to the estimated trend-cycle, seasonal and irregular components. These statistics gave information about the average percent changes in each component with and without regard to sign over different spans, relative contribution of components to percent changes in the original series, average duration of run etc.

The X-11 method was modified in 1973 at Statistics Canada to include two statistics, Q_1 and Q_2 which provided an indication of the amount and nature of the irregular and the seasonal component respectively. A description of these statistics and their basic assumptions are discussed by Huot and de Fontenay (1973).

Considerable research has been carried out since the first set of guidelines was developed and it has now been reduced to only one Q statistic which results from the combination of eleven other measures. Most of them are obtained from the summary measures in Table F.2. The values of the eleven statistics range from 0 to 3, low values indicating good quality with 1 being the cut-off point for the test. A weighted sum of the eleven statistics makes up the final Q value. If Q exceeds 1, the series fails the guidelines, i.e., the quality of the seasonal adjustment is considered unacceptable. The sections to follow give a detailed description of each of the statistics.

II. THE FIRST SEVEN QUALITY CONTROL STATISTICS, M1 TO M7

(1) The relative contribution of the irregulars over three months span (M1).

In the Summary Measures section of the X-11-ARIMA program, Table F2.B contains the relative contributions to the variance of the percent change (difference) in the components of the original series. In table F2.B under D13, D12 and D10, the contribution of the irregular, trend-cycle and seasonal components can be found over the spans 1 to 12 (or 4 for quarterly series).

For example, the value for span 1 under the heading D13 denoted here by $R_{I(1)}$ is calculated as follows:

$$\bar{I}(1)^2 = \left\{ \sum_{t=2}^{N} \{I_t - I_{t-1}\}^2 / (N-1) \right\} \qquad (2.1)$$

where I_t are the final irregulars from Table D13 and N is the number of points in the series, and

$$\bar{O}'(1)^2 = \bar{I}(1)^2 + \overline{TC}(1)^2 + \bar{S}(1)^2 \qquad (2.2)$$

where $\overline{TC}(1)^2$ and $\bar{S}(1)^2$ are calculated from Tables D12 and D10 respectively, according to the formula (2.1). The estimated relative contribution of the irregular to the variance of the percent change in the original series over span 1, equals

$$R_{I(1)} = \frac{\bar{I}(1)^2}{\bar{O}(1)^2} \times 100\% \qquad (2.3)$$

From the point of view of seasonal adjustment, it is important to know the proportion of the irregular contribution relative to the seasonal contribution. If the irregular variation is too high when compared to the variation in the seasonal component, the two components cannot be separated successfully.

Applying differencing (over span 1, span 2, etc.) has the effect of removing a linear trend from the original series in an attempt to make it stationary (it is necessary to have a stationary series otherwise the variance is not defined). Unfortunately, differencing effects the variance of the other components as well.

In order to find out how much of the cycle, seasonal and irregular is removed by lag one, lag two, etc. differencing, the transfer functions of the differencing operators were studied. The following assumptions were made when examining the effect of the transfer functions:

(a) The irregular component I_t's are independent identically distributed random variables, i.e. their contribution to the variance is constant at all frequencies.

(b) The seasonal component shows typical behaviour in the distribution of power over the fundamental and harmonic frequencies.

(c) The cycle is distributed evenly over the very low frequencies.

Using these assumptions, the following information was extracted from the transfer functions and tabulated in Table I.

TABLE I. Percentage of the power (variance) left after applying differencing

Over span	T	C	S	I
1	0	6	24	134
2	0	18	64	122
3	0	36	112	112
4	0	60	147	116

Renormalizing the above table into a form where the relative power of I equals 100%, we obtain Table II.

TABLE II. Percentage of power (variance) left relative to I equals 100, after differencing

Over span	T	C	S	I
1	0	4.5	17.9	100.0
2	0	14.8	52.5	100.0
3	0	32.1	100.0	100.0
4	0	51.7	126.7	100.0

Since the main concern is to get a clear idea of the relative variation of the seasonal versus that of the irregular component, obviously a first difference removes too much of the seasonal relative to the irregular component. A lag 3 difference, however, appears optimal because it preserves the original proportions between the seasonal and the irregular. It has the minor disadvantage of not removing the cycle completely. Still, differencing over span 3 provides the best measure for comparing the contribution of the irregular against that of the seasonal component. Three months span corresponds to one quarter, thus in quarterly series testing is carried out on lag 1 differences.

The maximum acceptable contribution of the irregular to the total variance was set at 10% in the lag 3 difference. Thus if

$$R_{I(3)} > 10\% \qquad (2.4)$$

the series fails the test statistics M1. Renormalizing $R_{I(3)}$ yields

$$M1 = \frac{R_{I(3)}}{10} \qquad (2.5)$$

Thus if M1 is greater than 1 the contribution of the irregular to the variance is considered too high.

II.2) The relative contribution of the irregular component to the variance of the stationary portion of the series (M2).

This measure is similar to M1. The only difference is in the trend remover used to make the series stationary. Instead of lag three differencing, a line is fitted to the trend-cycle values in Table D12 to obtain a trend estimate (or an exponential growth is fitted and all the components are logarithmically transformed if the series is multiplicative). This trend estimate is removed from the original series to obtain a stationary raw series B1'. Table D12 is transformed as well by removing the same trend from it to get a new Table D12'. The relative contribution of the components appearing in Table F2.F are calculated as follows:

$$\text{contribution of I} = \frac{\text{variance of table D13}}{\text{variance of table B1'}} \qquad (2.6)$$

$$\text{contribution of C} = \frac{\text{variance of table D12'}}{\text{variance of table B1'}} \qquad (2.7)$$

$$\text{contribution of S} = \frac{\text{variance of table D10}}{\text{variance of table B1'}} \qquad (2.8)$$

If the contribution of I is greater than 10%, the series fails the M2 test, where

$$M2 = 100 \times \frac{\text{contribution of I}}{10} \qquad (2.9)$$

Thus if M2 exceeds 1 the variation of the irregular component contributes too much to the total variation of the series.

II.3) The amount of month-to-month change in the irregular as compared to the amount of month-to-month change in the trend-cycle (M3).

The purpose of a seasonal adjustment procedure is to extract the seasonal component from the raw data in order to estimate a seasonally adjusted series. Because of the iterative nature of the X-11 program it is important that in the steps leading up to the final seasonal adjustment, not only must the seasonal be well identified, but the trend-cycle and irregular component be properly estimated as well. If the month-to-month movement of the irregulars is dominant in the CI series, it is difficult to separate these two components and the overall quality of the seasonal adjustment suffers.

The statistic measuring this relationship between the irregular and the trend-cycle is the $\bar{I}/\bar{C}$ ratio where $\bar{I}$ and $\bar{C}$ are the mean absolute change from tables D13 and D12 respectively. This $\bar{I}/\bar{C}$ ratio can be found at the top of Table D12, and also in Table F2.H. If it exceeds 3, the amount of irregular movement is considered too high. The corresponding test statistic is the following:

$$M3 = (\bar{I}/\bar{C} - 1)/2. \qquad (2.10)$$

The formula for quarterly series is:

$$M3 = (\bar{I}/\bar{C} - .33)/.67 \qquad (2.11)$$

If M3 exceeds 1, the series fails this test.

II.4) The amount of autocorrelation in the irregular as described by the average duration of run (M4).

One of the basic assumptions of the statistical F-tests in the X-11 method is that the irregular component is a purely random process with constant variance and zero covariance when the relationship among the trend-cycle, seasonal and irregular is additive (multiplicative).

The program uses the Average Duration of Run statistic (ADR) to test for the randomness in the final estimated residuals obtained from Table D13 and prints it in Table F2.D under I. This non-parametric test, developed by W.A. Wallis and G.H. Moore (1941), is constructed on the basis of the number of turning points (a turning point occurs in a time series when the sign of the month-to-month change reverses). It is designed to test the randomness of the residuals against the alternative hypothesis that the errors I_t follow a first order autoregressive process of the form $I_t = \rho I_{t-1} + e_t$, where ρ is the autocorrelation coefficient and e_t is a purely random process.

Given a purely random process of infinite length, the ADR statistic would equal 1.50. For a series of 120 observations, the 99% confidence interval for the ADR extends from 1.30 to 1.75. Values greater than 1.75 indicate positive auto-correlation and those smaller than 1.30, negative autocorrelation of the residuals. The test statistic M4 is based on the normal approximation formula given by Bradley (1968).

$$M4 = \frac{\left| \frac{N-1}{ADR} - \frac{2(N-1)}{3} \right|}{\left(\frac{16N-29}{90} \right)^{\frac{1}{2}}} \times \frac{1}{2.58} \qquad (2.12)$$

where the value 2.58 is the 1% limit value of the standard normal distribution in two-sided tests. If M4 is greater than 1, there is significant autocorrelation present in the residuals and the series fails this test.

II.5) The months (quarters) for cyclical dominance statistic (M5).

This statistic measures the number of months (quarters) it takes the average absolute change in the trend-cycle to dominate that in the irregular. This value is printed in Table F2.E. This measure is similar to M3, namely it examines the relative size of the changes in the irregular and trend-cycle components. The $\bar{I}(k)/\bar{C}(k)$ ratios are computed for spans k equal to 1 to 12 (or 1 to 4 for quarterly series), and the MCD is derived from them:

$$MCD = k \quad \text{if} \quad \bar{I}(k)/\bar{C}(k) \leq 1$$
$$\text{and} \quad \bar{I}(k-1)/\bar{C}(k-1) > 1 \qquad (2.13)$$

For example, given the following $\bar{I}/\bar{C}$ ratios for spans 1, 2, 3, 4:

$$\bar{I}(1)/\bar{C}(1) = 1.82$$
$$\bar{I}(2)/\bar{C}(2) = 1.10$$
$$\bar{I}(3)/\bar{C}(3) = 0.81$$
$$\bar{I}(4)/\bar{C}(4) = 0.72$$

then the MCD equals 3, indicating that it takes 3 months on average for the absolute change in the trend-cycle to become higher than that of the irregular component.

The MCD statistic takes integer values only. Therefore, it is not capable of distinguishing between an $\bar{I}/\bar{C}$ ratio that just fell below 1 after, say, 3 months span and one that exceeded 1 by only a minimal amount after 2 months span and became much less than 1 after a 3 months span. To remedy this problem, a new statistic MCD' was calculated by linearly interpolating the $\bar{I}/\bar{C}$ ratio to find its intersection with 1. In the example quoted above, the MCD' value is the following:

$$MCD' = 2 + \frac{1.10 - 1.00}{1.10 - 0.81} = 2.34 \qquad (2.14)$$

An MCD statistic of 6 or over has been traditionally considered to be unacceptable. Thus the final M5 statistic takes the form:

$$M5 = \frac{MCD' - 0.5}{5.0} \qquad (2.15)$$

The quarterly equivalent of this test is as follows:

$$M5 = \frac{QCD' - 0.17}{1.67} \qquad (2.16)$$

M5 values greater than 1 fail the test for months (quarters) for cyclical dominance.

II.6) The amount of year-to-year change in the irregular as compared to the amount of year-to-year change in the seasonal (M6).

As mentioned earlier, it is very important from the point of view of seasonal adjustment that the seasonal factors are properly identified. One of the steps in the X-11 seasonal adjustment is the application of the 7-term moving average (3 x 5) weights to the SI ratios (differences) in order to separate the irregular from the seasonal component. Experience has shown that when the year-to-year change in the irregular is too small, compared to the year-to-year change in the seasonal factors, as described by a low $\bar{I}/\bar{S}$ ratio, the (3 x 5) moving average is not flexible enough to follow the seasonal movement. On the other hand, when the $\bar{I}/\bar{S}$ ratio is too high, the (3 x 5) seasonal filter proves too flexible and the resulting seasonal factors are contaminated with some of the irregular movement.

Studies with 421 series presently seasonally adjusted by X-11 at Statistics Canada indicated (see Lothian (1978)) that when the $\bar{I}/\bar{S}$ ratio fell between 1.5 and 6.5, the (3 x 5) moving average worked relatively well. Beyond that range, the use of a shorter seasonal filter (for too low $\bar{I}/\bar{S}$ values) or a longer moving average (for too high ratios) would have been necessary to separate the two components correctly. Incidentally, of the 421 series 2% had $\bar{I}/\bar{S}$ values less than 1.5 and 2% had values exceeding 6.5. The M6 measure is based on the cut-off points 1.5 and 6.5 and is formulated the following way:

$$M6 = \left| \frac{\bar{I}/\bar{S} - 4.0}{2.5} \right| \qquad (2.17)$$

If it exceeds 1.0, the statistics fails, but the problem may be remedied by adjusting the series with a (3 x 1) moving average if the $\bar{I}/\bar{S}$ ratio as shown in Table F2.H is less than 1.5 or using the stable seasonality option if this ratio is greater than 6.5.

II.7) The amount of stable seasonality present relative to the amount of moving seasonality (M7).

The 1967 version of the X-11 program contained a one-way-analysis F-test applied to the final SI ratios in Table D10 to measure the amount of stable seasonality present. At Statistics Canada, a companion F-test was developed by J. Higginson (197[illegible]) to signal if there is moving seasonality in the series.

These two F-values were combined into one statistic denoted by T that was designed to indicate whether the seasonality present in the series is 'identifiable' by X-11 or not. Here the seasonality is called identifiable if the absolute error (or distortion) introduced in the final seasonal factor estimates is not too high. It was found that this distortion depended on both the F_S (F-value from the stable seasonality test) and F_M values (F-value from the moving seasonality test). Low F_S values suggested high distortion, while high F_M-values indicated further distortion was introduced due to movement. The cut-off point was based on 10-year monthly series and it corresponds to a combination of F_S and F_M values that indicate 50% distortion in the seasonal factor estimate. Thus the test statistic took the following form:

$$M7 = T = \sqrt{\frac{1}{2}\left(\frac{7}{F_S} + \frac{3F_M}{F_S}\right)} \qquad (2.18)$$

For a detailed description on how the test was derived, the reader is referred to Lothian and Morry (1978). M7 values exceeding 1 indicate that the seasonality in the series is not identifiable.

III. THE LAST FOUR QUALITY CONTROL STATISTICS DESCRIBING THE YEAR-TO-YEAR MOVEMENT IN THE SEASONAL COMPONENT, M8 TO M11

The seasonal filters of X-11 work well only on constant seasonals in the first and last three years of the series, while in the middle years they can reproduce a line or a constant. Thus only a constant seasonal component can be optimally estimated for the whole length of the series. If the original seasonals contain year-to-year movement, the seasonal factor estimates will have considerable error.

We distinguish between two types of movement; one that exhibits quasi random fluctuations and the other where changes appear in the same direction throughout the years. The size of the first type of movement can be measured from the average absolute year-to-year change in the seasonal factors, while the simple arithmetic mean of the changes gives an indication of the size of systematic (linear) movement. Random fluctuations are measured by statistics M8 and M10. Statistics M9 and M11 describe the size of linear movement. M8 and M9 are calculated using all the data in Table D10.

Since users are mostly interested in the quality of seasonal adjustment in the recent years, statistics M10 and M11 were introduced to describe the seasonal movement at the end of the series. It is especially important to know if there is significant linear movement in the seasonal factors of the last years because the seasonal factor estimates will then be considerably distorted by the end-weights of the seasonal filters. It is this same distortion that prevents the use of the last three years' seasonal factors to measure the amount of seasonal movement. Instead, the changes of the three years before the last three years are examined in the hope that the seasonal movement remains unaltered in the end years.

Obviously an average absolute change of .5 in multiplicative seasonal factors ranging from 98.0 to 102.0 within a year does not have the same significance as an average change of .5 coming from additive seasonal differences in the range −165 to +200. Therefore, it is important to normalize the values in Table D10 before proceeding to calculate the statistics. Thus the measures M8 to M11 were based on the normalized seasonal factors:

$$S'_t = \frac{S_t - \bar{S}}{\text{standard deviation of } S_t} \tag{3.1}$$

III.1) The size of the fluctuations in the seasonal component throughout the whole series (M8).

As mentioned before, the fluctuations are measured by the average absolute change.

$$|\overline{\Delta S'}| = \frac{1}{J(N-1)} \sum_{j=1}^{J} \sum_{i=2}^{N} |S'_{Ji+j} - S'_{J(i-1)+j}| \tag{3.2}$$

where N is the number of years and J equals 4 or 12 (for quarterly or monthly series). The maximum acceptable change was set at 10%. Thus the M8 measure took the following form:

$$M8 = 100 \times |\overline{\Delta S'}| \times \frac{1}{10} \tag{3.3}$$

III.2) The average linear movement in the seasonal component throughout the whole series (M9).

Averaging the year-to-year changes for each month, measures the amount of systematic movement. If there are only random fluctuations from year-to-year this average will be very close to zero. If most of the changes are in the same direction per month, the average absolute change will be very close to the average arithmetic change (3.2).

Using the formula:

$$\sum_{i=1}^{N-1} \Delta S'_{Ji+j} = S'_{J(N-1)+j} - S'_{0,j} \qquad (3.4)$$

and setting the acceptance limit at 10%, we obtain:

$$M9 = 100 \times \frac{\sum_{j=1}^{J} \left| S'_{J(N-1)+j} - S'_{0,j} \right|}{J(N-1)} \times \frac{1}{10} \qquad (3.5)$$

III.3) The size of the seasonal component fluctuations in the recent years (M10).

This statistic is equivalent to M8 except that only year N–2, N–3, N–4, and N–5 are involved in the calculations.

Thus formula (3.2) becomes:

$$\overline{|\Delta S'|}_R = \frac{1}{3J} \sum_{j=1}^{J} \sum_{i=N-4}^{N-2} \left| S'_{Ji+j} - S'_{J(i-1)+j} \right| \qquad (3.6)$$

The final statistic M10 is of the form:

$$M10 = 100 \times \overline{|\Delta S'|}_R \times \frac{1}{10} \qquad (3.7)$$

III.4) The average linear movement in the seasonal component in recent years (M11).

This measure corresponds to M9 using data from year N–2, N–3, N–4 and N–5. The formula for calculating M11 is the following:

$$M11 = 100 \times \frac{\sum_{j=1}^{J} \left| S'_{J(N-2)+j} - S'_{J(N-5)+j} \right|}{3J} \times \frac{1}{10} \qquad (3.8)$$

When M11 exceeds 1, there is strong indication that the seasonal factors for recent years are highly distorted due to the flattening effect of the end weights on linear movements.

IV. GENERAL COMMENTS ON THE OVERALL QUALITY OF THE ADJUSTMENT

The eleven statistics each examine a different facet of the adjustment and no one statistic can judge the overall quality of the adjustment. Also, each statistic has been developed for an average series and thus might break down for an unusual series.

It is possible that the series fails the M1 or M2 statistic and the adjustment does not necessarily suffer. These two statistics measure the irregular variation in proportion to the seasonal variation. The average series adjusted has a cycle which contributes about 5 to 10% to the stationary portion of the variance. The threshold level for the M1 and M2 statistics is based on this assumption. If the series contains no cycle, the irregular can contribute 13 to 14% to the total variation (resulting in M1 and M2 values exceeding 1) and still be acceptable. Similarly, if the cycle contributes more than 10%, the threshold level should be lowered.

If a series has a flat (i.e. almost constant) trend-cycle, it is possible to have an $\overline{I}/\overline{C}$ ratio exceeding 3 and thus failing the M3 test, without jeopardizing the quality of the adjustment. Actually, the X-11 program compensates for the lack of trend by applying a 23-term Henderson moving average to estimate the trend-cycle. However, if the user's main objective is business cycle analysis a high M3 value signals a serious problem. It indicates that the final seasonally adjusted series contains a very high proportion of irregular movement that will prevent users from properly identifying the trend-cycle component.

Finding significant autocorrelation in the final irregulars as indicated by an M4 value greater than 1, can signal, for example, that the user should have applied trading-day regression and thus the adjustment is not valid. At the same time it is possible that the original irregulars were autocorrelated due to the sampling design. This will not affect the X-11 seasonal adjustment that is based on recognizing characteristic seasonal and trend-cycle behaviour in a series and obtains the irregulars as the residuals of the procedure. Thus the correct seasonal factor can still be well identified. It was found that the measure M4 moved rather independently from the other measures and quite often it was the only statistic that failed or one of the very few that did not fail. Consequently, it was not as related to the quality of seasonal adjustment as the others and was assigned a minimum weight.

In the case of M5, what was said about M3 applies again. It is possible that the irregulars are too high but it is also conceivable that the series contains an almost constant trend-cycle which does not prevent X-11 from isolate out the right seasonal movement.

As pointed out before, M6 is the only statistic where failure can be corrected. The user is advised to rerun X-11 and apply the appropriate seasonal moving average to the SI series in order to improve the quality of seasonal adjustment.

Any series that fails the statistic M7 has either no seasonality or the seasonal estimates are so distorted that the seasonal component is not identifiable as indicated in the message after Table D8. This measure is the most important one in the set of quality control statistics and is, therefore, assigned the highest weight. If the series fails M7, the user is strongly advised not to adjust the series. However, there are exceptions even here. It is possible that due to using an additive option in adjusting a series where the components are related multiplicatively and that has a rapidly growing trend, the F_M-value from the test for moving seasonality is very high. This can result in an M7 value exceeding 1. If the adjustment is rerun multiplicatively, the F_M-value will be reduced significantly and M7 passes the guidelines.

Failing statistics M8 and M10 might not be crucial if M9 and M11 pass the guidelines (their value being less than 1) and the user is only worried about bias in the current seasonal estimates. Similarly, if M9 and M11 exceed 1, but the user is only interested in the historical seasonal factors, those estimates can still be accurate because the central weights of the seasonal moving average can follow any linear movement. However, if one is interested in the current seasonal factors, high M9 and M11 values indicate the presence of significant distortion in the estimates.

From the above discussion, it is obvious no one statistic can assess the quality of the adjustment. If all eleven fail, the adjustment is unacceptable. But what if some fail and others do not? A quality control statistic was developed that is a weighted sum of the eleven statistics. Each statistic was assigned a weight according to its relative importance to the overall quality of the adjustment. One statistic cannot cause the adjustment to be rejected, rather it must be a composite effect of all the statistics. The weights assigned to the eleven statistics appear in Table III.

TABLE III. The Standard Eleven M Weights

Statistics (Mi)	Weight (w_i)
M1	13
M2	13
M3	10
M4	5
M5	11
M6	10
M7	16
M8	7
M9	7
M10	4
M11	4

The eleven statistics can sometimes take values less than zero or greater than three. If this happens the statistic is set to be zero or three respectively. Thus the quality control statistic Q is defined as:

$$Q = \frac{\sum_{i=1}^{11} w_i M i}{\sum_{i=1}^{11} w_i} \qquad (4.1)$$

If the user selects a seasonal moving average different from a (3 x 5) for estimating the seasonal factors, the statistic M6 is not relevant. Thus under these conditions:

$$w_6 = 0. \qquad (4.2)$$

If the series is less than 6 years long, or the stable seasonal option is chosen, the statistics M8, M9, M10 and M11 cannot be calculated and the weights are redefined as displayed in Table IV

TABLE IV. Modified M Weights

Statistics(Mi)	Weight (w_i)
M1	17
M2	17
M3	10
M4	5
M5	11
M6	10
M7	30
M8	0
M9	0
M10	0
M11	0

This combination of the eleven statistics were very successful in assessing the quality of adjustment of 421 series tested by the authors. These series were adjusted at Statistics Canada with the X-11-ARIMA program and varied in length from 5 to 30 years. The average value for the eleven M statistics and the Q statistics for the 421 series are given in Table V.

TABLE V. Average Values for the Statistics

Statistics	Monthly Series	Quarterly Series	All Series
M1	0.719	0.556	0.680
M2	0.605	0.332	0.540
M3	0.485	0.304	0.442
M4	0.424	0.662	0.481
M5	0.593	0.465	0.563
M6	0.380	0.516	0.412
M7	0.403	0.362	0.393
M8	0.640	0.562	0.619
M9	0.394	0.393	0.393
M10	0.724	0.685	0.714
M11	0.684	0.669	0.680
Q	0.529	0.461	0.513

If the Q statistic is greater than 1, the adjustment of the series is declared to be unacceptable. The adjustment is also rejected if the test for identifiable seasonality fails. For quarterly series 11.0% of the series failed the Q statistic and an additional 1% failed the test for identifiable seasonality. For monthly series, 8.4% had Q-values higher than 1 and an additional 3.7% were rejected because they did not pass the test for identifiable seasonality. Overall 12.1% of the 421 seasonal adjustments were rejected. The 51 series that failed were examined in detail by the authors and for all of them, the adjustment was deemed to be unacceptable. The quality control statistics presented here will enable users with a large number of series to quickly assess the quality of all their adjustments as well as enable people with little knowledge of seasonal adjustment to make judgements on the acceptability of the results. The Q statistic provides a general assessment of the quality of the adjustment, but the users should beware of attaching significance to small changes in the statistic. This especially holds for aggregate adjustments as shown in Appendix A.

At the back of each printout produced by the X-11-ARIMA program, appears a summary of the Q statistics for all the series adjusted in that run. Thus, the quality of large numbers of series can be quickly judged. Immediately after the Q statistics are copies of the F2 and F3 tables of all the series run. If any series has produced an unacceptable adjustment, the user can turn to the F2 and F3 tables for that series and further identify the problem. Following this procedure, hundreds of series can be assessed in less than an hour.

APPENDIX A

QUALITY ASSESSMENT OF AGGREGATE SERIES

The new X-11-ARIMA program can automatically produce direct and indirect aggregate seasonal adjustments of several component series. The preceding quality control statistics are produced for both the direct and indirect adjustment. The Q statistics for the two methods can be used to assess the acceptability of the adjustment for both methods. Unfortunately, the Q statistics cannot be used to judge which of the two methods gives a superior adjustment. The Q's for the two types of adjustments are usually very close to each other and small differences in the Q's cannot be interpreted as being significant. The M8 and M10 statistics for the indirect method will generally be greater than for the direct. This will tend to make the Q for the indirect method greater than for direct. This creates a bias in the Q statistic against indirect adjustments.

Additional summary statistics are produced if an aggregate adjustment is requested. These statistics are printed on the same page as the summary of all the Q statistics for the series adjusted. This Summary appears right after the printout of the last series.

The comparison test for the direct versus the indirect seasonal adjustment method is based on a paper by Lothian and Morry (1977). The statistic tests the degree of smoothness of the seasonally adjusted series. The standard deviation of the month-to-month (or quarter-to-quarter) changes in the two seasonally adjusted series is computed for the whole series and for the last three years of the series. The standard deviation of the direct differences is then subtracted from the indirect for both the whole series and the last three years. If the resulting differences are positive, the indirect method gives a smoother adjustment than the direct method. If they are negative, the direct method results in a smoother seasonally adjusted series. It is also possible that the results for the last three years disagree with those of the full series. The differences are normalized by dividing by the average value of the seasonally adjusted series and multiplying by 100 to get the percentage difference between the direct and indirect methods.

ebruary 6, 1978

KOLMOGOROV-SMIRNOV SIGNIFICANCE TEST FOR THE FINAL IRREGULARS(1)

This test was incorporated into the program in order to provide information about the independence of the irregulars in the series. It examines the normalized cumulative periodogram of the residuals and compares it to the theoretical cumulative periodogram of a random process.

The periodogram decomposes the total variance of a series according to frequencies.

The normalized cumulative periodogram(2) takes values from zero to one. It gives the contribution to the total variance of the series up to the frequency in question.

An infinite series of independent random variables is expected to have equal contributions to the total variance at all frequencies. Thus, if the residuals are the estimates of a sample realization of a purely random process and if the size of the sample tends to infinity, the normalized cumulative periodogram tends to coincide with the diagonal of the square in which it is drawn.

The Kolmogorov-Smirnov test(3) is applied to assess whether the normalized periodogram of the estimated residuals from Table D13 deviates significantly (at the 5% level) from that of a purely random process.

If the test fails, the hypothesis of a purely random process is rejected against the alternative that periodic processes are present in the residuals.

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(1) For a more detailed description of this test, the reader is referred to E.B. Dagum, J. Lothian, M. Morry, A Test of Independence of the Residuals based on the Cumulative Periodogram 1975, Statistics Canada - working paper.

(2) For further information see G.E.P. Box and G.M. Jenkins, Time Series Analysis, Forecasting and Control, Holden Day, San Francisco, 1970 pages 36-44, 294-298.

(3) See James V. Bradley, Distribution-free Statistical Tests, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1968 p. 296.

Caution must be exercised, however, when interpreting the test results. The filters (or moving averages) of X-11 are known to extract components of low and seasonal frequencies from the irregulars thus introducing a lack of low and mid-frequencies and a predominance of high frequencies in the cumulative periodogram. This type of behaviour is expected and does not necessarily indicate the presence of non-random irregulars in the original series nor necessarily indicate a problem with the seasonal adjustment.

It should also be pointed out that even if the test gives evidence of positively autocorrelated irregulars (predominance of low frequencies in the periodogram), it is most likely not due to faulty seasonal adjustment but rather it is an indication that the raw series contains autocorrelated irregulars. Seasonal adjustment was carried out on simulated series containing autocorrelated irregulars. X-11 had no trouble extracting the proper seasonal factors from these series. It also reproduced the autocorrelated irregulars as attested by the Kolmogorov-Smirnov test.

Thus the above test provides information about the quality of the raw series as well as about the quality of the seasonal adjustment.

A Test for the Presence of Seasonality and a Model Test

by John Higginson

Seasonal Adjustment Methods

Statistics Canada, September 1976

The user may wonder whether a certain series contains a significant amount of seasonality and if so, whether an additive or multiplicative model provides the better fit. The following procedure, developed by Bell Canada, attempts to answer these questions.

The Model Test

First the raw series is checked to see if it contains any zeros or negative terms. If it does, the multiplicative option of the X-11 will not work, so there is no point in performing a model test.

If the series has all positive terms, additive and multiplicative models are fitted to the raw series, and the residuals are squared, summed and printed. The user can choose the model having the smaller sum. The models are fitted as follows. First a 2 by 12 moving average is applied to the raw series, O, if it is monthly (if it is quarterly, replace 12 by 4 throughout), in order to obtain an estimate of the trend-cycle $\hat{C}$. $\hat{C}$ is both divided into the raw to estimate the SI ratios for the multiplicative model, $S_m I_m = O/\hat{C}$, and is also subtracted from the raw to estimate the S + I differences for the additive model, $S_a + I_a = O - \hat{C}$. These SI ratios and differences are then averaged (a simple average is taken) to estimate the seasonal factors for both models, $S_m = \overline{S_m I_m}$ and $S_a = \overline{S_a + I_a}$. The multiplicative model fitted to the data is $\hat{C}S_m$, so the residuals are $O - \hat{C}S_m$; similarly the residuals after the additive fit are $O - \hat{C} - S_a$. The program prints out the raw data, a graph of the raw data, and the sums of the squared residuals.

Test for the Presence of Seasonality

A one-way analysis of variance test is applied twice — once to the S + I differences of the additive model and once to the SI ratios (for details of analysis of variance methods, see Mendenhall, "Introduction to Probability and Statistics", 3rd edition, 1971, Wadsworth Publishing, page 341). This test is identical to that used in the X-11 program for stable seasonality. The between-months variance is divided by the residual variance and the ratio is printed, for each model. If both ratios are less than 25, there is probably not enough seasonality to justify using the X-11 program. If at least one ratio is 25 or more, there is probably significant seasonality present in the raw series, and the model yielding the larger ratio may be the better model. This comparison of the variance ratios usually yields the same conclusion as the model test.

The model test can be applied to several series in one run, as long as the data for all the series to be tested are located on the same file.

The job control cards needed to run the test are:

//ƀEXECƀPGM=BCMTEST,REGIØN=60K
//STEPLIBƀDDƀDSN=~~HIGGNSON~~.BCMTEST.LØAD,DISP=SHR
//FT03F001ƀDDƀSYSØUT=A,DCB=RECFM=UA
//FT01F001ƀDDƀ *

Control Cards (described in detail later)

//FT02F001ƀDD ... (to be completed as described below)
numerical data

//

If the numerical data are on cards, the FT02 card should be //FT02F001ƀDDƀ*. If the data are on tape or disk the FT02 card must contain the name of the dataset and the requisite information about UNIT, DISP, DCB, and VØL = SER. For example if the data are on tape, the cards might read:

//FT02F001ƀDDƀDSN= ..., UNIT=2400-3,DISP=ØLD,
//ƀDCB=(RECFM=FB,LRECL= ...,BLKSIZE= ...),
//ƀVØL=SER= ...

If the data are on a disk the card might be:

//FT02F001ƀDDƀDSN=NAME.NA631274.DATA,DISP=SHR

The Control Cards

The first control card needed to run the test must contain in columns 1 and 2 the number of series to be tested in the run; this number must be between 01 and 99 and must be right-justified. There is only one of this type of card present in any job deck.

The second control card needed should contain three pieces of information. In columns 3 to 5 punch the number of data points in the series, right justified. The maximum number of data points permitted is 480; this means no more than 40 years of monthly data. It is advisable to have at least 3 years of data. The first year of data must start in January or the first quarter but the last year need not end in December or the last quarter.

In columns 9 and 10 punch the number of data points per year, right justified. This number is 12 for monthly series and 4 for quarterly series. In columns 12 to 15 punch the year of the first data point, right justified.

The third control card describes the format of the yearly numerical data on the input file. Other fields, if any, must be described using the X format, e.g. (8X, 12F6.0). Here 8X may refer to a field of 8 bytes containing the series identifier, and 12F6.0 refers to fields that contain a year of monthly data. If a year of data requires two or more logical records (for example two cards), separate the records by a slash (/) in the format statement, e.g. (8X, 6F12.0,/8X,6F 12.0)

The left parenthesis in the format statement must be punched in column one.

The fourth control card contains the title of the series. Any title of up to 80 characters is permitted.

Order of the Control Cards

After the first control card a set of second, third, and fourth control cards must be included for each series to be run in the test. So the order of the control cards is

First control card

Second " " for the 1st series

Third " " " " " "

Fourth " " " " " "

Second " " for the 2nd series

Third " " " " " "

Fourth " " " " " "

Second control card for the Nth series

Third " " " " " "

Fourth " " " " " "

where N is the integer punched on the first control card.

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Some consideration of decomposition of a time series

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SUMMARY

Suppose that an observable Gaussian time series Z_t can be written as the sum of an unobservable signal component T_t and a white noise component e_t. This paper proposes a procedure to estimate the T_t component uniquely by maximizing the variance of e_t with respect to a known model for Z_t. Properties of this procedure are discussed and a comparison is made with a number of smoothing and filtering procedures commonly used in practice.

Some key words: Autoregressive-moving average model; Canonical decomposition; Covariance generating function; Smoothing spline; Symmetric moving average filter; Time series.

1. INTRODUCTION

Let Z_t be a discrete observable Gaussian time series. In many problems it is appropriate to suppose that Z_t is the sum of an unobservable signal or trend component T_t contaminated by a white noise component e_t,

$$Z_t = T_t + e_t. \tag{1·1}$$

For example, T_t may be the solar radiation hitting the earth at time t and e_t the measurement error. Again, T_t may be the 'intrinsic' value of a stock while e_t is the irregular shock reflecting the 'temporary' condition of the market. When the stochastic structures of the T_t and the e_t series are known, estimates of T_t can be readily obtained (Whittle, 1963). However, in practice, such information is often not available and only the stochastic structure of Z_t can be verified from the observed data. It is therefore of interest to investigate the amount of information contained in Z_t about the structures of T_t and e_t.

In this paper, we suppose that Z_t follows the model (Box & Jenkins, 1970)

$$\phi(B) Z_t = \theta(B) a_t, \tag{1·2}$$

where B is the backshift operator, $BZ_t = Z_{t-1}$, $\phi(B)$ is a polynomial in B of degree p having its zeros on or outside of the unit circle, $\theta(B)$ is a polynomial in B of degree q having its zeros outside of the unit circle, $\phi(B)$ and $\theta(B)$ have no common zero, and the a_t's are identically and independently distributed as $N(0, \sigma_a^2)$. We assume that all the parameters in (1·2) are known. Based on this model and on the assumption that e_t is a Gaussian white noise process independent of T_t, we propose a procedure to estimate T_t uniquely by maximizing the variance of e_t. Properties of this procedure will be explored and a comparison of the result with a number of filtering and smoothing procedures used in practice will be presented.

2. A CANONICAL DECOMPOSITION OF Z_t

2·1. *Admissible decomposition*

We here discuss the range of possible stochastic structures for the components T_t and e_t given that Z_t follows the model (1·2). We assume that the e_t's are identically and independently distributed as $N(0, \sigma_e^2)$, where σ_e^2 is unknown. Then, T_t must be a Gaussian process. Now from (1·1) and (1·2)

$$\phi(B) Z_t = u_t + \phi(B) e_t, \tag{2·1}$$

where $u_t = \phi(B) T_t$. Thus, $\theta(B) a_t = u_t + \phi(B) e_t$, so that the covariance generating function of u_t is

$$C_u(B) = \theta(B)\theta(F)\sigma_a^2 - \phi(B)\phi(F)\sigma_e^2, \tag{2·2}$$

where $F = B^{-1}$, provided that σ_e^2 is such that $C_u(e^{-i\lambda}) \geqslant 0$ $(0 \leqslant \lambda \leqslant \pi)$. Note that $C_u(e^{-i\lambda})$ is proportional to the spectral density function of u_t. From a result given by Anderson (1971, p. 224), the model for T_t can be written as

$$\phi(B) T_t = \eta(B) b_t, \tag{2·3}$$

where $\eta(B)$ is a polynomial in B of degree $r \leqslant \max(p, q)$ having all its zeros on or outside of the unit circle, the b_t's are identically and independently distributed as $N(0, \sigma_b^2)$, and

$$\theta(B)\theta(F)\sigma_a^2 = \eta(B)\eta(F)\sigma_b^2 + \phi(B)\phi(F)\sigma_e^2. \tag{2·4}$$

Clearly, given the model of Z_t, the models for T_t and e_t are not unique. Any choice of $\eta(B)$, $\sigma_b^2 \geqslant 0$ and $\sigma_e^2 \geqslant 0$ satisfying (2·4) will be called an 'admissible' decomposition. For $\sigma_e^2 = 0$, we have the degenerate case that, with probability one, $T_t = Z_t$. Also, since $|\eta(e^{-i\lambda})|^2 \sigma_b^2 \geqslant 0$, the range of admissible σ_e^2 is $0 \leqslant \sigma_e^2 \leqslant \bar{\sigma}_e^2$, where

$$\bar{\sigma}_e^2 = \sigma_a^2 |\theta(e^{-i\lambda^*})/\phi(e^{-i\lambda^*})|^2 = \min_{0 \leqslant \lambda \leqslant \pi} \sigma_a^2 |\theta(e^{-i\lambda})/\phi(e^{-i\lambda})|^2. \tag{2·5}$$

Suppose that data $(Z_1, \ldots, Z_n)$ are available. For a given admissible decomposition and for t not close to the endpoints, the asymptotic formula for the minimum mean squared error estimates of T_t and e_t are given respectively by

$$W(B)\hat{T}_t = Z_t, \quad \hat{e}_t = Z_t - \hat{T}_t, \tag{2·6}$$

with $W(B) = 1 - \sigma_e^2 \sigma_a^{-2}\{\phi(B)\phi(F)\}/\{\theta(B)\theta(F)\}$, where the coefficient of B^j is the weight applied to Z_{t-j}. Equivalent forms of this result are given, for example, by Whittle (1963, p. 57) when Z_t is stationary, and by Cleveland & Tiao (1976) when Z_t is nonstationary. Modifications of the formula for t near the endpoints are given by G. E. P. Box, S. C. Hillmer & G. C. Tiao in an unpublished conference paper. Note that $\hat{T}_t$ and $\hat{e}_t$ can be readily calculated from the data and the known model for Z_t once σ_e^2 is chosen.

The polynomial $W(B)$ in (2·6) has the following two basic properties. First

$$0 \leqslant W(e^{-i\lambda}) \leqslant 1 \quad (0 \leqslant \lambda \leqslant \pi). \tag{2·7}$$

Secondly, if and only if $\sigma_e^2 = \bar{\sigma}_e^2$

$$\min_{0 \leqslant \lambda \leqslant \pi} W(e^{i\lambda}) = 0. \tag{2·8}$$

2·2. *Decomposition by maximizing σ_e^2*

In the absence of prior knowledge on σ_e^2, all the information about the stochastic structure of $\phi(B) T_t$ and e_t supplied by the known model for Z_t is contained in the relation (2·4). When decomposing a time series Z_t, it seems sensible to make the variance σ_e^2 as large as possible subject to (2·4). Intuitively, this extracts the most white noise and thus yields the strongest signal one can recover from the observed series. We shall call the decomposition corresponding to $\sigma_e^2 = \bar{\sigma}_e^2$ the canonical decomposition. Some properties of this procedure are now discussed.

For the variance and autocorrelations of T_t, suppose Z_t is stationary, i.e. that all the zeros of $\phi(B)$ lie outside of the unit circle. Then, since $\text{var}(Z_t) = \text{var}(T_t) + \sigma_e^2$, maximizing σ_e^2 is equivalent to minimizing $\text{var}(T_t)$. In addition, let $\rho_Z(K)$ and $\rho_T(K)$ be, respectively, the Kth autocorrelation coefficient of Z_t and T_t. Then,

$$\rho_T(K) = c\rho_Z(K) \quad (K = 1, 2, \ldots), \tag{2·9}$$

where $c = \text{var}(Z_t)/\{\text{var}(Z_t) - \sigma_e^2\}$. It follows that $|\rho_T(K)|$ is maximized when $\sigma_e^2 = \bar{\sigma}_e^2$. Thus, for the canonical decomposition, we achieve the strongest possible autocorrelation structure for T_t.

Secondly, we consider the structure of $\eta(B)$. From (2·4) and (2·5), when $\sigma_e^2 = \bar{\sigma}_e^2$, $\eta(e^{-i\lambda^*}) = 0$. That is, for the canonical decomposition, the moving average polynomial $\eta(B)$ for the model of T_t in (2·3) has at least one zero on the unit circle. To see one implication of this result, consider the n random variables $T = (T_1, \ldots, T_n)$. It is well known that, for large n, the eigenvalues of the covariance matrix of T approach $2\pi f_T(-\pi + 2\pi j\lambda/n)$ $(j = 1, \ldots, n)$, where $f_T(\lambda) = |\eta(e^{-i\lambda})/\phi(e^{-i\lambda})|^2 \sigma_b^2$. Thus, asymptotically, at least one of the eigenvalues will approach zero, implying a linear dependence among $T_1, \ldots, T_n$.

The innovation variance of T_t is considered next. For the model (2·3) the quantity σ_b^2 is sometimes called the variance of the innovations b_t. Irrespective of whether T_t is stationary or not, using a result of Hannan (1970, p. 137), we have from (2·4) that

$$\log \sigma_b^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log\{f(\lambda, \sigma_e^2)\} \, d\lambda, \tag{2·10}$$

where $f(\lambda, \sigma_e^2) = |\theta(e^{-i\lambda})|^2 \sigma_a^2 - |\phi(e^{-i\lambda})|^2 \sigma_e^2$. Now $f(\lambda, \sigma_e^2)$ does not depend on σ_e^2 when $\phi(e^{-i\lambda}) = 0$, and is otherwise strictly decreasing as σ_e^2 increases. It follows that, consistent with the model (1·2), σ_b^2 is minimized when $\sigma_e^2 = \bar{\sigma}_e^2$. This is an intuitively pleasing results since the randomness of the component T_t stems from the innovation process b_t. By minimizing the variance σ_b^2, we are thus making T_t as nearly deterministic as possible. In other words, T_t is the strongest signal process which could lead to the known model (1·2).

The minimum variance properties concerning T_t discussed above carry directly over to the estimates $\hat{T}_t$. First, suppose that Z_t is stationary. Then from (2·6) the covariance generating function of $\hat{T}_t$ is

$$C_{\hat{T}}(B) = \{W(B)\}^2 \{\phi(B)\phi(F)\}^{-1} \theta(B)\theta(F)\sigma_a^2, \tag{2·11}$$

which is nonincreasing in σ_e^2 for $|B| = 1$ and $0 \leqslant \sigma_e^2 \leqslant \bar{\sigma}_e^2$. Thus the variance

$$\text{var}(\hat{T}_t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} C_{\hat{T}}(e^{-i\lambda}) \, d\lambda \tag{2·12}$$

is minimized when $\sigma_e^2 = \bar{\sigma}_e^2$.

Next consider the variance of the innovations of $\hat{T}_t$. Whether Z_t is stationary or nonstationary, (2·6) implies that the covariance generating function of $\hat{u}_t = \phi(B)\hat{T}_t$ is

$$C_{\hat{u}}(B) = \{\theta(B)\theta(F) - (\sigma_e^2/\sigma_a^2)\phi(B)\phi(F)\}^2 \{\theta(B)\theta(F)\}^{-1} \sigma_a^2. \tag{2·13}$$

Thus, the model for $\hat{T}_t$ can be written as

$$\phi(B)\theta(B)\hat{T}_t = \Psi(B)\alpha_t, \tag{2·14}$$

where $\sigma_\alpha^2 \Psi(B)\Psi(F) = \theta(B)\theta(F)C_{\hat{u}}(B)$, the α_t's are identically and independently distributed as $N(0, \sigma_\alpha^2)$, and $\Psi(B)$ is a polynomial in B of degree r^2 having its zeros lying on or outside of the unit circle. Now, the variance σ_α^2 of the innovations α_t's is

$$\log \sigma_\alpha^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log C_{\hat{u}}(e^{-i\lambda})\, d\lambda. \tag{2·15}$$

Since $C_{\hat{u}}(e^{-i\lambda})$ is nonincreasing in λ for $0 \leqslant \sigma_e^2 \leqslant \bar{\sigma}_e^2$, σ_α^2 is minimized when $\sigma_e^2 = \bar{\sigma}_e^2$.

Further, for the structure of $\Psi(B)$, it is easy to see from (2·13) and (2·14) that when $\sigma_e^2 = \bar{\sigma}_e^2$, $\Psi(e^{-i\lambda^*}) = 0$, so that the moving average polynomial $\Psi(B)$ in (2·14) has at least one zero on the unit circle.

3. SYMMETRIC SMOOTHING FILTERS

3·1. *General considerations*

Over the years, various symmetric moving average filters of the form

$$\delta(B) = \delta_0 + \sum_{j=1}^{\infty} \delta_j(B^j + B^{-j}) \tag{3·1}$$

have been proposed to 'smooth' an observed series Z_t in one way or another (Kendall & Stuart, 1966, p. 366; Whittaker & Robinson, 1944, p. 285). The basic purpose is to damp out the influence of the noise, or irregular fluctuations, in the data so as to discern the 'underlying trend'. Two basic questions may now be raised. (i) Since the use of a filter of the form (3·1) presupposes that Z_t contain nondeterministic components, for otherwise regression techniques would be appropriate, it is of interest to inquire about the existence of an underlying stochastic structure for which such a filter would be optimal. (ii) More important, one would want to know whether or not the use of specific filters is consistent with information verifiable from the data.

Clearly, if a given filter $\delta(B)$ can be written in the form $W(B)$ and satisfies the property (2·7), then the operation $\delta(B)Z_t$ is optimal in the sense that it gives the minimum mean squared error estimate of the component T_t with respect to the model (1·2) for Z_t. If, in addition, $\delta(B)$ satisfies (2·8), then the estimate $\delta(B)Z_t$ corresponds to the canonical decomposition. Further, if the model $\phi(B)Z_t = \theta(B)a_t$ is judged appropriate from the data, then the use of the filter $\delta(B)$ is consistent.

With the above considerations, we now turn to discuss several filters which have been employed in practice.

3·2. *The two-fold iterated moving average*

We first consider the filter

$$\delta(B) = m^{-2}\left\{m + \sum_{j=1}^{m-1} (m-j)(B^j + B^{-j})\right\}. \tag{3·2}$$

The special case $m = 3$ giving the weights $(\frac{1}{9}, \frac{2}{9}, \frac{3}{9}, \frac{2}{9}, \frac{1}{9})$ is a filter employed, for example, in the Census X-11 program to obtain a preliminary estimate of the seasonal component (Shiskin, Young & Musgrave, 1967). The filtering operation can be written as

$$\delta(B)Z_t = m^{-2}\frac{(1-B^m)(1-F^m)}{(1-B)(1-F)}Z_t, \tag{3·3}$$

which is a two-fold iterated moving average of Z_t. It is readily seen that $0 \leqslant \delta(e^{-i\lambda}) \leqslant 1$ with $\delta(1) = 1$ and $\delta(e^{-i2\pi/m}) = 0$. Thus, $\delta(B)$ can be expressed as

$$\delta(B) = 1 - \bar{\sigma}_e^2 \sigma_a^{-2} \phi(B)\phi(F), \tag{3·4}$$

and $\delta(B)Z_t$ corresponds to the canonical decomposition with respect to the model $\phi(B)Z_t = a_t$, where $\phi(B)$ is a polynomial of degree $m-1$. It can be verified that $\phi(B)$ must be of the form $\phi(B) = (1-B)\phi_1(B)$, where $\phi_1(B)$ is a polynomial of degree $m-2$ having all its zeros lying outside of the unit circle. For $m = 2$, 3 and 4, the specific models for Z_t and $\bar{\sigma}_e^2\sigma_a^{-2}$ are, respectively,

$$(1-B)Z_t = a_t, \quad \bar{\sigma}_e^2\sigma_a^{-2} = \tfrac{1}{4},$$
$$(1-B)(1+0{\cdot}268B)Z_t = a_t, \quad \bar{\sigma}_e^2\sigma_a^{-2} = \tfrac{1}{9},$$
$$(1-B)(1+0{\cdot}120B+0{\cdot}428B)Z_t = a_t, \quad \bar{\sigma}_e^2\sigma_a^{-2} = \tfrac{1}{16},$$

3·3. *Henderson's* 13-*term trend filter*

We next discuss Henderson's 13-term trend filter (Macauley, 1931), which is also employed in the Census X-11 seasonal adjustment program. The weights δ_j are given in Table 1. It can be verified that

$$\min_{0\leqslant\lambda\leqslant\pi} \delta(e^{-i\lambda}) = -0{\cdot}054,$$

so that the condition (2·7) is violated. Strictly speaking, we cannot interpret this filter in terms of the additive decomposition considered in this paper. However, a close approximation satisfying (2·7) can be given. Specifically, suppose the model for Z_t is

$$(1-B)^2(1+0{\cdot}978B+0{\cdot}604B^2+0{\cdot}256B^3+0{\cdot}063B^4)Z_t = a_t, \tag{3·5}$$

and suppose that $\sigma_e^2/\sigma_a^2 = 0{\cdot}3338$. Then the estimate $\hat{T}_t = W(B)Z_t$ is a 13-term moving average with weights w_j as given in Table 1. The weights w_j's are, for practical purposes, the same as the δ_j's. Thus the use of Henderson's filter would be approximately consistent with the model (3·5). Further, it can be verified that

$$\min_{0\leqslant\lambda\leqslant\pi} W(e^{-i\lambda}) = 0{\cdot}0002,$$

so that the filtering operation corresponds very nearly to the canonical decomposition.

Table 1. *Exact and approximate weights for Henderson's* 13-*term filter*

j	0	1	2	3	4	5	6
δ_j	0·240	0·214	0·147	0·066	0	−0·028	−0·019
w_j	0·261	0·214	0·140	0·059	0·001	−0·022	−0·021

3·4. *Smoothing splines*

As a final example, we consider the smoothing spline functions proposed by Schoenberg (1964). Given the series Z_t and an arbitrary positive constant ω, the problem is to find a function of time, $\hat{T}(t')$, having a square integrable mth derivative over an interval $I = [a, b]$ such that the quantity

$$f = \sum_{t=1}^{n}\{Z_t - \hat{T}(t)\}^2 + \omega\frac{1}{(m!)^2}\int_I \{\hat{T}^m(t')\}^2\,dt' \tag{3·6}$$

is minimized. The solution is a spline of order $2m$. In particular, writing $\hat{T}(t) = \hat{T}_t$ for integral values of t, then, given n data values $Z = (Z_1, \ldots, Z_n)'$,

$$\hat{T} = (I+\omega K_m' A_m^{-1} K_m)^{-1} Z \tag{3·7}$$

where $\hat{T} = (\hat{T}_1, \ldots, \hat{T}_n)'$, K_m is the $(n-m)\times n$ matrix such that $K_m\hat{T} = (\Delta^m\hat{T}_1, \ldots, \Delta^m\hat{T}_{n-m})'$ and A_m is an $(n-m)\times(n-m)$ positive-definite symmetric matrix. The elements a_{ij} of A_m

are obtained from the function $a_{ij} = M_{2m}(i-j)$, where

$$M_{2m}(x) = \frac{1}{(2m-1)!}(F-2+B)^m(x_+)^{2m-1}, \quad x_+ = \begin{cases} x & \text{if } x \geqslant 0, \\ 0 & \text{if } x < 0. \end{cases} \tag{3·8}$$

The nonzero values of a_{ij} for $m = 1, \ldots, 4$ are given in Table 2. We note that A_m is proportional to the covariance matrix of a moving average process of order $m-1$ obtained by Tiao (1972). Let $\beta_m \eta_{m-1}(B) \eta_{m-1}(F)$ be the generating function of the elements of A_m, where $\eta_0(B) = 1$ and $\eta_{m-1}(B) = (1-\eta_1 B - \ldots - \eta_{m-1} B^{m-1})$. For $m = 2, \ldots, 5$ the values of the η's and the β's can be found in that paper. In terms of the additive decomposition (1·1), it is readily seen that for t not close to the endpoints, $\hat{T}_t$ is optimal with respect to the model

$$(1-B)^m T_t = \eta_{m-1}(B)\, b_t \tag{3·9}$$

and $\omega = \beta_m \sigma_e^2 \sigma_b^{-2}$. Now $\hat{T}_t$ can be written as $\hat{T}_t = \delta(B) Z_t$, where

$$\delta(B) = 1 - \sigma_e^2 \sigma_a^{-2} \frac{(1-B)^m(1-F)^m}{\theta(B)\,\theta(F)}, \tag{3·10}$$

and where $(\omega\sigma_a^2 \sigma_e^{-2})\,\theta(B)\,\theta(F) = \beta_m \eta_{m-1}(B)\,\eta_{m-1}(F) + \omega(1-B)^m(1-F)^m$. This then implies that for a smoothing spline of order $2m$ to be consistent, the model for Z_t must of the form $(1-B)^m Z_t = \theta(B)\,a_t$, where $\theta(B) = (1-\theta_1 B - \ldots - \theta_m B^m)$.

Table 2. *Values of a_{ij} for $m = 1, \ldots, 4$*

m	$\lvert i-j\rvert = 0$	$\lvert i-j\rvert = 1$	$\lvert i-j\rvert = 2$	$\lvert i-j\rvert = 3$
1	1	.	.	.
2	4/6	1/6	.	.
3	66/120	26/120	1/120	.
4	2416/5040	1191/5040	120/5040	1/5040

Finally, one can readily verify that $\eta_{m-1}(e^{i\lambda})\,\eta_{m-1}(e^{-i\lambda}) > 0$, so that $\hat{T}_t$ does not correspond to the canonical decomposition.

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SPECTRAL ANALYSIS OF SEASONAL ADJUSTMENT PROCEDURES*

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This paper discusses one of the uses to which two powerful techniques of modern time series analysis may be put in economics: namely, the study of the precise effects of seasonal adjustment procedures on the characteristics of the series to which they are applied. Since most economic data appearing at intervals of less than a year are to a greater or lesser extent "manufactured" from more basic time series, the problem of assessing the effects of the "manufacturing" processes upon the essential characteristics of the raw material to which they are applied is not unimportant. Perhaps the most common type of adjustment applied to raw economic time series is that designed to eliminate so-called seasonal fluctuations. The precise nature of seasonality is not easy to define, but an attempt is made in Section 2.1 below.

The techniques employed to study the effects of seasonal adjustment procedures are those of *spectral* and *cross-spectral* analysis. In somewhat oversimplified terms the basic idea behind these types of analysis is that a stochastic time series may be decomposed into an infinite number of sine and cosine waves with infinitesimal random amplitudes. Spectral analysis deals with a single time series in terms of its frequency "content"; cross-spectral analysis deals with the relation between two time series in terms of their respective frequency "contents." The two techniques are discussed in both theoretical and practical terms.

Spectral analyses have been made for about seventy-five time series of United States employment, unemployment, labor force, and various categories thereof. Cross-spectral analyses have been made of the relations between these series and the corresponding series as seasonally adjusted by the procedures used by the Bureau of Labor Statistics. Two major conclusions regarding the effects of the BLS seasonal adjustment procedures emerge from these analyses. First, these procedures remove far more from the series to which they are applied than can properly be considered as seasonal. Second, if the relation between two seasonally adjusted series in time is compared with the corresponding relation between the original series in time, it is found that there is a distortion due to the process of seasonal adjustment itself. Both defects impair the usefulness of the seasonally adjusted series as indicators of economic conditions, but, of the two, temporal distortion is the more serious defect. Examples of some of these results are discussed below in Section 3.3.

1. THE NATURE OF SPECTRAL ANALYSIS

1.1. *Description of a Time Series in the Time Domain*

THE QUESTION of assessing the effects of a method of seasonal adjustment, or any other adjustment procedure, may be answered in terms of an accurate description of the original series, the modified series, and the relation between the two. In order to implement this approach, one must have a statistical theory of the description of cyclical time series. Interest in formal techniques of describing time series

* For author's acknowledgements and some additional references to the literature, see page 284.

goes back to the early years of this century. The periodogram and the correlogram are examples of techniques which were employed very early. Let $x(t)$ be a time series, with zero mean, observed at points in time $t=\ldots, -2, -1, 0, 1, 2, \ldots$. We generally suppose that $x(t)$ is actually a continuous function of time, although we observe it at discrete intervals. The *autocovariance* of $x(t)$, if it exists, is

$$(1.1) \qquad \gamma(t, \tau)=\mathscr{E}x(t)x(t+\tau) \qquad (\tau=0, \pm 1, \pm 2, \ldots).$$

In general, γ will be a function of t as well as of the lag τ. But if $x(t)$ is what is known as a covariance-stationary stochastic process, the autocovariance will be a function of the lag alone and not of the point in time at which x is measured.[1] For such a process, we may normalize the autocovariances by dividing each by the variance, $\gamma(0)$; the result is the *autocorrelation*. If $x(t)$ is covariance stationary, we may estimate the autocovariance for lag τ, $\gamma(\tau)$, by the mean lagged product

$$(1.2) \qquad C(\tau) = \frac{1}{2T} \sum_{t=-T+M}^{T-M} \{x(t-\tau)x(t)+x(t)x(t+\tau)\} ,$$

where we assume we have a finite number of observations $2T$ and where $\tau=0, 1, \ldots, M$, M being the maximal lag autocovariance computed.[2] The estimated autocorrelation for lag τ is then

$$(1.3) \qquad R(\tau) = \frac{C(\tau)}{C(0)} \qquad (\tau=1, \ldots, M).$$

A plot of $R(\tau)$ as a function of τ is called the *correlogram* of the time series. One way to describe a time series statistically is to compute and plot its correlogram or its estimated autocovariance function.[3]

An example of an estimated autocovariance function for an economic time series is graphed in Figure 1. The function for the weighted second difference of total United States unemployment, based on data for July, 1947, through December, 1961, is plotted as the solid line; the function for the same series seasonally adjusted is plotted as the dashed line. The weighted second differences of both series were of the form

[1] The term "covariance stationarity" is used by Parzen [18]. Doob [5] and others use such terms as "weak stationarity," "stationarity in the wide sense," or "second-order stationarity." Note that covariance stationarity implies that $\gamma(\tau) = \gamma(-\tau)$.

[2] Blackman and Tukey [2] suggest adjusting the divisor, 2T, for loss of degrees of freedom; unfortunately, although this procedure leads to unbiased estimates under general conditions, it appears to have unfortunate consequences in the estimation of power spectra. The estimate (1.2) is used to avoid these difficulties. This estimate was suggested orally by E. Parzen. The particular form of (1.2), based on numbering the observations from $-T$ to T rather than from 0 to T and using both positive and negative lags, is introduced in order to facilitate a comparison with the definitions of cross-lag-covariances presented in Section 3.2 below.

[3] Alternatively, of course, one might "describe" the time series by plotting the values $x(t)$ against time. Such a graph, called the *trace* of the series, does not involve abstracting any of the essential properties of the series and is not highly useful for the purpose here envisaged.

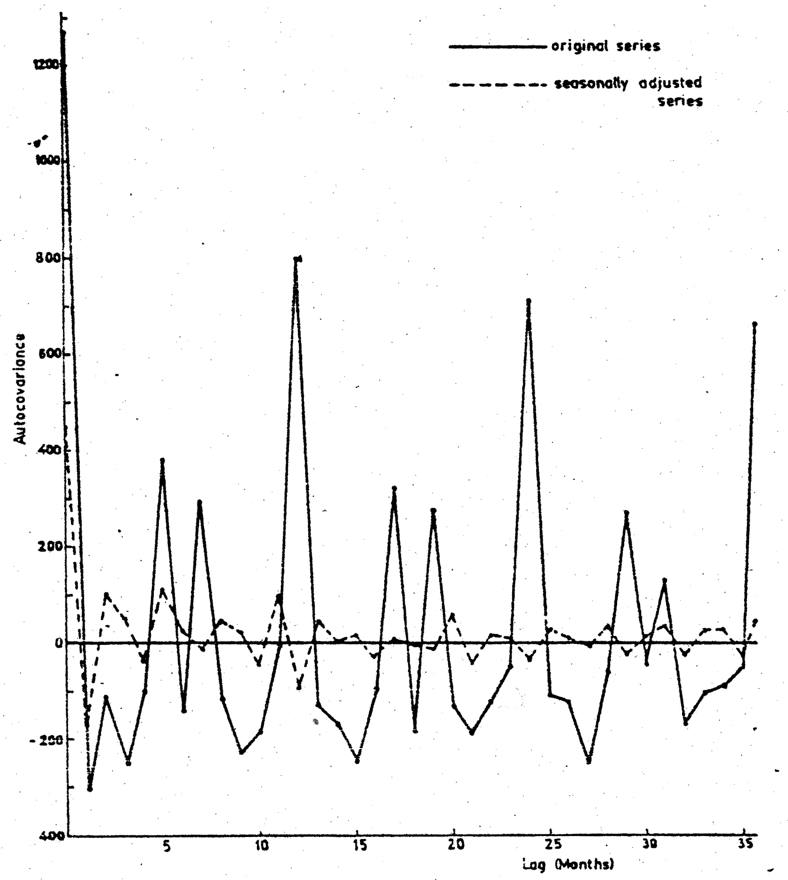

FIGURE 1

$$x(t)-1.5x(t-1)+0.5625x(t-2)\,. \tag{1.4}$$

The reasons for applying this particular differencing scheme are discussed more fully below. The sharp spikes at lags of 12 months, 24 months, and 36 months in the autocovariance function of the original series are due to seasonal effects, as are some of the other regularities. Some of the latter are also due to the differencing procedure which was applied. This is clearly not the place for an extensive discussion of the properties and uses of the autocovariance function of economic time series.[4] Suffice it to say that such a description is of considerable use if we wish to determine an autoregressive scheme or test for randomness; however, as will soon appear, it is not as useful in the analysis of adjustment procedures as an alternative description of the time series in terms of its frequency content.

[4] Some of the practical difficulties involved in interpreting autocovariance functions are illustrated in Kendall [11] and Quenouille [21]. Wold [27] is the classic work on the subject.

1.2. *Description of a Time Series in the Frequency Domain*

Let us begin by considering a simple deterministic function of time:

(1.5) $x(t) = A \sin \lambda t$.

Except for its origin in time (phase) this function is completely described by a pair of numbers, namely the amplitude A and the frequency λ. An alternative, but much more complicated, description of the same function is in terms of its *average lag product function*:[5]

$$
\begin{aligned}
(1.6) \quad & \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t)x(t+\tau)\,dt \\
&= \lim_{T\to\infty} \frac{A^2}{2T}\int_{-T}^{T} \sin\lambda t \sin\lambda(t+\tau)\,dt \\
&= \lim_{T\to\infty} \frac{A^2}{2T}\{\cos\lambda\tau\int_{-T}^{T}\sin^2\lambda t\,dt + \sin\lambda\tau\int_{-T}^{T}\sin\lambda t\cos\lambda t\,dt\} \\
&= \lim_{T\to\infty} \frac{A^2}{2\pi}\{\cos\lambda\tau\int_{-\pi}^{\pi}\sin^2 mt'\,dt' + \sin\lambda\tau\int_{-\pi}^{\pi}\sin mt'\cos mt'\,dt'\} \\
&= \frac{A^2}{2}\cos\lambda\tau ,
\end{aligned}
$$

where $t' = \pi t/T$ and $m = \lambda T/\pi$. The average lag product function is quite analogous to the autocovariance function introduced in the previous section. Of the two descriptions it is apparent that the one in terms of the pair (λ, A) is by far the simpler and more revealing.

The above considerations lead one to ask whether or not it might also be possible to describe stochastic time functions in such terms as frequency and amplitude. The answer is indeed yes; to see how and why, it is useful to take up a particular case of a Gaussian, or normal, time series.[6]

A time series, $x(t)$, is called Gaussian if for any n and every choice of times $t_1 < t_2 < \ldots < t_n$, the values of the function $x(t_1), \ldots, x(t_n)$ have an n-variate normal distribution. To simplify, let the means in this distribution all be zero. In this case, as is well known, the distribution, i.e., the probability law governing the process, is specified by the autocovariances, $\mathscr{E}x(t_i)x(t_k)$, $i, k = 1, \ldots, n$. If the process is stationary in addition,[7] these autocovariances will depend only on the time difference $\tau = t_i - t_k$. Thus zero-mean, stationary Gaussian time series have probability structures which are entirely described by their autocovariance functions $\gamma(\tau)$ defined earlier.

As an example of a zero-mean Gaussian random process consider the following:

[5] This might loosely be called an autocovariance function. To use such an expression here, however, would be improper as we are dealing with a non-stochastic function.

[6] The discussion which follows in the next three paragraphs is based in part on Goodman [8, pp. 221–23].

[7] For Gaussian processes, covariance stationarity implies strict stationarity. The latter, of course, implies the former in all cases.

Let $0 \leqslant \lambda_1 < \ldots < \lambda_N \leqslant \pi$ be N discrete frequencies, and let a_i and b_i, $i=1, \ldots, N$ be $2N$ independent, zero-mean, normal variates with common variance σ_i^2 for each i, i.e.,

$$(1.7) \qquad \mathscr{E}a_i a_k = \mathscr{E}b_i b_k = \begin{cases} \sigma_i^2 & \text{if} \quad i=k, i=1, \ldots, N \\ 0 & \text{if} \quad i \neq k \end{cases}$$
$$\mathscr{E}a_i b_k = 0, \qquad i, k=1, \ldots, N .$$

The time series

$$(1.8) \qquad x(t) = \sum_{i=1}^{N} (a_i \cos \lambda_i t + b_i \sin \lambda_i t)$$

is a zero-mean, stationary Gaussian process.

Since $x(t)$ is a linear combination of normal variates, it is itself normal. Indeed the variables $x(t_k)$, $k=1, \ldots, n$, have a multivariate normal distribution. The means are obviously zero. We have only to prove the process is stationary, i.e., to show $\mathscr{E}x(t)x(t+\tau)$ depends only on τ and not on t. Using (1.7), we find

$$(1.9) \qquad \mathscr{E}x(t)x(t+\tau) = \mathscr{E}\Big\{\sum_{i=1}^{N}\sum_{k=1}^{N} [a_i a_k \cos \lambda_i t \cos \lambda_k(t+\tau) +$$
$$a_i b_k \cos \lambda_i t \sin \lambda_k(t+\tau) + a_k b_i \cos \lambda_k(t+\tau) \sin \lambda_i t +$$
$$+ b_i b_k \sin \lambda_i t \sin \lambda_k(t+\tau)]\Big\}$$
$$= \sum_{i=1}^{N} \sigma_i^2 [\cos \lambda_i t \cos \lambda_i(t+\tau) + \sin \lambda_i t \sin \lambda_i(t+\tau)]$$
$$= \sum_{i=1}^{N} \sigma_i^2 \cos \lambda_i \tau .$$

Thus the process $x(t)$ is stationary with autocovariance function $\gamma(\tau)$ equal to the above.

The variance σ_i^2, as a function of frequency (i.e., of the index i), is called the *spectrum* of the time series $x(t)$. The autocovariance function, by (1.9) is just the Fourier cosine transform of the spectrum and, from the theory of such transforms, it follows that, conversely, the spectrum may be represented as

$$(1.10) \qquad \sigma_i^2 = \frac{1}{\pi}\int_{-\pi}^{\pi} \gamma(\tau) \cos \lambda_i \tau \, d\tau .^{8}$$

[8] Since

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \gamma(\tau) \cos \lambda_k \tau \, d\tau = \frac{1}{\pi}\int_{-\pi}^{\pi} \Big\{\sum_{k=1}^{N} \sigma_i^2 \cos \lambda_i \tau \cos \lambda_k \tau\Big\} d\tau$$
$$= \frac{1}{\pi}\sum_{k=1}^{N} \sigma_i^2 \Big\{\int_{-\pi}^{\pi} \cos \lambda_i \tau \cos \lambda_k \tau \, d\tau\Big\}$$
$$= \frac{1}{\pi}\sum_{k=1}^{N} \sigma_i^2 \begin{Bmatrix} \pi, & \text{for} \quad \lambda_i = \lambda_k \\ 0, & \text{for} \quad \lambda_i \neq \lambda_k \end{Bmatrix} = \sigma_i^2 .$$

Thus the spectrum and the autocovariance function are Fourier transform pairs; one may just as well consider one as the other.

Since the process $x(t)$ is assumed to be Gaussian either $\gamma(\tau)$ or σ_i^2, $i=1, \ldots, N$, gives us a complete description of the probability law of the process: the autocovariance function describes it in the time domain; the spectrum describes it in the frequency domain.[9] If the coefficients a_i and b_i in (1.8) had not been assumed to be normal, of course $x(t)$ would not be a Gaussian process; nevertheless, the spectrum and the autocovariance function are still descriptions of the process, albeit imperfect ones, and each contains exactly the same information as the other.

It is natural at this point to ask whether any stationary random process can be equally well described in frequency and time domains as above. The answer for stationary processes is again in the affirmative provided we do not restrict ourselves to finite parameter schemes such as (1.8). It can be shown that any covariance stationary random process $x(t)$ can be expressed in the form

$$(1.11) \qquad x(t) = \int_0^\infty \cos \lambda t \, dU(\lambda) + \int_0^\infty \sin \lambda t \, dV(\lambda) ,$$

where $dU(\lambda)$ and $dV(\lambda)$ are random variables with properties described below.[10] The reason for using a Stieltjes rather than an ordinary integral will become apparent in a moment. Regarding the integrals as sums and the variables $dU(\lambda)$ and $dV(\lambda)$ as amplitudes, however, we see that (1.11) shows that the time series $x(t)$ is "decomposed" into a superposition of sine and cosine waves of different frequencies and random amplitudes $dU(\lambda)$ and $dV(\lambda)$. No special significance should be attached to the various sinusoidal variations; they are merely components of a description of the time series $x(t)$ in the frequency domain.

The random variables $dU(\lambda)$ and $dV(\lambda)$, by means of which (1.11) expresses the stochastic time series $x(t)$, have particularly simple properties. First, they are orthogonal; that is, their covariances at different frequencies vanish:

$$(1.12) \qquad \mathscr{E} dU(\lambda) dU(\lambda') = \mathscr{E} dV(\lambda) dV(\lambda') = 0 , \quad \text{all} \quad \lambda \neq \lambda' ,$$

where "$\lambda \neq \lambda'$" means "λ and λ' belong to non-overlapping intervals." Second, the variables $dU(\lambda)$ and $dV(\lambda)$ are orthogonal at any frequencies:

$$(1.13) \qquad \mathscr{E} dU(\lambda) dV(\lambda') = 0 , \quad \text{all } \lambda \text{ and } \lambda' .$$

And third, the random variables $dU(\lambda)$ and $dV(\lambda)$ have a common variance at the same frequency:

$$(1.14) \qquad \mathscr{E}[dU(\lambda)]^2 = \mathscr{E}[dV(\lambda)]^2 = dF(\lambda) .$$

The function $dF(\lambda)$ is often called the *power spectrum* of $x(t)$, and $F(\lambda)$ the *cumulative*

[9] As Tukey [25] points out, estimation of the spectrum of a time series is analogous to components-of-variance analysis in more classical applications of statistics.

[10] See Doob [5, pp. 481–86].

power spectrum. $F(\lambda)$ is a monotonically increasing function which can be written in the form

(1.15) $$F(\lambda)=F_1(\lambda)+F_2(\lambda)+F_3(\lambda),$$

where F_1, F_2, and F_3 are nondecreasing functions of frequency. F_1 is an absolutely continuous function with derivative $f(\lambda)$; F_2 is a step function, and F_3 is the continuous singular component of F (i.e., a function which is constant almost everywhere in the mathematical sense).[11] In economic applications, the component F_3 may be neglected. The interpretation of the step function is simple in the case of the time series defined in (1.8); in this case, the cumulated power spectrum consists of a step function with steps of σ_i^2 at frequency λ_i. The function $F_2(\lambda)$ thus corresponds to a component of the time series which is perfectly periodic (although not for that reason perfectly deterministic).[12] The component $F_1(\lambda)$ is the most interesting from an economic point of view. It is absolutely continuous and we may write $dF_1(\lambda)/d\lambda=f(\lambda)$.[12a] It is, however, the possibility of components F_2 that leads us to write (1.11) in Stieltjes integral form.

The *spectral representation* of the time series $x(t)$ given by (1.11) is a representation in the frequency domain of the stochastic process. The great simplicity of the properties of the random variables $dU(\lambda)$ and $dV(\lambda)$ suggests that in many types of problems involving time series it will be more useful to consider these variables and their properties than to consider the statistical properties of the time series itself.[13] The Gaussian or normal time series defined earlier in this section has a particularly simple definition in terms of the spectral representation of a process. The time series $x(t)$ is Gaussian if and only if the random variables $dU(\lambda)$ and $dV(\lambda)$ are distributed as independent univariate normal variables with common variance at the same frequency. In this case, the probability law governing $x(t)$ is completely specified by specifying the power spectrum $dF(\lambda)$; i.e., the common variances of $dU(\lambda)$ and $dV(\lambda)$ at every frequency; or, alternatively, by specifying the autocovariance function $\gamma(\tau)$.

Note, as before, that the spectral representation of a time series suppresses all information concerning its origin in time, i.e., all *phase* information; however, if the process is stationary this information is irrelevant.

Since, in the case of a Gaussian process, either the power spectrum or the auto-

[11] Doob [5, p. 488].

[12] We argue below that it is most unlikely that any realistic economic time series will possess such a component.

[12a] Hannan [9, p. 12].

[13] For example, in regression problems involving serial correlation in the residuals, properties of various estimates may be better described in terms of the spectrum of the residuals than in terms of their autocovariance function. The point is that the spectral representation takes serial dependence as central, whereas in the time domain lack of independence is essentially a complication.

covariance function completely specifies the probability law, they both contain exactly the same information about the process. This is true irrespective of whether the process is Gaussian or not, since it may be shown that the power spectrum and the autocovariance function form a Fourier transform pair and are therefore uniquely related to one another.[14] This fact is simply derived making use of the properties (1.12–1.14) of the random variables $dU(\lambda)$ and $dV(\lambda)$:

$$(1.16)\quad \begin{aligned}\gamma(\tau) &= \mathscr{E}x(t)x(t+\tau)\\ &= \mathscr{E}\Big\{\int_0^\infty\int_0^\infty \cos\lambda t\cos\lambda'(t+\tau)\,dU(\lambda)\,dU(\lambda')\\ &\quad+\int_0^\infty\int_0^\infty \cos\lambda t\sin\lambda'(t+\tau)\,dU(\lambda)\,dV(\lambda')\\ &\quad+\int_0^\infty\int_0^\infty \cos\lambda'(t+\tau)\sin\lambda t\,dU(\lambda')\,dV(\lambda)\\ &\quad+\int_0^\infty\int_0^\infty \sin\lambda t\sin\lambda'(t+\tau)\,dV(\lambda)\,dV(\lambda')\Big\}\\ &= \int_0^\infty \{\cos\lambda t\cos\lambda(t+\tau)+\sin\lambda t\sin\lambda(t+\tau)\}\,dF(\lambda)\\ &= \int_0^\infty \cos\lambda\tau\,dF(\lambda)\,.\end{aligned}$$

A particularly important special case of (1.16) occurs when τ is set equal to zero; for in this case we have

$$(1.17)\quad \gamma(0) = \int_0^\infty (\cos 0)\,dF(\lambda) = \int_0^\infty dF(\lambda)\,,$$

where $\gamma(0)$ is just the variance of $x(t)$. *Thus the spectrum of a time series may be thought of as a decomposition of the variance at different frequencies.* Equation (1.17) expresses the variance of $x(t)$ as the "sum" of variances at different frequencies.

Equation (1.11) gives a representation of a real time series in terms of a superposition of cosines and sines, real numbers, with random amplitudes, also real numbers. The entire expression may be represented in complex form more compactly. The complex form is also convenient for other reasons, as will become clear. As we know

$$e^{j\lambda t} = \cos\lambda t + j\sin\lambda t\,,$$

where $j=\sqrt{-1}$. Thus

$$\cos\lambda t = \frac{e^{j\lambda t}+e^{-j\lambda t}}{2}\,,$$

$$\sin\lambda t = \frac{e^{j\lambda t}-e^{-j\lambda t}}{2j}\,.$$

[14] Doob [5, pp. 473–76]; Wilks [26, pp. 517–21].

Substituting in (1.11)

$$(1.18)\quad x(t) = \int_0^\infty \left(\frac{e^{j\lambda t}+e^{-j\lambda t}}{2}\right) dU(\lambda) + \int_0^\infty \left(\frac{e^{j\lambda t}-e^{-j\lambda t}}{2j}\right) dV(\lambda)$$

$$= \int_{-\infty}^{\infty} \tfrac{1}{2}e^{j\lambda t} dU(\lambda) - \int_{-\infty}^{\infty} \tfrac{1}{2}e^{j\lambda t} j dV(\lambda) = \int_{-\infty}^{\infty} e^{j\lambda t} dZ(\lambda),$$

where $dZ(\lambda)$ is the complex random variable

$$(1.19)\quad dZ(\lambda) = \tfrac{1}{2}[dU(\lambda) - jdV(\lambda)],$$

and where we define $dU(\lambda)$ and $dV(\lambda)$ for negative frequencies by

$$(1.20)\quad \begin{aligned} dU(\lambda) &= dU(-\lambda), \\ -dV(\lambda) &= dV(-\lambda). \end{aligned}$$

If the variance of a complex random variable is defined as the expected value of its squared modulus we have from (1.14):

$$(1.21)\quad \mathscr{E}\, dZ(\lambda)\overline{dZ(\lambda)} = \tfrac{1}{2}dF(\lambda),$$

where the bar denotes the complex conjugate. There are two possible covariances for a complex random variable, $\mathscr{E}\, dZ(\lambda)\overline{dZ(\lambda')}$ and $\mathscr{E}\, \overline{dZ(\lambda)}dZ(\lambda')$, but by (1.12) and (1.13) these are both zero.

It is now possible to write the autocovariance of $x(t)$ in most suggestive form: From (1.16)

$$(1.22)\quad \gamma(\tau) = \int_0^\infty \cos \lambda\tau\, dF(\lambda) = \int_0^\infty \frac{e^{j\lambda\tau}+e^{-j\lambda\tau}}{2} dF(\lambda) = \int_{-\infty}^{\infty} e^{j\lambda\tau} \frac{dF(\lambda)}{2},$$

since by (1.20) and (1.21) $dF(-\lambda) = \overline{dF(\lambda)} = dF(\lambda)$. Thus the autocovariance function is just the complex Fourier transform of the spectrum. By the theory of such transforms,[15] the spectrum in turn is

$$(1.23)\quad dF(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} e^{-j\lambda\tau}\gamma(\tau)\, d\tau.$$

Thus the autocovariance function and the power spectrum are just a Fourier transform pair; knowledge of one is equivalent to knowledge of the other. The values of the spectrum (as estimated) will, however, normally have simpler statistical properties since they are essentially variances of normal independent variables in the case of Gaussian processes.

It should be noted that real stationary time series have a spectrum which is symmetric about the origin. For such time series it is possible to prove an important theorem: a necessary and sufficient condition that the spectrum of a time series equal a constant, i.e., be "flat," is that all autocovariances except that for $\tau = 0$ be

[15] Stuart [24, p. 46].

zero.[16] Thus if successive values of $x(t)$ are independent, all the autocovariances except at $\tau=0$ will be zero, and the spectrum of $x(t)$ will be flat. Conversely, if $x(t)$ is a Gaussian time series and has a flat spectrum then successive values of $x(t)$ are independent. Stationary time series with flat spectra are called *white noise.*

1.3. *Practical Problems in the Estimation of Spectra*

One problem in practical spectral analysis of economic time series is what has been called *aliasing*.[17] Economic time series, though they may be continuous in theory, are in fact observed only at discrete intervals of time. This means that harmonic components of the series with periods shorter than twice the period of observation, or with frequencies greater in absolute value than ½ cycles per period, cannot be directly discerned. Such harmonic components will, however, be reflected in the estimated power spectrum, since the effects of such components will be combined with the effects of harmonics of lower frequencies which are observed directly. The rationale behind this observation is illustrated in Figure 2. The solid

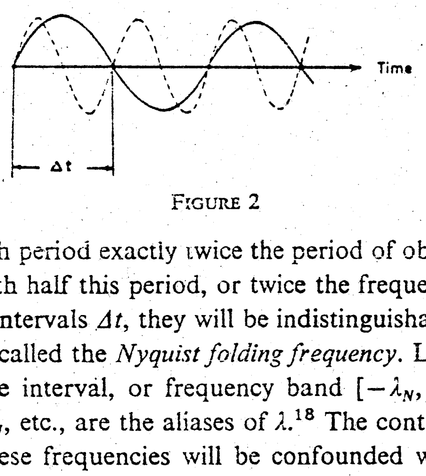

FIGURE 2

line is a sine wave with period exactly twice the period of observation; the dashed line is a sine wave with half this period, or twice the frequency; if the two waves are observed only at intervals Δt, they will be indistinguishable. The frequency of ½ cycles per period is called the *Nyquist folding frequency*. Let this be λ_N and let λ be a frequency in the interval, or frequency band $[-\lambda_N, \lambda_N]$; then frequencies $\lambda \pm \lambda_N$, $\lambda \pm 2\lambda_N$, $\lambda \pm 3\lambda_N$, etc., are the aliases of λ.[18] The contributions of harmonic components at all these frequencies will be confounded with the effects of the harmonic component of frequency λ. Thus, in the interpretation of a spectral analysis of an economic time series, one must be careful not to identify the effects at an observed frequency necessarily with that frequency alone. However, in the analyses reported below, aliasing does not appear to have been a severe problem.[19]

[16] See Wilks [26, p. 521].

[17] Blackman and Tukey [2, pp. 31–33, 117–20].

[18] From this point onward in this section, frequency will be expressed in terms of cycles per unit time rather than in fractions of 2π.

[19] The practical significance of aliasing is that peaks in the power spectrum at frequencies less than ½ cycles per unit time may be due to causes operating well above the observable frequency range. The only criterion one can use to decide whether aliasing is a problem is whether observed peaks have plausible explanations; if not, one may have to seek an explanation at higher frequencies. Since all peaks in the analyses reported below had one simple explanation, seasonality, there appeared to be no problem.

A second problem in the analysis of economic time series is the severe limitation on the number of observations available. When I first began to explore the possibilities of using spectral techniques in economics, I had in mind that they might be applied in the study of livestock cycles. Livestock inventory figures are available, for example, annually since 1867. Thus somewhat under 100 observations are available for analysis, which would seem to be a large number. However, a complete cattle cycle takes between 13 and 17 years, which means that less than seven such cycles have been observed. The effective number of observations, then, for cycles of this length is, at most, seven, a small number indeed for any sort of statistical analysis! It follows that, in the study of economic time series, the power of discrimination at low frequencies is severely limited by the length of the record. Note that this is true regardless of how short the unit observation period; while we can increase the maximum observable frequency by reducing the interval between observations, we cannot improve discrimination at low frequencies by such means. Not all fluctuations in economic time series of interest, however, are of such long duration. In particular, most of the operations performed on quarterly, monthly, or weekly economic time series are designed to remove high frequency components and to make these series more usuful for short term economic forecasts or for assessment of the current economic situation. Whether or not existing procedures perform this function adequately is a problem of some importance. Furthermore, it is a problem in which high frequency components are of special interest, and, therefore, in which the short length of most economic time series does not pose a severe handicap on the use of spectral techniques.

Suppose that we have $2T$ observations on a particular time series available. How do we go about estimating the power spectrum of the series? First note that if $2T$ is the total number of observations, autocovariances of, at most, order $M=2T-1$ can be computed. Indeed, we shall generally wish to make M much less than $2T$ in order to increase the statistical reliability of the estimated autocovariances of high order. The general rule of thumb that has been adopted in the analyses reported below is that the maximal lag should be no more than about 20 per cent of the total number of observations available. Suppose, then, that we have computed M autocovariances according to 1.2; the next step is to make use of the discrete analogue of equation (1.23) above. This turns out to be

$$(1.23') \quad I_M(\lambda_i) = \frac{1}{\pi}[c(0)+2 \sum_{\tau=1}^{M} \cos 2\pi\lambda_i\tau c(\tau)],$$

where $\lambda_i=i/2M$, $i=0, 1, \ldots, M$.[20] Note that we can measure the power only at

[20] Since

$$\sum_{\tau=-M}^{M} e^{-j2\pi\lambda_i\tau}c(\tau) = \sum_{\tau=1}^{M} (\cos 2\pi\lambda_i\tau - j \sin 2\pi\lambda_i\tau)c(\tau)$$
$$+ \sum_{\tau=-M}^{1} (\cos 2\pi\lambda_i\tau - j \sin 2\pi\lambda_i\tau)c(\tau) + c(0) = c(0) + 2 \sum_{\tau=1}^{M} \cos 2\pi\lambda_i\tau c(\tau),$$

since $\sin i\pi = 0$ and $\sin(-x) = -\sin x$. It is usual to make the end correction which gives $c(M)$ half the weight of $c(\tau)$, $0 < \tau < M$.

certain frequencies equal in number to the number of lags which have been chosen. Since all economic time series are real, their spectra will be symmetric about the origin; hence, we need only consider positive frequencies. When the frequencies are equally spaced, as they are in (1.23′), the expression is just the classical periodogram due to Schuster.[21]

The classical periodogram, as described in the preceding paragraph, is the technique one would normally use for investigating the harmonics of fixed, known frequencies under the assumption that the time series under consideration is strictly periodic, repeating itself exactly every M periods. If this were the case, the true cumulative power spectrum would be a pure step function; i.e., we would have only F_2 in the decomposition (1.15). If the time series is not strictly periodic, however, it will have a continuous power spectrum. The most that one can hope to do with a finite number of observations is to measure the average power, in some sense, in a band about the frequencies λ_i. These averages may be defined in a variety of ways by specifying weights in either the time domain or the frequency domain. As the number of observations, and hence the number of lags, is increased, it would be possible to reduce the width of the band and so obtain finer discrimination of frequencies and better measurement of the power at each frequency. It can be shown, however, that as the width of the frequency band over which average power is measured is reduced, the variance of any estimate of this power is increased, *no matter how many observations are taken*.[22] The limiting case where one attempts to measure power at a particular frequency is precisely the classical periodogram analysis applied to time series which are not strictly periodic, and no series of relevance in economics is strictly periodic.

Although it can be shown that the periodogram ordinates are asymptotically unbiased estimators of the power spectrum at corresponding frequencies for non-strictly periodic time series, it can also be shown that they are not consistent estimators; indeed the variance of the periodogram ordinates tends to $[\gamma(0)f(\lambda_i)]^2$, where, assuming $F_2(\lambda)\equiv 0$, $dF(\lambda_i)/d\lambda = f(\lambda_i)$.[23] It is this fact, for example, which in part accounts for the spiked appearance of a periodogram estimated from a finite sample of white noise (which should have a flat spectrum). Consequently, use of the classical periodogram in economics is not only inappropriate but will lead to results which are nearly impossible to interpret. Perhaps it is this that has led to the disrepute into which harmonic analysis of economic time series has fallen.

Rather, then, than estimate the power spectrum at a specific frequency, it is necessary, in order to achieve results which are statistically consistent, to estimate average power about the frequency in question. Naturally, we would like to estimate an average which reflects most heavily the power at frequencies near the one at which we center the estimate. Thus, we consider weighted averages of power in

21 Schuster [23]

22 Parzen [17, p. 180].

23 Jenkins [10, p. 150]; Hannan [9, pp. 52–55].

the frequency domain; these averages correspond to weighting the raw autocovariances of the time series before computing the spectral estimates. In the frequency domain a particular weighting scheme is called a *spectral window*; the inverse Fourier transform of a spectral window is a weighting scheme in the time domain for the raw autocovariances and is called a *lag window*. Ideally, one would like to choose a spectral window that would weight power equally at all frequencies $\pm\frac{1}{2}$ the distance from the chosen frequency point to the next frequency point (assuming the centers of our averages to be equally spaced) and give power at frequencies outside this interval a zero weight. Unfortunately, it is mathematically impossible to find a finite lag window corresponding to this spectral window. Consequently, a great deal of the literature of spectral analysis deals with alternative choices of spectral and lag windows and with the statistical properties of the resulting estimates.[24]

It would serve no useful function in this paper to go into the technical details of different sorts of windows and their statistical properties. Suffice it to say that in the analyses reported below a window recommended by Parzen has been used. This corresponds to forming weighted averages of the raw autocovariances

$$(1.24) \quad C(\tau) = \sum_{k=1}^{M} w(k)c(k),$$

where the weights are given by

$$(1.25) \quad w(k) = \begin{cases} 1 - 6\dfrac{k^2}{M^2}\left(1 - \dfrac{k}{M}\right), & 0 \leqslant k \leqslant \dfrac{M}{2}, \\ 2\left(1 - \dfrac{k}{M}\right)^3, & \dfrac{M}{2} \leqslant k \leqslant M, \\ 0, & M < k. \end{cases}$$

M is the maximal lag. Alternatively, in the frequency domain, the lag window (1.25) corresponds to the spectral window

$$(1.26) \quad A(\omega) = \frac{3}{4\pi M^3}\left[\frac{\sin 2\pi \dfrac{M(\lambda-\omega)}{4}}{\sin 2\pi \dfrac{(\lambda-\omega)}{4}}\right]^4 .$$[25]

Thus, the estimates obtained are of spectral averages about a series of designated

[24] Blackman and Tukey [2]; Jenkins [10]; Parzen [17].

[25] Parzen [17, p. 176, eq. 5.10]. One of the reasons for using Parzen's window is that it is everywhere positive. Another window, due to Tukey, has small negative side lobes. Thus, the high power at low frequencies present in most economic time series can actually produce *negative* estimates of the spectrum at higher frequencies. Since the spectrum is essentially a variance, negative estimates are not easy to interpret. See the discussion of leakage below.

frequencies λ; as one can see from the form of the window (1.26), they are averages of the power at all frequencies, but the main weight of the average is concentrated in the close vicinity of the designated frequency. The window is symmetric about the frequency λ. A graph of this function for $M=36$ and positive values of $\lambda-\omega$ is plotted in Figure 3. Fewer lags would have produced a window less sharply focused on the designated frequency; more lags, a window of lesser width.

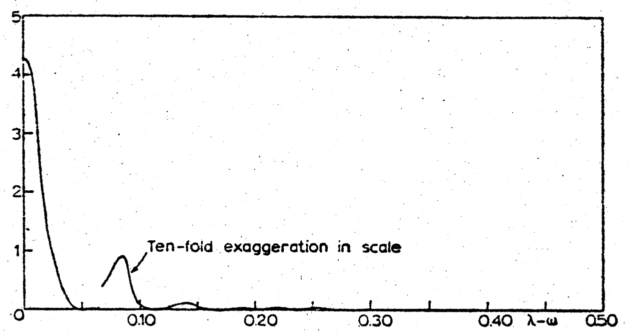

FIGURE 3

The fact that only spectral averages have the statistical property of consistency, and that, therefore, we must look at the spectrum of a time series through a window, has important consequences for the estimation of spectra. The issue is closely connected with the problem of the non-stationarity of many economic time series. The window through which we look at a spectrum does concentrate its main weight very near a designated frequency, but not all the weight is concentrated there; there is some weight, however small, at all other frequencies. This means that very high power at some frequencies will distort spectral estimates at other frequencies, some of them distant from those at which high power is present. This phenomenon might be called "leakage through the edges of the window." Two important forms of non-stationarity in time series that are Gaussian, or nearly so, are: (a) a variance that changes with time, and (b) a mean that changes with time. Changing variance has not appeared to be a serious problem in those time series studied here.[26] Changes in the mean with time are better known under the designation "trends." In any finite realization of a process, trends will be indistinguishable from very low frequency components. Indeed, the sample mean itself may be regarded as a cycle of zero frequency or infinite period. Thus, since most

[26] Various devices, such as dividing the observations by a moving sample variance, could be employed if variance non-stationarity appeared to be present. Over short periods of under 200 observations, however, it would be difficult to detect changes in variance. Changing variance may present problems in longer economic time series.

Other, more complicated, forms of non-stationarity are, of course, possible, e.g., trending autocovariances. However, it is most improbable that such forms would occur without a trending variance.

economic time series do show trends of one sort or another, the power spectrum of a typical economic time series will show very high power concentrated at frequencies near zero, and gradually diminishing power at higher frequencies. This shape is illustrated in Figure 4 by the power spectrum of the Federal Reserve Board Index of Industrial Production.[27]

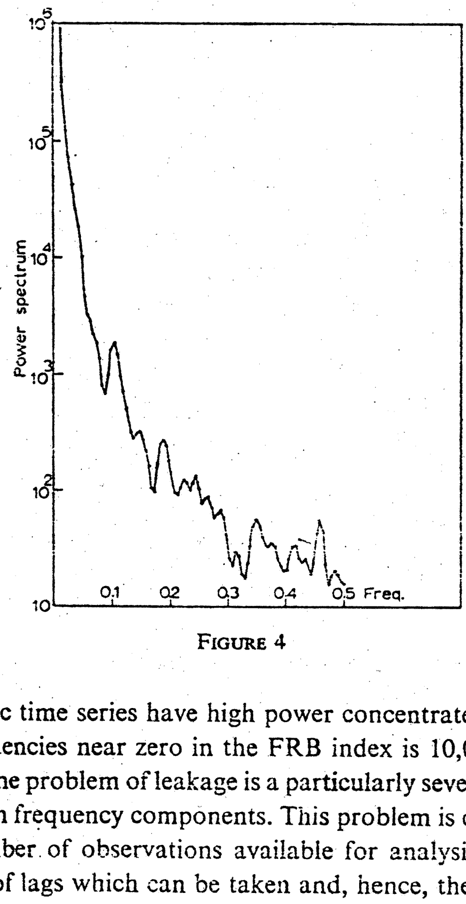

FIGURE 4

Because economic time series have high power concentrated at low frequencies (the power at frequencies near zero in the FRB index is 10,000 times as great as the power near ½), the problem of leakage is a particularly severe one in any attempt to analyse their high frequency components. This problem is compounded because of the limited number of observations available for analysis, for this limitation limits the number of lags which can be taken and, hence, the degree to which the width of the window may be reduced. To circumvent this difficulty, it is necessary to "filter" out some of the power at low frequencies. In communications or electrical engineering, in connection with which much of the theory of spectral analysis has been developed, filters are often actual physical entities. For our purposes, however, a filter may be defined as any series of arithmetical operations

[27] The estimates were computed on the basis of 486 monthly observations. These were differenced, for reasons to be described. An estimate of the power spectrum for the undifferenced series was recovered from an estimate of that of the differenced series by a procedure called "recoloring." The seasonally adjusted FRB Index of Industrial Production is an unusually smooth series which accounts for the very great concentration of power at the low end of the frequency scale.

used to transform the data prior to its analysis. The use of such operations to filter out power at low frequencies is called *prewhitening*.[28]

Many sorts of arithmetical operations could be employed to remove low frequency components; for example: (a) fit a polynomial in time of low degree to the data and subtract the appropriate calculated value from each observation; (b) subtract a weighted or unweighted moving average from each observation, or (c) difference the series. Filters of types (b) or (c) are all of the general form

$$(1.27)\qquad y(t)=\sum_{i=-p}^{q}\delta_i x(t+i)\,,$$

where the δ_i are constants independent of time. The series $x(t)$ is the original, or *input*, series; $y(t)$ is the transformed, or *output*, series. For example, for ordinary first differences of the input, we have $\delta_0=1$, $\delta_{-1}=-1$, and $\delta_i=0$ for all $i\neq 0,-1$; for a centered, three-term, unweighted, moving average, we have $\delta_1=\frac{1}{3}$, $\delta_0=\frac{1}{3}$, $\delta_{-1}=\frac{1}{3}$, $\delta_i=0$, $i\neq 1,\ 0,\ -1$. Filters of the general form given in (1.27) are called time-invariant *linear filters*. Subtraction of a polynominal trend, determined by a regression, from the original series is a non-time invariant filter.[29] Any one of the standard methods of seasonal adjustment is a nonlinear filter, and usually not time invariant as well. The importance of linear filters in spectral analysis arises because the spectrum of the output of a linear filter is very simply related to the spectrum of the input series. Let u be the shift operator such that

$$(1.28)\qquad \begin{aligned} ux(t)&=x(t+1)\\ u^i x(t)&=x(t+i)\,. \end{aligned}$$

Then (1.27) may be written as

$$(1.29)\qquad y(t)=\sum_{i=-p}^{q}\delta_i u^i x(t)\,.$$

The frequency response function of a linear filter, which is discussed at greater length below, is obtained by setting the input equal to $e^{-j2\pi 0}$ (recall that $j=\sqrt{-1}$):

$$(1.30)\qquad l(\lambda)=\sum_{i=-p}^{q}\delta_i e^{-j2\pi\lambda i}.\ ^{30}$$

The frequency response function is generally a complex-valued function of frequency; the square of its modulus at each frequency is called the transfer function of the filter and denoted by $|l(\lambda)|^2$. It can be shown that the spectrum of an output from a linear filter, $g(\lambda)$, is related to the spectrum of the input, $f(\lambda)$, by

$$(1.31)\qquad g(\lambda)=|l(\lambda)|^2 f(\lambda)\,.^{31}$$

[28] Blackman and Tukey [2, pp. 39–43].

[29] A fitted *moving* polynomial would fall into the category of time invariant linear filters, since the "time" in the polynomial is then essentially an index different from real time.

[30] Lanning and Battin [13, pp. 192–93]; see also Section 3.2 below.

[31] Bartlett [1, p. 174].

Thus, an estimate of the power spectrum of a series which has been prewhitened by application of a linear filter can be obtained from the estimate of the power spectrum of the prewhitened series by dividing the latter, at each frequency, by the transfer function of the filter. Although the term does not appear in general usage, I have called such a procedure "recoloring", and the resulting estimate the *recolored power spectrum.*

To minimize the effects of leakage through the edges of a spectral window, one would ideally like to prewhiten the series in such a way that the power spectrum of the result was as flat as possible. Of course, if one could do this perfectly, it would be tantamount to perfect knowledge of the spectrum of the original series and so quite unnecessary. Some leakage effects are thus unavoidable. As far as economic time series are concerned, it would seem that the best practice is to remove considerable power only at the lowest frequencies. However, care must be taken not to remove so much that much higher power occurs at high frequencies than at low, for then leakage of this power would seriously distort our estimates at the lower frequencies. In order to eliminate trends and consequent high power at low frequencies, repeated differences of the form

(1.32) $\Delta_k x(t) = x(t) - k\,x(t-1)$

were employed.[32] In the case $k=1$, the effect is to give first-, second-, or higher-order differences as the operation defined in (1.32) is repeated. When, however, ordinary first or second differences were applied to the series discussed below, it was found that too much power was removed near the zero frequency. The resulting estimated spectra tended to have high power concentrated in the upper frequencies and relatively little at lower frequencies, which meant that estimates at some of the lower frequencies of interest were likely to have been distorted. For want of a better term, I call differences of the form (1.32) "quasi-differences" when $k \neq 1$.

This result appeared to dictate a choice of k different from one. But what other value? In particular, should k be less or greater than one? It can be shown that the transfer function for the pth quasi-difference is

(1.33) $|l(\lambda)|^2 = [1 - 2k \cos 2\pi\lambda + k^2]^p$.[33]

[32] Durbin [6] gives several reasons for preferring difference filters to moving averages or trend elimination by the subtraction of a polynomial in time.

[33] By equation (1.28) the first-order quasi-difference may be written as $(1 - ku^{-1})x(t)$. Hence, the pth order quasi-difference of this form is $(1 - ku^{-1})^p x(t)$ which may be expanded as

(i) $$(1 - ku^{-1})^p x(t) = \sum_{i=0}^{p} \binom{p}{i} (-k)^i u^{-i} x(t).$$

Inserting this result in (1.30), we obtain

(ii) $$l(\lambda) = \sum_{i=0}^{p} \binom{p}{i} (-k)^i e^{j2\pi\lambda i} = \sum_{i=0}^{p} \binom{p}{i} (-k)^i [e^{j2\pi\lambda}]^i = [1 - ke^{j2\pi\lambda}]^p.$$

Thus

(iii) $$|l(\lambda)|^2 = [1 - ke^{-j2\pi\lambda}]^p [1 - ke^{j2\pi\lambda}]^p = [(1 - ke^{-j2\pi\lambda})(1 - ke^{j2\pi\lambda})]^p$$
$$= [1 - k(e^{j2\pi\lambda} + e^{-j2\pi\lambda}) + k^2]^p = [1 - 2k \cos 2\pi\lambda + k^2]^p.$$

At frequencies near zero this function is approximately $(1-k)^{2p}$; and, at frequencies near $\frac{1}{2}$, it is approximately $(1+k)^{2p}$. It follows that by making k less than one we shall raise power at higher frequencies less and also lower power at lower frequencies by a smaller amount. Beyond this observation, however, there seems to be no very good method of determining k.[34] Hence, it was decided to try a number of different values of k and p and to choose that combination yielding the flattest estimated spectrum. In all the analyses performed $k=0.75$ and $p=1, 2$, or 3 proved to give satisfactory results. Thus the procedure employed amounted to setting $k=0.75$, computing power spectra of the series differenced one, two, or three times according to (1.32), selecting the flattest spectrum, then recoloring the estimates at each frequency by dividing by the transfer function at that frequency for the appropriate value of p.[35]

2. SEASONALITY AND SEASONAL ADJUSTMENT

2.1. *The Nature of Seasonality*

When one examines a number of economic time series observed at quarterly, monthly, or even weekly intervals, one is particularly struck by the highly regular within-year movements in many series. For example, the regular swing of the monthly time series of U.S. employment from a low point in midwinter to a peak in midsummer is one of the most persistent and characteristic features of this

[34] In an oral discussion, Parzen suggested prewhitening by means of autoregressive schemes estimated by least squares. This would mean estimating the appropriate values for δ_i in (1.27) directly. In some ways this technique would appear to produce results similar to those obtained by the procedure actually employed. Unfortunately it has not been possible to date to explore the potentialities of Parzen's suggestion.

[35] I am indebted to H. Rosenblatt for pointing out that the suggestion for using a quasi-difference, rather than an ordinary first difference, to avoid difficulty at zero frequency occurs in Blackman and Tukey [2, pp. 52–3], where a value, $k = 0.6$, is given. What seems to be novel in the present approach is the use of higher order differences and iteration until the estimated spectrum becomes flat.

It may be that this approach has wider implications. For example, in the estimation of seasonal factors it is necessary first to remove trend. This is usually accomplished by deducting a moving average of the series from itself. When such a procedure is followed, however, it is not immediately clear that all the trend has been removed. (It is this difficulty which in the BLS method leads to the iterations there employed.) If a time series is regarded as trend-cycle plus seasonal plus white noise, then any filter which reduces the series to white noise plus seasonal (see below for a definition of seasonal) clearly removes the undesired trend-cycle. What more natural way is there to determine this point empirically?

Durbin has pointed out that quasi-differencing amounts to removal, over a finite period, of an exponentially damped polynomial and that, therefore, the beginning and end of the series are treated asymmetrically. For this reason, he suggests $(1 - ku^{-1})(1 - ku)x(t) = -kx(t+1) + (1+k^2)x(t) - kx(t-1)$, which is a more symmetrical filter. I have not had an opportunity to explore this suggestion.

series.[36] Almost all economic time series contain such regular movements to a greater or lesser degree. These movements are perhaps most prominent in agricultural series, less so in some financial series, but surprisingly widespread in series covering every facet of economic life.[37] Their very regularity suggests that, in the main, they are due to the round of the climatic seasons, conventional holidays, and repetitive institutional practices, if not directly affecting the particular series under consideration, then indirectly through other related economic phenomena. However, these movements, which we call seasonal, are not perfectly regular, i.e., do not repeat themselves exactly from year to year. The pattern of weather is never exactly the same from one year to another; the dating of conventional holidays, the number of weekends in a month, institutional conventions and economic relationships change, sometimes gradually and systematically, sometimes abruptly and irregularly.

It is the close but not perfect approach to regularity which renders the precise definition of seasonality so difficult. If there were no regularity at all, there would be no problem; on the other hand, if such movements were perfectly regular, seasonality could be defined as precisely those regularities. The notion of the power spectrum of a time series enables us to give a more precise definition of seasonality even in the presence of irregularity and systematic change.

Suppose first that some part of the time series is perfectly periodic within a one year period. The strict periodicity need not rule out a stochastic series, however. Such a series may be represented by a finite sum of sine and cosine terms with random amplitudes. If the fundamental period of the series is one year and monthly observations (1/12 of the fundamental period) are taken, it suffices to include precisely eleven such terms in addition to a constant representing the mean of the series. The representation is thus

$$(2.1) \qquad y(t)=\alpha_0+\sum_{i=1}^{6}(\alpha_i \cos 2\pi\lambda_i t+\beta_i \sin 2\pi\lambda_i t),$$

where $\lambda_i=i/12$. Note that since $\sin(2\pi\cdot 6/12)t=0$ for all t, the last sine term in the sum vanishes identically, so that there are only eleven α's and β's in addition to the constant term required to determine a particular annual pattern.[38] The frequencies λ_i and the corresponding periods of their cycles are shown in the accompanying table. The frequencies λ_i listed below may be called *seasonal frequencies*.[39]

[36] [19, p. 162].

[37] Kuznets [12]; Moore [14, pp. 517–51]; [16, pp. 185–92].

[38] That is, β_6 is perfectly arbitrary.

[39] It is rather important not to attach any particular meaning to the individual seasonal frequencies. All six frequencies are required to represent an arbitrary twelve month pattern as the sum of sinesoidal variations. If fewer are required or if, for example, the variance of α_i and β_i is much larger for one particular frequency than for all the others, it simply means that the twelve month pattern is quite like a sine wave of that frequency.

(1) Frequency (cycles/month)	(2) Period (months)	(3) Number of times a cycle is completed in one year
0.0833	12	1
0.1667	6	2
0.2500	4	3
0.3333	3	4
0.4167	2.4	5
0.5000	2	6

Suppose that $\alpha_1, \ldots, \alpha_6$ and $\beta_1, \ldots, \beta_6$ are independent random variables with zero means and variances:

$$\mathscr{E}\alpha_i^2 = \mathscr{E}\beta_i^2 = \sigma_i^2 \qquad (i=1, \ldots, 6)\,. \tag{2.2}$$

The constant α_0 may be any sort of stationary process so long as it is independent of α_i and β_i, $i=1, \ldots, 6$. Then, as we saw in Section 1.2, the cumulative spectrum of the term in the summation in (2.1) consists of a series of jumps at the seasonal frequencies λ_i equal in height to the variances σ_i^2. The spectrum of the process $y(t)$ is thus the superposition of six spectral "lines" on the spectrum of α_0, where the "lines" are such that the integrated power spectrum has jumps of σ_i^2 at the seasonal frequencies λ_i.[40] In this simple and rather unrealistic case a precise definition of seasonality can be given: it is just that part of the series which gives rise to the spectral lines at the seasonal frequencies. Note that the seasonal part is regular without being deterministic, i.e., nonstochastic.

It is doubtful whether seasonal variations found in economic time series, however, conform to the simple model outlined above. Instead we must expect that the amplitudes and phases of the harmonic terms in (2.1) will be subject to some systematic as well as random changes over time. We may immediately dispose of systematic phase changes since they may always be translated into corresponding changes in the amplitudes of the sine and cosine terms at the same frequency. Let us consider, then, what effect systematic, but still stochastic, changes in amplitude and perfectly random phase shifts will have on the spectrum of the process (2.1). We may do this effectively by considering the following simple process:

$$y(t) = x(t)\cos(\lambda t + \varphi)\,, \tag{2.3}$$

where $x(t)$ is a stationary process with zero mean, autocovariance function $\gamma_x(\tau)$, and continuous spectrum

$$f_x(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} e^{-j\lambda\tau}\gamma_x(\tau)\,d\tau\,. \tag{2.4}$$

[40] If the spectrum of α_0 is continuous, the spectrum of $y(t)$ is said to be *mixed*. The analysis of processes with mixed spectra has been extensively discussed by Priestley [20].

For simplicity we suppose that φ is a random variable, independent of $x(t)$ and uniformly distributed in the interval $[-\pi, \pi]$.[41] We may suppose that the spectrum of $x(t)$ is roughly the same shape as that of, say, the Federal Reserve Board index of industrial production pictured in Figure 4; that is, it is symmetric about the origin with high power at and near zero frequency, the power falling off rapidly in either direction away from zero frequency.

Since $x(t)$ and φ are independent and $x(t)$ has zero mean, it is easily seen that $y(x)$ has zero mean. Furthermore, $y(t)$ is a stationary process with autocovariance function

$$
\begin{aligned}
(2.5)\quad \gamma_y(\tau) &= \mathscr{E}y(t)y(t+\tau) \\
&= \frac{\gamma_x(\tau)}{2\pi}\int_{-\pi}^{\pi}[\cos\lambda t\cos\varphi-\sin\lambda t\sin\varphi]\times \\
&\quad [\cos\lambda(t+\tau)\cos\varphi-\sin\lambda(t+\tau)\sin\varphi]\,d\varphi \\
&= \frac{\gamma_x(\tau)}{2\pi}\pi[\cos\lambda t\cos\lambda(t+\tau)+\sin\lambda t\sin\lambda(t+\tau)]=\tfrac{1}{2}\gamma_x(\tau)\cos\lambda\tau\,.
\end{aligned}
$$

Writing $\cos\lambda\tau$ in complex form, we may easily show that the spectrum of $y(t)$ is

$$
\begin{aligned}
(2.6)\quad f_y(\mu) &= \frac{1}{\pi}\int_{-\infty}^{\infty} e^{-j\mu\tau}\gamma_y(\tau)\,d\tau \\
&= \frac{1}{\pi}\int_{-\infty}^{\infty} e^{-j\mu\tau}\left(\frac{e^{j\lambda\tau}+e^{-j\lambda\tau}}{2}\right)\frac{\gamma_x(\tau)}{2}\,d\tau \\
&= \tfrac{1}{4}\left\{\frac{1}{\pi}\int_{-\infty}^{\infty}[e^{-j(\mu-\lambda)\tau}\gamma_x(\tau)+e^{-j(\mu+\lambda)\tau}\gamma_x(\tau)]\,dt\right\} \\
&= \tfrac{1}{4}\{f_x(\mu-\lambda)+f_x(\mu+\lambda)\}\,.
\end{aligned}
$$

Thus the spectrum of $y(t)$ is approximately the spectrum of $x(t)$ with origin shifted to the frequency λ and attenuated by a factor of 4.[42]

Given the supposed shape of the spectrum of the random amplitudes α_i and β_i in (2.1), it is easily seen that the spectrum of $y(t)$ will no longer consist of a series of spectral lines superimposed on the continuous spectrum of α_0 at the frequencies λ_i, but rather of a series of more or less broad peaks at those frequencies. The narrower the peaks, the more pronounced and regular will be the seasonal variation.[43]

[41] $\cos(\lambda t+\varphi)$ is stationary if and only if φ has the rectangular distribution on the interval $[-\pi,\pi]$, so that if this were not assumed, $y(t)$ could not be treated as a stationary series.

[42] Processes of this sort are called *narrow band* processes; see Davenport and Root [4, p. 158]. Note that $f_x(\mu+\lambda)$ will be small compared with $f_x(\mu-\lambda)$ if the power in $x(t)$ is concentrated near the origin.

[43] In the limiting case in which $x(t)$ tends to a constant $\bar{x}$ (a non-zero mean for $x(t)$ would have added a spectral line at the origin in (2.6)), $y(t)$ becomes

$$
\begin{aligned}
\bar{x}\cos(\lambda t+\varphi) &= (\bar{x}\cos\varphi)\cos\lambda t+(-\bar{x}\sin\varphi)\sin\lambda t \\
&= a_t\cos\lambda t+b_t\sin\lambda t\,.
\end{aligned}
$$

Under the assumption that φ is uniformly distributed in $[-\pi,\pi]$ we have $\mathscr{E}a_t=\mathscr{E}b_t=0$, $\mathscr{E}a_tb_t=0$, $\mathscr{E}a_t^2=\mathscr{E}b_t^2=\bar{x}^2/2$, precisely the characteristics specified for α_t and β_t in (2.1), apart from normality.

In the more general case, then, we may define seasonality as that characteristic of a time series that gives rise to spectral peaks at seasonal frequencies.

Two series with exceedingly marked seasonal variations are the monthly series of federally inspected hog and cattle slaughter which go back to 1907. Graphs of the estimated positive halves of the spectra of these two series, based on recolored second quasi-differences of the form described in Section 1.3, are given in Figures 5 and 6. Both series exhibit extremely marked peaks at the frequency 0.0833 corre-

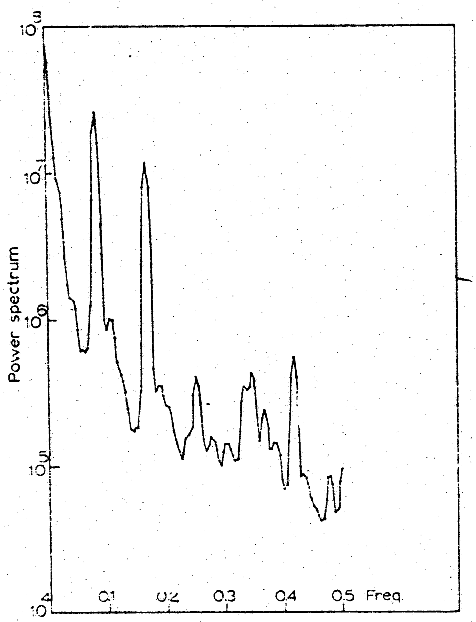

FIGURE 5

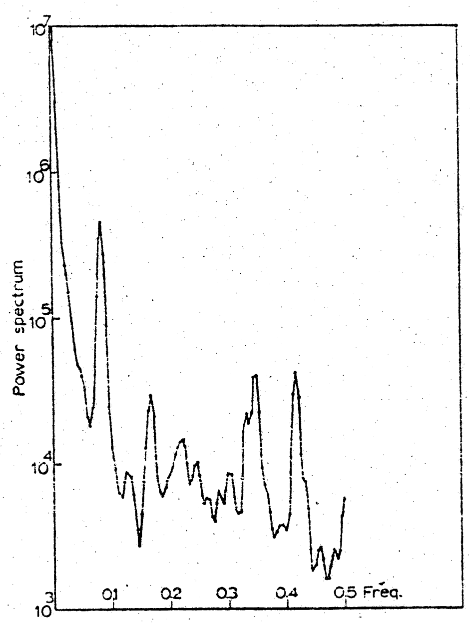

FIGURE 6

sponding to a twelve month cycle. The hog slaughter series has in addition a very marked peak at 0.1667 corresponding to a six month cycle. There are smaller peaks at all the other seasonal frequencies, the ones at 0.3333 in both series being largely obscured by a nearby peak at or near 0.3500.[44] Thus, the definition given above appears to correspond well with intuition.

[44] In their forthcoming book, Granger and Hatanaka give an example which probably contains the explanation for this peak: "Taking into account leap years, the average number of days in a month is 30.437 and so there are 4.348 weeks in the average month. If a . . . series . . ., measured every month, contained an important weekly cycle, it would thus correspond to the frequency 0.348." Thus since slaughter occurs only on working days, which have more or less a weekly cycle, we would expect a peak at the point nearest 0.348 at which the average power spectrum is measured. Essentially, then, this peak occurs because of the cycle in the number of working days per month.

2.2. *Seasonal Adjustment*

In many economic time series, for example, the monthly series on United States unemployment, that part of the month to month variation that can be called seasonal is so important as to obscure other movements in the series. It is probably true that the single most important—or, at least, politically most important—statistic published by the Federal Government is the seasonally adjusted unemployment rate. The reason that the seasonally adjusted rate, and not the rate as estimated directly from the Bureau of the Census Current Population Survey, is most important is not hard to explain. During the mild recession of 1961 it was exceedingly important to the President and his advisors to assess correctly the general movement of unemployment, to know, for example, whether the change from May to June, 1961, could reasonably have been attributed to the sort of factors which normally cause unemployment to rise between May and June, or whether the general tendency of unemployment was up, which could be interpreted as a deepening of the recession calling for remedial actions. It would, of course, have been easy enough to compare the unemployment rates in May and June with those of the year before, but that would not have revealed whether unemployment rates had *risen* to their then high levels or *fallen* to them from still higher levels in the past few months. In terms of the concepts introduced in the previous section, the high power at seasonal frequencies was making interpretation of the information contained in the series difficult. The process of seasonal adjustment which was applied to the series in order to make it more readily interpretable was designed to smooth the series by filtering out some of the high power at seasonal frequencies.[45]

In one sense, the whole problem of seasonal adjustment of economic time series is a spurious one. Seasonal variations have causes (for example, variations in the weather), and insofar as these causes are measurable they should be used to explain changes that are normally regarded as seasonal. Indeed, seasonality does not occur in isolated economic series, but seasonal and other changes in one series are related to those in another. Hence, ideally one should formulate a complete econometric model in which the causes of seasonality are incorporated directly in the equations. Even if one cannot measure all the causes of seasonality, it may be possible to allow for seasonal shifts in economic relationships through the use of shift variables. To implement the sort of approach to seasonality consistent with this point of view would require the formulation and estimation of a monthly econometric model, and this is subject to a number of practical and conceptual

[45] Alternatively one might argue that the only *new* information in an observation on an economic time series is that part which could not have been predicted on the basis of past knowledge of the series. Since unemployment "usually" rises in June, the fact that it does is not news. Seasonal adjustment can then be viewed as a kind of prediction designed to remove a certain type of "non-information" from the series.

difficulties.[46] On the practical side the problems include the lack of availability of many relevant series, the non-measurability of key items, and the lack of appropriate statistical methodology for estimation of simultaneous equation systems relating highly temporally disaggregated series. In addition, the precise structure of the model will very much affect the analysis of seasonal effects, i.e., the results will be much affected by specification error in the model. On the conceptual side the problem is basically one of continuing structural change, which is essentially the sort of thing which causes seasonality to show up in the form of spectral peaks rather than spectral lines.

Considering these difficulties, the more simple-minded approach of attempting to find a filter which will remove spectral peaks at seasonal frequencies and alter the remaining characteristics of the series as little as possible makes considerable practical sense. It is not my purpose here to develop such filters, by no means an easy task, but rather to analyze the extent to which existing procedures succeed in this task with particular reference to the U.S. employment and unemployment statistics.

While a considerable number and variety of seasonal adjustment methods have been proposed, the vast bulk of those in current use are very closely related to the technique used by the Bureau of Labor Statistics to adjust the monthly series of employment and unemployment estimates issued by that agency.[47] As a detailed description of this method has been given elsewhere [**19**, Appendix G], it is only necessary to present a bare and incomplete outline here. The model underlying this method is the familiar one, each value of a series being the product (or sum)[48] of three factors: (1) trend-cycle, (2) seasonal, and (3) irregular. The problem is to disentangle these three components, then to eliminate the seasonal by division (or subtraction). In the terminology introduced earlier in this paper, trend-cycle consists largely of the low frequency components of the time series; seasonal of components at seasonal frequencies; and irregular of all the rest.

The first step in the procedure is an attempt to isolate the components at seasonal frequencies. This is done by computing a twelve month moving average of the series and finding the ratio of series for each month to the average centered on that month. The forming of ratios to moving average is a highly nonlinear filter designed to remove the low frequency components, leaving only the higher frequency seasonal and irregular components.

[46] T. C. Liu is attempting to construct a monthly model of the U. S. economy. Not only does this involve a formidable amount of interpolation but, for a variety of reasons, the model must be constructed with seasonally *adjusted* data, thus rendering it inapplicable for the purpose of seasonal adjustment.

[47] A comprehensive survey of proposed methods and those in use has been made by S. N. Marris [**16**, pp. 35–78].

[48] Multiplicative models can always be converted into additive ones by transforming the data to logarithms; conversely an additive model can be converted to a multiplicative one by taking exponentials.

The next step in the procedure attempts to separate seasonal components from the other high frequency components with which they are mixed. This is done in the BLS method by computing a five term moving average of the ratios to moving average for each month (i.e., five Januaries, five Februaries, etc.). The reason a five term moving average is used rather than a longer one (in the extreme all the ratios available) is to allow for the possibility of slowly varying seasonals (the narrow band versus the spectral line nature of seasonality). The length of the average reflects a judgment on the speed of this variation; the shorter the average, the faster the seasonal factor may vary and the broader the band of frequencies attributed to seasonality is supposed to be.

At this point in the procedure there is available a series of estimates of the seasonal component of the time series. If this is now divided into the original series, the result will be an estimate of the trend-cycle and irregular components, i.e., a preliminary seasonally adjusted series. However, the procedure does not stop with this preliminary adjustment, because it is felt that the original attempt to remove the low frequency components, without distorting the higher frequency ones by the formation of ratios to moving average, may not have been completely successful. Consequently, the resulting estimate of the trend-cycle and irregular components is again averaged, this time with a weighted seven term formula, in order to arrive at a new estimate of the trend-cycle component. If the ratios of the original series to this new estimate of the trend-cycle component are formed, the result will be a new estimate of the seasonal and irregular components. The second step in the procedure can then be repeated to arrive at a new estimate of the seasonal components, and so on indefinitely. The BLS procedure, however, is to repeat this operation only twice more after the initial estimate of the trend-cycle components by means of a twelve-month moving average.

The most recent complete set of seasonal factors obtained by the method is used to filter the series for the current year.

The above description does not attempt to do justice to the details of the procedure, which often involve considerable sophistication in the treatment of outliers, end-point corrections, and so forth. One further detail, however, peculiar to the U.S. employment and unemployment statistics, deserves special mention. The employment, unemployment, and labor force series, as estimated from a sample survey, actually consist of a large number of components. Distinctions are made on the basis of age and sex, major industry attachment (agricultural *versus* non-agricultural), and, in recent years, color. In the unemployment statistics there is also a classification on the basis of length of unemployment. Thus, in seasonally adjusting these series, it is possible either to adjust them all individually, including such major aggregates as total unemployment, employment, and labor force, or to adjust certain components, deriving seasonally adjusted values for the aggregates by implication. In the published series for the U.S. all aggregates and components are adjusted for seasonality separately except total unemployment. This series is seasonally adjusted by adjusting four major age-sex categories of unemployed (male and female, age 14–19, and age 20 and over) and summing the resulting four seasonally adjusted components to arrive at the adjusted total. The reason for this refinement is not hard to explain. If one examines the power spectra of the four categories presented in Figures 8–11, one finds that the degree of seasonality, as measured by the height of the peaks at seasonal frequencies, is much greater for males and females under 20 than it is for the older unemployed. The seasonal patterns, either as conventionally measured or by implication of the power spectra, are also quite different for the four groups. Since the relative contribution of the four groups to the total volume of unemployment has been changing over time, separate adjustment is a way of narrowing the frequency bands about seasonal frequencies in which the seasonal adjustment procedure must reduce power in the aggregate series, and thereby increasing the supposed accuracy of the adjustment.[49] It is this same idea, turned on its head so to speak, which provides the rationale for the so-called "residual" method of obtaining the seasonally adjusted total unemployment series.[50] According to this rationale, the way to obtain the seasonally adjusted unemployment series is first to adjust total employment and total labor force by the usual BLS procedure and then to derive the adjusted unemployment series by subtraction. There is a variety of reasons adduced for preferring this procedure to the one actually employed; closer examination does not suggest any strong a priori reason for preferring it to the one in use. In Sections 3.1 and 3.3 the effects of the residual method and those of the standard method are compared; the results suggest that, in terms of the criteria developed below, neither method is significantly superior to the other.

3. ANALYSIS OF THE EFFECTS OF SEASONAL ADJUSTMENT PROCEDURES

3.1. *Comparison of Power Spectra*

Existing methods of seasonal adjustment, of which the BLS method described above is typical, are non parametric in character. There are no statistically adequate tests for removal of seasonal effects or for the changes in the seasonal patterns which are induced by averaging the monthly ratios to moving averages for only a

[49] Examination of the spectra of other components of unemployment and of various components of employment and labor force also reveals some striking differences in the degree and pattern of seasonality. As the relative contributions of these components to the corresponding aggregates has also been changing over time, it would also seem that some such component by component analyses should be incorporated in the seasonal adjustment of other aggregates. However, this is not done by the BLS. Too fine a disaggregation would of course tend to increase the sampling variability of both adjusted and unadjusted estimates, which provides some reason for not attempting to build up estimates from a very large number of components.

[50] Although this idea had been discussed in government circles some years previously, it appears to have been first brought to public attention by Brittain [3]. For its advocacy in a more political context see Samuelson [22].

few past years. Furthermore, all these methods are highly nonlinear filters, so that it is difficult to specify a priori what the effects of seasonal adjustment on the series will be; possible effects include: (a) removal of more than strictly seasonal components; (b) introduction of non-seasonal movements; and (c) distortion of the relationship between the seasonally adjusted series and other related series. Although it is possible, at least in a partial way, to assess effects (a) and (b) by comparing the power spectrum of a series with its seasonally adjusted counterpart, it is not possible to assess effect (c). The reason, as has already been mentioned in Section 1, is that both the spectrum and the autocovariance function of a time series suppress all phase information. Such information is, of course, neither interesting nor relevant when considering a single stationary time series, but it is both when two such time series are considered.

Figures 7–16 give the estimated power spectra for total unemployment and unemployment in the four major age-sex categories. The spectrum of the original series is given as a solid line. The dashed line in Figures 7–11 shows the estimated power spectrum of the corresponding seasonally adjusted series as computed by the ordinary BLS procedure; the dashed line in Figures 12–16 shows the power spectrum of the corresponding seasonally adjusted series as computed by the residual method, i.e., by subtracting seasonally adjusted employment from seasonally adjusted labor force. Figures 7 and 12 are the spectra for total U.S. unemployment estimated from monthly data July, 1947–December, 1961. The remaining four figures in each group refer to the major age-sex categories of unemployment. In order, males, 14–19 years of age; males, 20 years of age and over; females, 14–19 years of age; and females, 20 years of age and over. The data from which the spectra for the categories have been estimated cover the period July, 1948–December, 1961.

Perhaps the most striking finding is the great loss in power which occurs at all frequencies in the seasonally adjusted unemployment of both sexes regardless of whether the ordinary BLS method or the residual method is used; in all four cases the power spectra of the seasonally adjusted series lie well below the corresponding spectra for the original series. For the older age groups and the aggregate figure, the loss of power at non-seasonal frequencies is less marked, though still quite noticeable, especially at frequencies between 0.0833 and 0.3333 where a large amount of power is present in the original series. It is also apparent that in no case does the use of the residual method improve the situation much, if at all.[51]

[51] It is possible to construct asymptotic confidence intervals based on the chi-square distribution for the individual points on the power spectrum. The degrees of freedom are based on considerations related to the "width" of the spectral window used. Connecting these points, however, does not give a confidence bound for the spectrum as a whole. Furthermore, in order to test whether the estimated spectrum for the seasonally adjusted spectrum lies significantly below the estimated spectrum for the original series we would need *joint* confidence bands. These have not been developed, though confidence bounds for the *gain*, to be introduced below, are available for one type of spectral window.

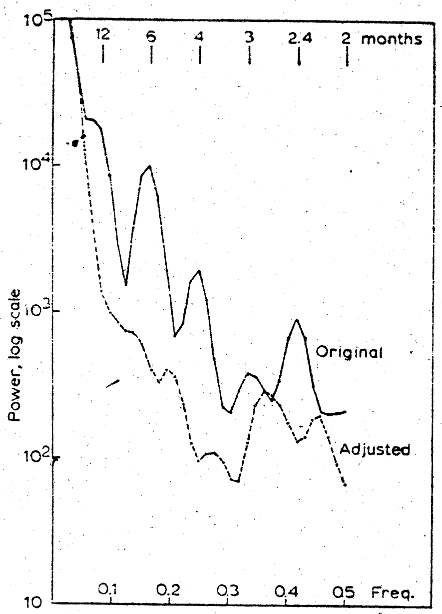

FIGURE 7

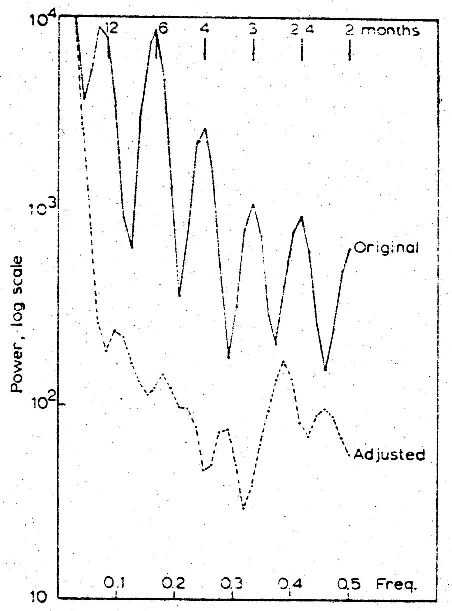

FIGURE 8

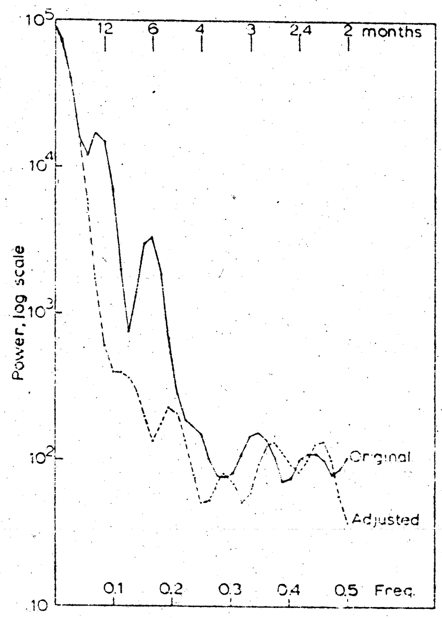

FIGURE 9

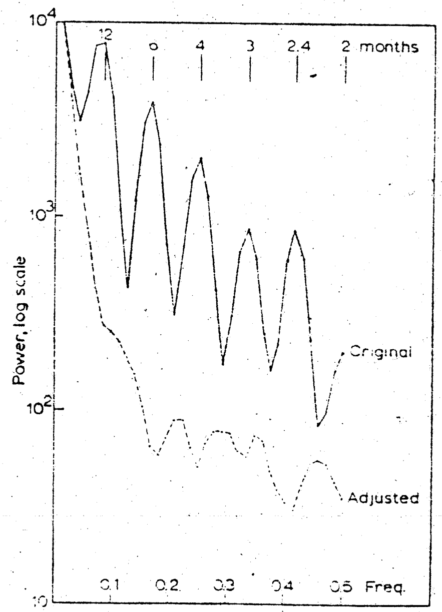

FIGURE 10

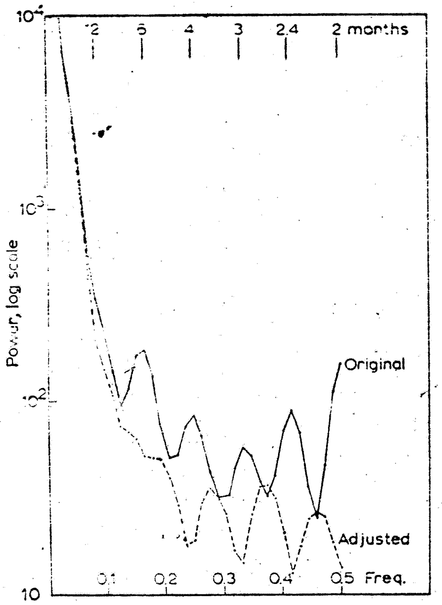

FIGURE 11

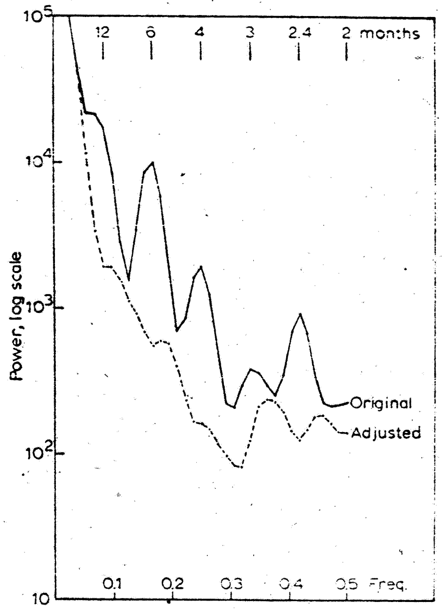

FIGURE 12

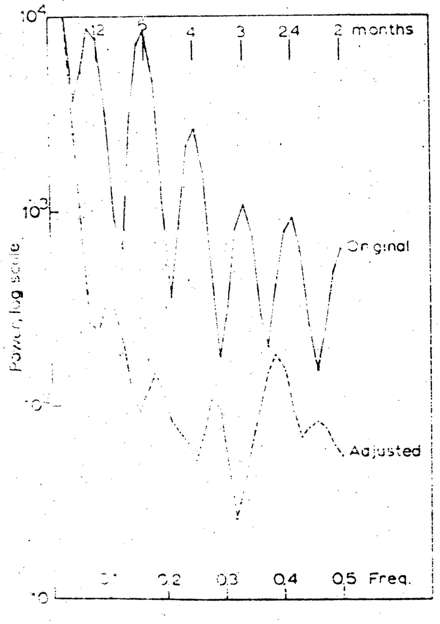

FIGURE 13

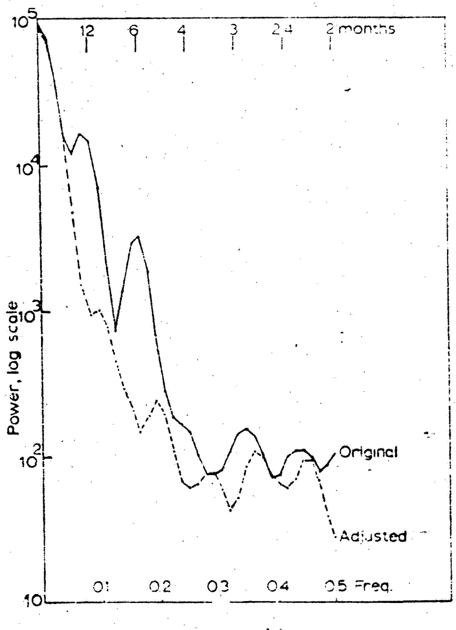

FIGURE 14

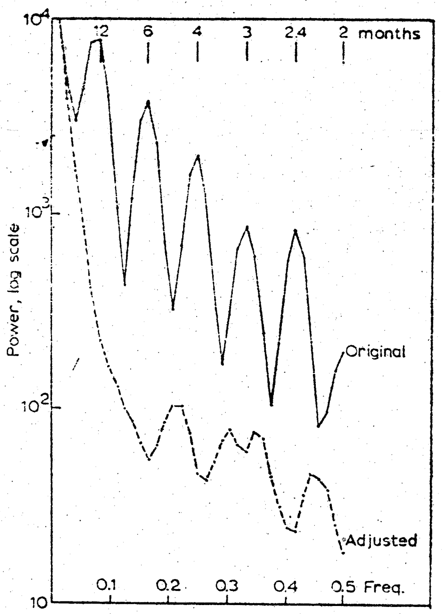

FIGURE 15

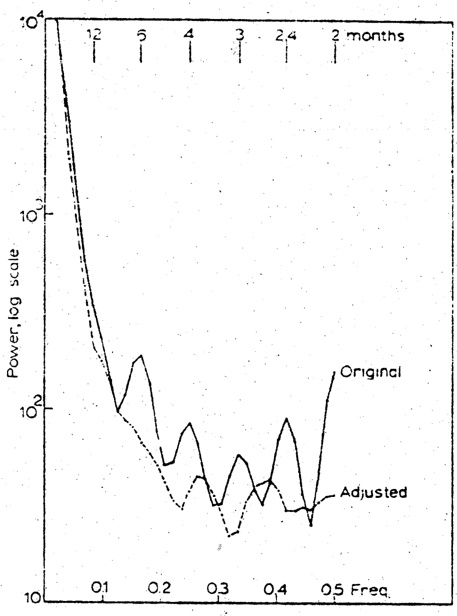

FIGURE 16

The conclusion which may be drawn from the relation between the pairs of estimated spectra is that either technique of seasonal adjustment removes more than strictly seasonal components. There is no evidence that non-seasonal components have actually been introduced by either method. The fact that more than strictly seasonal components are removed is due perhaps to the great flexibility of the seasonal factors as computed by the BLS. This flexibility arises from the use of relatively short, five term, moving averages to separate the seasonal from the irregular component in the second step of each iteration in the procedure. This high degree of flexibility is most serious when the seasonal patterns are most regular, as in the case of the two younger age groups, for the BLS seasonal factors then tend to incorporate more of the randomness as supposed, but actually nonexistent, change. Comparison of the power spectra of the original and seasonally adjusted series for other categories of unemployment, employment, and labor force tends to support this conclusion but also suggests that other factors in the make-up of the series, such as the relative amount of power at low and high frequencies, may play a role in determining whether or not seasonal adjustment will remove excessive power at non-seasonal frequencies.[52]

[52] One of the referees has pointed out that in drawing this conclusion I tacitly assume that the spectrum of the time series free of seasonality will be a smooth function of frequency free of dips at seasonal frequencies. There are two reasons for this assumption, one theoretical, the other empirical.

Theoretically, it is reasonable to regard the time series as composed of a trend-cycle component, The preceding conclusions are drawn in a rather rough fashion from visual comparison. To achieve a more quantitative measure of the loss of power at non-seasonal frequencies, as well as to answer the question of whether any temporal distortions (phase shifts) have been introduced by the adjustment processes, it is necessary to make use of another technique known as cross-spectral analysis.

3.2. *Filters, Frequency Response Functions, and Cross-Spectra*

In Section 1.3 we defined a linear, time-invariant filter in connection with the discussion of prewhitening. The frequency response function of a filter was defined, and the transfer function of the filter was defined as the squared modulus of the frequency response function. This discussion was couched in discrete terms, however, and the continuous formulation is easier to work with. In the continuous formulation, the linear, time-invariant filter L is defined by

$$y(t)=Lx(t) = \int_{-\infty}^{\infty} K(\tau)x(t-\tau)d\tau \,, \tag{3.1}$$

which includes the discrete case as a special case when the weights, $K(\tau)$, are appropriately defined.[53] The weight function, $K(\tau)$, is called the *kernel* of the filter

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a seasonal component, and white noise. Thus, the spectrum is built up out of a flat piece corresponding to the white noise, a series of narrow band peaks at seasonal frequencies, and the spectrum of the trend-cycle component. The assumption is thus essentially that all frequencies are needed to represent the trend-cycle component, and that although the power at low frequencies may be very high, this power as a function of frequency does not drop abruptly to zero at some point below the lowest seasonal frequency. If one assumes that the trend-cycle component can be approximated by some auto-regressive scheme, then precisely this conclusion emerges. For example, suppose that the pth order quasi-differences of the series $x(t)$, as defined in the previous section, are white noise, i.e., that the auto-regression is

$$\sum_{i=0}^{p} \binom{p}{i} (-k)^i x(t-i) = \varepsilon(t) \,,$$

where

$$\mathscr{E}\varepsilon(t) = 0 \,; \quad \mathscr{E}\varepsilon(t)\varepsilon(t') = \sigma^2 \,, \text{ if } t = t' \,, \text{ and } = 0 \,, \text{ if } t \neq t' \,.$$

Then the spectrum of $x(t)$ is just

$$f_x(\lambda) = \frac{1}{[1 - 2k \cos 2\pi\lambda + k^2]^p} \,,$$

which has exactly the shape which (we argued above) is typical of economic time series: it has high power at low frequencies, and low at high frequencies, but it has some power everywhere in the interval $[-\frac{1}{2}, \frac{1}{2}]$.

The empirical reason for the tacit assumption that the spectrum of a seasonality-free time series does not have dips at seasonal frequencies is that spectra of precisely this sort are produced when more reasonable methods of seasonal adjustment, such as that suggested by Hannan, are employed.

[53] For example, as $a\delta(\tau_1 - \tau_2)$ where $\delta(\tau_1 - \tau_2)$ is the Dirac delta function such that

$$\int_{-\infty}^{\infty} \delta(\tau_1 - \tau_2)d\tau_2 = 1, \quad \delta(\tau_1 - \tau_2) = 0 \quad \text{for } \tau_1 \neq \tau_2 \,.$$

L which operates on the input $x(t)$ to produce the output $y(t)$. The *frequency response function* of the filter, $l(\lambda)$, is the Fourier transform of the kernel:

$$l(\lambda) = \int_{-\infty}^{\infty} K(\tau)e^{-j\lambda\tau}\,d\tau = u(\lambda) + jv(\lambda)\,, \tag{3.2}$$

and is, in general, a complex-valued function of frequency. We have already introduced the notion of the transfer function of a filter, the squared modulus of $l(\lambda)$, and indicated the relation between the input and output power spectra in terms of this function in (1.31). A more complete description of the relationship between the input and output of a linear, time-invariant filter is given by the frequency response function of the filter. For linear filters, we have

$$\begin{aligned} Le^{j\lambda t} &= \int_{-\infty}^{\infty} K(\tau)e^{j\lambda(t-\tau)}\,d\tau = \left[\int_{-\infty}^{\infty} K(\tau)e^{-j\lambda\tau}\,d\tau\right]e^{j\lambda t} = l(\lambda)e^{j\lambda t} \\ &= [u(\lambda)+jv(\lambda)][\cos\lambda t + j\sin\lambda t] = [u(\lambda)\cos\lambda t - v(\lambda)\sin\lambda t] \\ &\quad + j[u(\lambda)\sin\lambda t + v(\lambda)\cos\lambda t]\,. \end{aligned} \tag{3.3}$$

Since $Le^{j\lambda t} = L\cos\lambda t + jL\sin\lambda t$, we see that the real part of $Le^{j\lambda t}$ is the output resulting from the input of a cosine wave of frequency λ and unit amplitude; the imaginary part is the output resulting from the input of a sine wave of frequency λ and unit amplitude.

Let the angle $\varphi(\lambda)$ be defined as

$$\varphi(\lambda) = \arg\,[u(\lambda)+jv(\lambda)] = \arctan\frac{v(\lambda)}{u(\lambda}\,; \tag{3.4}$$

so that

$$\begin{aligned} \cos\varphi(\lambda) &= \frac{u(\lambda)}{\sqrt{u(\lambda)^2+v(\lambda)^2}}\,, \\ \sin\varphi(\lambda) &= \frac{y(\lambda)}{\sqrt{u(\lambda)^2+v(\lambda)^2}}\,. \end{aligned} \tag{3.5}$$

Thus the effect of L on a sine wave of frequency λ and unit amplitude is to transform it to

$$\begin{aligned} L\sin\lambda t &= u(\lambda)\sin\lambda t + v(\lambda)\cos\lambda t \\ &= [u(\lambda)^2+v(\lambda)^2]^{\frac{1}{2}}[\cos\varphi(\lambda)\sin\lambda t + \sin\varphi(\lambda)\cos\lambda t] \\ &= [u(\lambda)^2+v(\lambda)^2]^{\frac{1}{2}}\sin[\lambda t + \varphi(\lambda)]\,, \end{aligned} \tag{3.6}$$

that is, another sine wave with amplitude $[u(\lambda)^2+v(\lambda)^2]^{\frac{1}{2}}$ and non-zero phase $\varphi(\lambda)$. The filter thus increases or attenuates the amplitude by a factor of $[u(\lambda)^2+v(\lambda)^2]^{\frac{1}{2}}$ and shifts the origin of the sine wave in time by $\varphi(\lambda)/\lambda$. Similarly

(3.7) $$L \cos \lambda t = u(\lambda) \cos \lambda t - v(\lambda) \sin \lambda t$$
$$= [u(\lambda)^2 + v(\lambda)^2]^{\frac{1}{2}} [\cos \varphi(\lambda) \cos \lambda t - \sin \varphi(\lambda) \sin \lambda t]$$
$$= [u(\lambda)^2 + v(\lambda)^2]^{\frac{1}{2}} \cos [\lambda t + \varphi(\lambda)] .$$

These effects on sine and cosine waves can be conveniently summarized, in terms of the frequency response function, for this, in polar form, is just

(3.8) $$l(\lambda) = G(\lambda) e^{j\varphi(\lambda)} ,$$

where $G(\lambda) = [u(\lambda)^2 + v(\lambda)^2]^{\frac{1}{2}}$ is called the *gain* of the filter and $\varphi(\lambda)$ its *phase angle*.

We are now in a position to show that the operation of a linear, time-invariant filter on any time series can be completely described in terms of its frequency response function. Let $x(t)$ be such a time series; then by (1.18) we can write

(3.9) $$Lx(t) = \int_{-\infty}^{\infty} K(\tau) x(t-\tau) d\tau = \int_{-\infty}^{\infty} K(\tau) \left[\int_{-\infty}^{\infty} e^{j\lambda(t-\tau)} dZ_x(\lambda) \right] d\tau$$
$$= \int_{-\infty}^{\infty} e^{j\lambda t} \int_{-\infty}^{\infty} K(\tau) e^{-j\lambda\tau} d\tau \, dZ_x(\lambda) = \int_{-\infty}^{\infty} e^{j\lambda t} \, l(\lambda) dZ_x(\lambda) .$$

Thus if we write

(3.10) $$y(t) = \int_{-\infty}^{\infty} e^{j\lambda t} dZ_y(\lambda) ,$$

we see that the new random variables $dZ_x(\lambda)$ and $dZ_y(\lambda)$ into which the input and output series of the filter can be decomposed are simply related by

(3.11) $$dZ_y(\lambda) = l(\lambda) dZ_x(\lambda) .$$

Multiplying on both sides by the complex conjugate of $dZ_x(\lambda)$ and taking expected values, we have

(3.12) $$l(\lambda) = \frac{\mathscr{E} dZ_y(\lambda) \overline{dZ_x(\lambda)}}{\mathscr{E} dZ_x(\lambda) \overline{dZ_x(\lambda)}} .$$

Thus, $l(\lambda)$ is the slope of the regression line between the complex random variables $dZ_y(\lambda)$ and $dZ_x(\lambda)$.

The interpretation of the values of a frequency response function as a series of regression slopes, one for each frequency in the representation of the input and output series of a filter, suggests an interesting application. In ordinary statistical work on economic time series we frequently relate two variables by means of a least squares regression line even though we know the relation between the two variables is not truly linear; the idea is to obtain some measure of the "average" relationship. In the same way, even though we know a filter is not truly linear or time-invariant, we can relate the input and output series by means of (3.12); then the regression slope obtained at each frequency represents the "average" frequency response function. In other words, if we take a sample input and output series from any filter and form an estimate of the quantity on the right hand side of (3.12), this will estimate the frequency response function of a linear, time-invariant filter *which is an approximation to the actual filter valid over the range of the random amplitudes which characterize the input series as it occurred in the sample.* The results then give a way of characterizing any filter in terms of gain and phase angle at frequencies for which the estimates are made.

But how do we form estimates of the complex quantities, i.e., the "regression slopes,"

$$\frac{\mathscr{E}\, dZ_y(\lambda)\, \overline{dZ_x(\lambda)}}{\mathscr{E}\, dZ_x(\lambda)\, \overline{dZ_x(\lambda)}}\ ?$$

The denominator is easily recognized by (1.21) as 1/2 the power spectrum at frequency λ of the input series $x(t)$. In Section 1.2 we showed this was the complex Fourier transform of the autocovariance function $\gamma_x(\tau)$ of the input series. Since this autocovariance function is even, it followed that its power spectrum $f_{xx}(\lambda)$ could be represented by a cosine transform of the autocovariance function. Thus we formed an estimate of the power spectrum by taking, essentially, the finite cosine transform of weighted averages of the sample autocovariance of $x(t)$.[54]

The numerators of our "regression slopes" are covariances of the complex random variables $dZ_y(\lambda)$ and $dZ_x(\lambda)$ at the same frequency. Twice these covariances as a function of frequency are called the *cross-spectrum* of the two time series $x(t)$ and $y(t)$, and denoted by

$$(3.13)\qquad dF_{xy}(\lambda) = dC_{xy}(\lambda) - j\, dQ_{xy}(\lambda) = 2\mathscr{E}\, dZ_y(\lambda)\, \overline{dZ_x(\lambda)}\,.$$

The real part of the cross-spectrum (which is generally a complex valued function) is $dC_{xy}(\lambda)$; it is called the co-spectrum of $x(t)$ and $y(t)$. The complex part of the cross-spectrum, $dQ_{xy}(\lambda)$, is called the *quadrature spectrum*. The meaning of these two parts of the cross-spectrum becomes clearer if we rewrite the two series in real form, following (1.11):

$$(3.14)\qquad \begin{aligned} x(t) &= \int_0^\infty \cos \lambda t\, dU_x(\lambda) + \int_0^\infty \sin \lambda t\, dV_x(\lambda)\,, \\ y(t) &= \int_0^\infty \cos \lambda t\, dU_y(\lambda) + \int_0^\infty \sin \lambda t\, dV_y(\lambda)\,. \end{aligned}$$

Since

$$(3.15)\qquad \begin{aligned} \mathscr{E}\, dZ_y(\lambda)\, \overline{dZ_x(\lambda)} &= \tfrac{1}{4}\mathscr{E}\{[dU_y(\lambda) - j\, dV_y(\lambda)][dU_x(\lambda) + j\, dV_x(\lambda)]\} \\ &= \tfrac{1}{4}[\mathscr{E}\, dU_x(\lambda)\, dU_y(\lambda) + \mathscr{E}\, dV_x(\lambda)\, dV_y(\lambda)] \\ &\quad - \tfrac{j}{4}[\mathscr{E}\, dU_x(\lambda)\, dV_y(\lambda) - \mathscr{E}\, dU_y(\lambda)\, dV_x(\lambda)]\,, \end{aligned}$$

[54] Equation (1.23′) with autocovariances modified by (1.25).

we see that $dC_{xy}(\lambda)$ is essentially the covariance of the amplitudes of the *in phase* components of $x(t)$ and $y(t)$; and $dQ_{xy}(\lambda)$ is essentially the covariance of amplitudes of cosines with sines which are 90° out of phase or *in quadrature*.

To see how to estimate the cross-spectrum we must first show that the covariance of the complex random variables $dZ_x(\lambda)$ and $dZ_y(\lambda)$ is zero at different frequencies; we already know from Section 1.2 that

(3.16) $$\mathscr{E}\, dZ_x(\lambda)\, \overline{dZ_x(\lambda')} = 0 = \mathscr{E}\, dZ_y(\lambda)\, \overline{dZ_y(\lambda')},$$

for λ and λ' in non-overlapping bands, i.e., "$\lambda \neq \lambda'$." What we wish to show is that

(3.17) $$\mathscr{E}\, dZ_x(\lambda')\, \overline{dZ_y(\lambda)} = 0 = \mathscr{E}\, \overline{dZ_x(\lambda')}\, dZ_y(\lambda),$$

for "$\lambda \neq \lambda'$," for all $l(\lambda) \neq 0$.

Unfortunately, it is not in general true that (3.17) holds for two arbitrary stationary processes; it will hold if and only if the two series are *jointly* stationary in the weak sense used throughout this paper; i.e., if and only if the lag covariance function is independent of time.[55] It is, however, easily seen that it is sufficient for joint stationarity that $x(t)$ be stationary and that $y(t)$ be the output of a linear, time-invariant filter applied to $x(t)$. For then (3.11) holds; multiplying by $\overline{dZ_x(\lambda')}$ and taking expected values, we have

(3.18) $$\mathscr{E}\, dZ_y(\lambda)\, \overline{dZ_x(\lambda')} = l(\lambda)\mathscr{E}\, dZ_x(\lambda)\, \overline{dZ_x(\lambda')} = 0, \qquad \text{for "}\lambda \neq \lambda'\text{" and } l(\lambda) \neq 0.$$

Similarly,

(3.19) $$\mathscr{E}\, \overline{dZ_y(\lambda)}\, dZ_x(\lambda') = \overline{l(\lambda)}\mathscr{E}\, dZ_x(\lambda)\, \overline{dZ_x(\lambda')} = 0, \qquad \text{for "}\lambda \neq \lambda'\text{" and } l(\lambda) \neq 0.$$

Conversely, it can be shown that if $x(t)$ is stationary and $y(t)$ is the output of a linear time-invariant filter applied to $x(t)$, then $\mathscr{E}x(t)\, y(t+\tau)$ will be independent of t. From this it also follows that (3.18) and (3.19) hold.[56] Joint stationarity is a very

[55] I am indebted to W. M. Gorman for pointing out my earlier mistake in this respect. He gave as a counterexample the two series,

$$x(t) = a_1 \cos t + b_1 \sin t + a_2 \cos 2t + b_2 \sin 2t,$$
$$y(t) = a_2 \cos t + b_2 \sin t + a_1 \cos 2t + b_1 \sin 2t,$$

which are each stationary if

$$\mathscr{E}a_i^2 = \sigma_i^2 = \mathscr{E}b_i^2,$$

and

$$\mathscr{E}a_i b_i = 0, \quad \mathscr{E}a_i a_j = 0 \quad \text{if} \quad i \neq j \quad \text{and} \quad \mathscr{E}b_i b_j = 0 \quad \text{if} \quad i \neq j,$$

but not jointly stationary since

$\mathscr{E}x(t)y(t+\tau) = \sigma_1^2 \cos[t + 2\tau] + \sigma_2^2 \sin[3t + \tau]$ is not independent of t.

[56] Doob [5, pp. 596–98].

strong condition. Nonetheless, it is clear that it is an entirely reasonable one when dealing with linear, time-invariant filters.

Equations (3.18) and (3.19) may be used to show that the cross-spectrum is the complex Fourier transform of the lag covariance function $\mathscr{E}y(t)x(t-\tau)$. For

$$(3.20)\qquad \mathscr{E}\, y(t)x(t-\tau) = \mathscr{E}\, y(t)\overline{x(t-\tau)}$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{j(\lambda-\lambda')t} e^{j\lambda'\tau} \mathscr{E}\, dZ_y(\lambda)\overline{dZ_x(\lambda')}$$
$$= \int_{-\infty}^{\infty} e^{j\lambda\tau} \frac{dF_{xy}(\lambda)}{2},$$

since $\mathscr{E}\, dZ_y(\lambda)\overline{dZ_x(\lambda')}=0$ for $\lambda\neq\lambda'$ by (3.18), and $=dF_{xy}(\lambda)/2$ for $\lambda=\lambda'$ by (3.13). It therefore follows that $\mathscr{E}y(t)x(t-\tau)$ and $dF_{xy}(\lambda)/2$ are Fourier transform pairs and

$$(3.21)\qquad dF_{xy}(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} e^{-j\lambda\tau}[\mathscr{E}\, y(t)x(t-\tau)]\, d\tau\, .$$

Equation (3.21) enables us to derive a way of estimating the real and complex parts of the cross-spectrum.

Just as before, the function F_{xy} has the classical decomposition in terms of three functions: the first absolutely continuous, the second a step function, and the third a singular function. If the last two are ruled out as essentially irrelevant for economic time series, we may write

$$(3.22)\qquad dF_{xy}(\lambda)=f_{xy}(\lambda)\, d\lambda = c_{xy}(\lambda)\, d\lambda - jq_{xy}(\lambda)\, d\lambda\, .$$

The discrete analogue of (3.21) for a finite number M of lags is then

$$(3.23)\qquad f_{xy}(\lambda) = c_{xy}(\lambda)-jq_{xy}(\lambda) = \frac{1}{\pi}\sum_{\tau=-M}^{M} e^{-j\lambda\tau}\mathscr{E}\, y(t)x(t-\tau)$$
$$= \frac{1}{\pi}\sum_{\tau=-M}^{M} (\cos\lambda\tau - j\sin\lambda\tau)\mathscr{E}\, y(t)x(t-\tau)$$
$$= \frac{1}{\pi}\Big\{\mathscr{E}\, y(t)x(t) + \sum_{\tau=1}^{M}\cos\lambda\tau\ \mathscr{E}[y(t)x(t-\tau)+y(t)x(t+\tau)]$$
$$-j\sum_{\tau=1}^{M}\sin\lambda\tau\ \mathscr{E}[y(t)x(t-\tau)-y(t)x(t+\tau)]\Big\}\, .$$

Note that if $y(t)$ is set equal to $x(t)$, the complex part vanishes and the expected values in the first and second terms become, respectively, $\gamma(0)$ and $2\gamma(\tau)$ so that (3.23) represents a generalization of (1.23′).

Let

$$(3.24)\qquad R_{xy}(\tau) = \frac{1}{2T}\sum_{t=-T+M}^{T-M}\{y(t)x(t-\tau)+y(t)x(t+\tau)\}$$

and

(3.25) $$S_{xy}(\tau) = \frac{1}{2T} \sum_{Mt=-T+}^{T-M} \{y(t)x(t-\tau) - y(t)x(t+\tau)\}$$

be the sample estimates of the expected values occurring in (3.23); then replacing the expected values by their sample estimates and equating real part to real part and complex part to complex part, we have estimates of the co- and quadrature spectra of $x(t)$ and $y(t)$

(3.26) $$\hat{c}_{xy}(\lambda) = \frac{1}{\pi}\left\{R(0) + \sum_{\tau=1}^{M} \cos \lambda\tau \, R_{xy}(\tau)\right\},$$

$$\hat{q}_{xy}(\lambda) = \frac{1}{\pi}\left\{\sum_{\tau=1}^{M} \sin \lambda\tau \, S_{xy}(\tau)\right\}.$$

These estimates, however, are not consistent.[57] To obtain consistent estimates, suitably weighted averages of the raw sample estimates $R_{xy}(\tau)$ and $S_{xy}(\tau)$ are employed; i.e., a lag window is used. In the results reported below, the window (1.25), which was used for spectral estimation, was also employed. In addition, both series were prewhitened by the quasi-difference method described in Section 1.3. The same order of quasi-differences was used to prewhiten both the original and seasonally adjusted series; after estimation of the cross-spectra, the estimates were recolored by dividing at each frequency by the transfer function of the quasi-difference filter.[58]

Given estimates $\hat{c}_{xy}(\lambda)$, $\hat{q}_{xy}(\lambda)$, and $\hat{f}_{xy}(\lambda)$ of the co-spectrum and quadrature spectrum of the input and output series, and the spectrum of the input series, the frequency response function is estimated by

(3.27) $$\hat{l}(\lambda) = \frac{\hat{c}_{xy}(\lambda)}{\hat{f}_{xx}(\lambda)} - j\,\frac{\hat{q}_{xy}(\lambda)}{\hat{f}_{xx}(\lambda)}.$$

Consequently, estimates of its gain and phase angle are given by

(3.28) $$\hat{G}(\lambda) = \frac{\sqrt{\hat{c}_{xy}(\lambda)^2 + \hat{q}_{xy}(\lambda)^2}}{\hat{f}_{xx}(\lambda)},$$

$$\hat{\varphi}(\lambda) = \arctan\left[-\frac{\hat{q}_{xy}(\lambda)}{\hat{c}_{xy}(\lambda)}\right].$$

[57] Murthy [15].

[58] The asymptotic distribution of the cross-spectrum and spectra of two series as estimated using a different lag window than the one used here has been worked out by Goodman [7]. Goodman shows how simultaneous confidence intervals may be obtained and works out several for combinations of the estimates such as the gain and phase angle of a filter. Unfortunately, in addition to being based on a different window, application of Goodman's results requires an a priori knowledge of the coherency between the two series which is analogous to a squared correlation coefficient and which is defined, for two series, as the squared modulus of the cross-spectrum divided by the product of the spectra. Since the procedure replacing the actual coherency by its estimated value seems somewhat doubtful, confidence intervals have not been given for the estimates presented below.

3.3. *Estimated Gains and Time Lags for Seasonal Filters*

The estimated gains and phase angles divided by frequency (to give an estimated time lag) are plotted for the five unemployment series and the BLS and residual methods of seasonal adjustment in Figures 17–26.

The gains which are graphed at the top of each chart give a picture entirely consistent with, but more accurate than, the comparisons of the spectra of the original and seasonally adjusted series which were made in Section 3.1. Except in the case of unemployed males, 20 and over, in which the gain is slightly and insignificantly larger than one at a few frequencies for the BLS method of adjustment, the estimated gains are uniformly between zero and one; and, of course, they tend to be lowest at seasonal frequencies. However, whereas before we were only able to note excessive removal of power at nonseasonal frequencies, we can now measure the extent of power removal at different frequencies in the same series and at the same frequency in different series. The first important point to be noted is that in the cases of both unemployed males and females over 20, which as already noted do not have as well defined seasonals as the two younger groups, power removal at different seasonal frequencies is highly uneven. There does not appear to be a consistent frequency pattern in the effectiveness of the seasonal adjustment, and the residual method offers no improvement on this score. The removal of excessive power at non-seasonal frequencies shows up in an exceedingly clear fashion in the gains. Cross-comparisons of the gains of the BLS and residual filters for seasonal adjustment at the same frequencies again yield no clearcut pattern and tend to support the contention that the effects of either method are about equally bad.

In comparing the spectra of the original and seasonally adjusted series, all information about phase shifts induced by the filter was suppressed. The possibility of phase shifts, however, particularly at low frequencies which contain the bulk of a series' information on the state of the economy, is a very serious one. Perhaps the most important finding of this paper is contained in the lower halves of Figures 17–26 in which the time lags (phase angle divided by frequency) have been plotted.[59] These show that the seasonal adjustment filters produce strong phase shifts particularly at low frequencies.[60] If all economic time series had the same harmonic

[59] Before computation of the time lags all negative phase angles were converted into positive ones by the addition of 2π (i.e., 360°). The rationale behind this adjustment is as follows: a positive phase angle of $\varphi(\lambda)$ at a frequency λ means that $\cos[\lambda t + \varphi(\lambda)]$ and $\sin[\lambda t + \varphi(\lambda)]$ are shifted *backward* in time by $\varphi(\lambda)/\lambda$. A negative phase angle of $\varphi(\lambda)$, however, means that they are shifted *forward*. Now $\cos[\lambda t + 2\pi + \varphi(\lambda)] = \cos[\lambda t + \varphi(\lambda)]$ and $\sin[\lambda t + 2\pi + \varphi(\lambda)] = \sin[\lambda t + \varphi(\lambda)]$. Thus a forward shift of $\varphi(\lambda)/\lambda$ is precisely equivalent to a backward shift of $2\pi/\lambda + \varphi(\lambda)/\lambda$ when $\varphi(\lambda)$ is negative. From the standpoint of the use of the seasonally adjusted values of a time series to indicate turning points and other major fluctuations, it is only backward shifts which make sense.

[60] It can easily be seen that any one-sided linear filter (i.e., one which operates only on past values of the data) must produce some phase shifts. For, in order that no phase shifts occur, the frequency response function *(See further p. 282)*

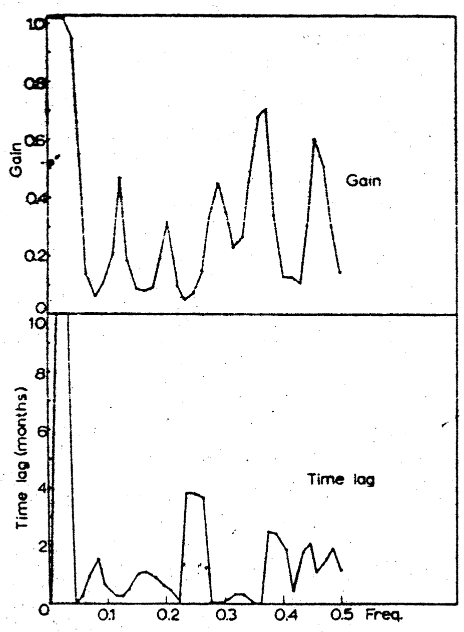

FIGURE 17

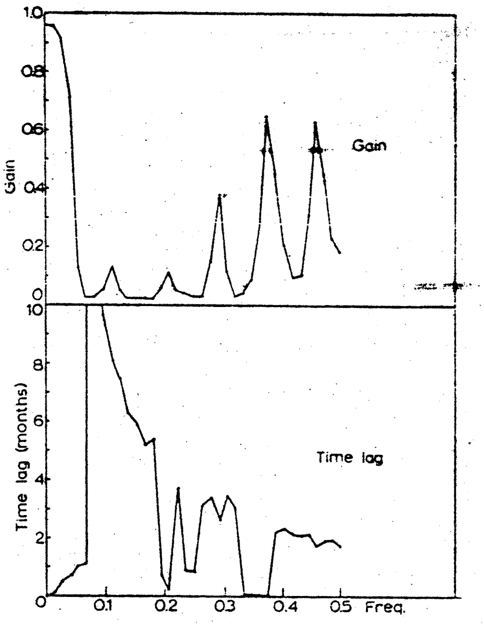

FIGURE 18

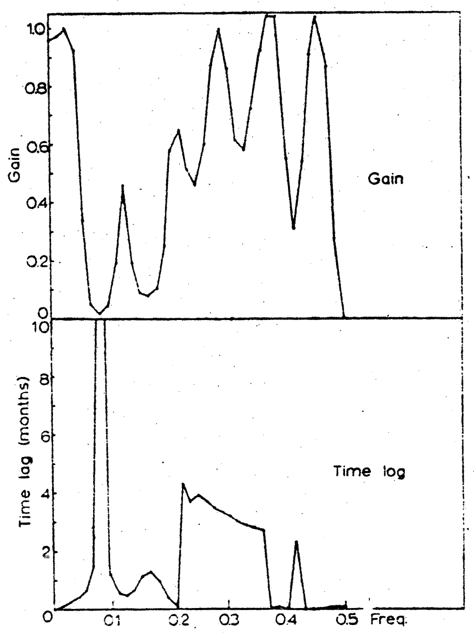

FIGURE 19

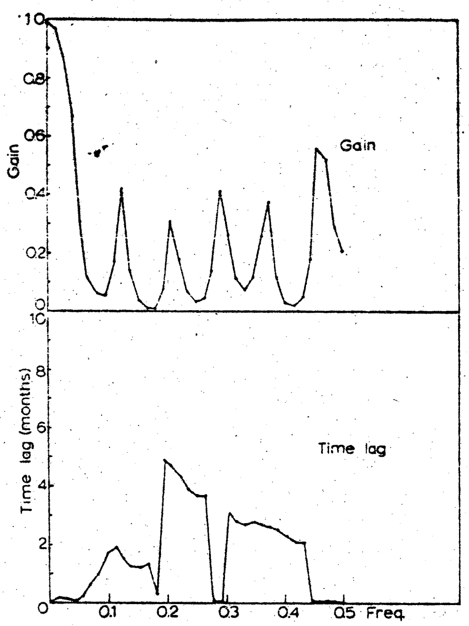

FIGURE 20

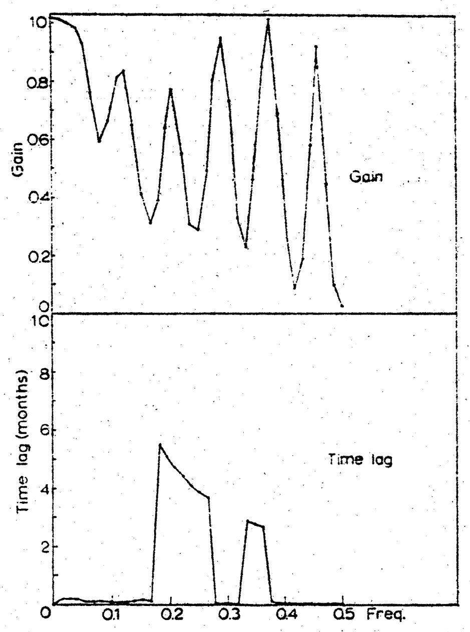

FIGURE 21

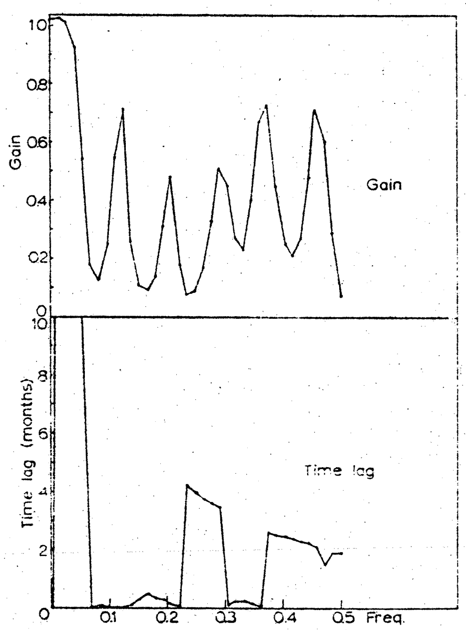

FIGURE 22

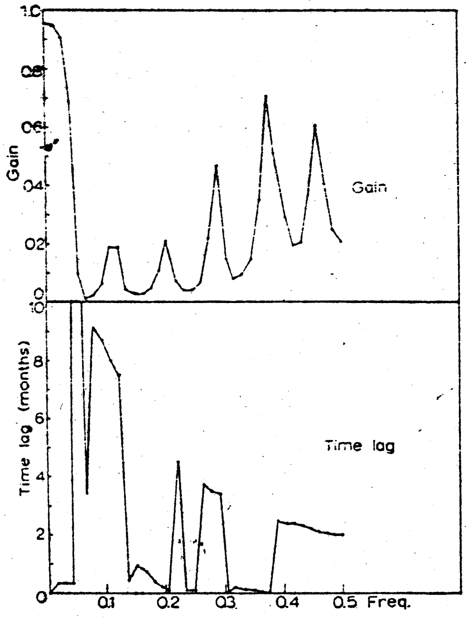

FIGURE 23

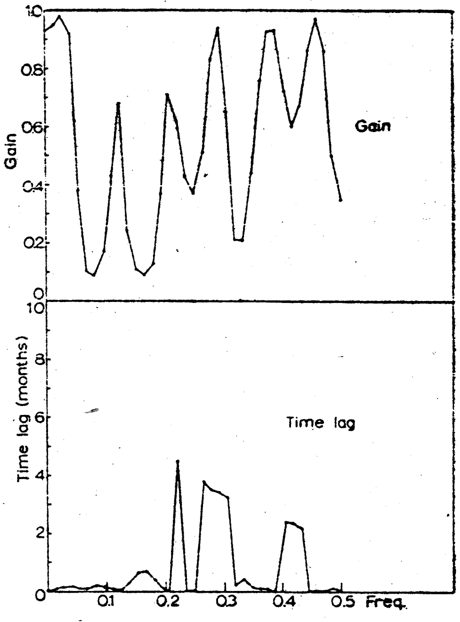

FIGURE 24

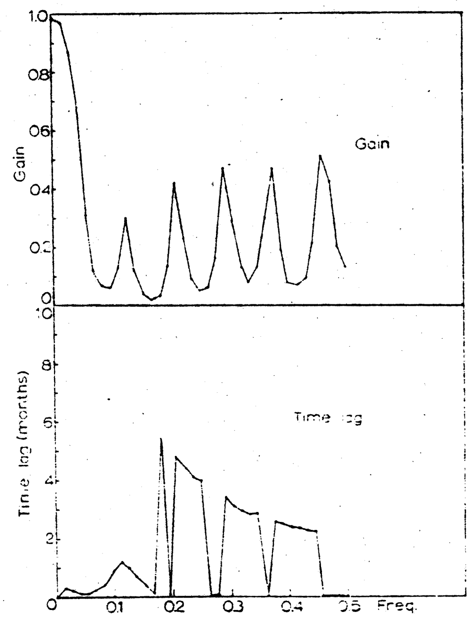

FIGURE 25

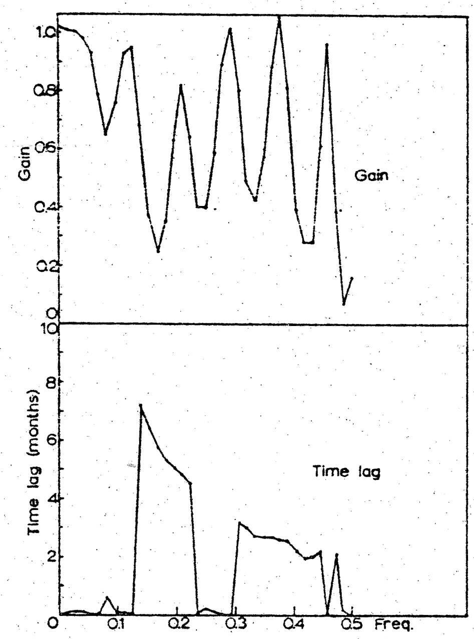

FIGURE 26

content and if the filters employed to remove seasonality produced exactly the same phase shifts at the same frequency, this phenomenon would be of little significance. It is apparent, however, from the analyses presented in this paper that both of these assumptions lie very far from the truth. Thus, the fact that different series have quite different harmonic compositions and that ordinary methods of seasonal adjustment produce severe and different phase shifts at different frequencies, means, in effect, that these methods of adjustment distort the lead-lag relationships which exist among the original series to which they are applied. Consequently, doubt is cast upon the standard practice of seasonally adjusting key economic indicators by the usual methods, although I would not argue with the position that some form of allowance for seasonality is necessary if the data are to be used with any intelligence.

$$l(\lambda) = \int_{-\infty}^{\infty} K(\tau)\, e^{-j\lambda\tau} d\tau = u(\lambda) + jv(\lambda)$$

must be real, i.e., $v(\lambda) = 0$, and this can only occur when $K(\tau)$ is symmetric about $\tau = 0$. Of course the BLS and residual filters are not linear, but they are one-sided, which is a plausible reason for the phase shifts which they produce. It would seem desirable in future research to attempt to design filters that would minimize phase shifts or, at least, confine large phase shifts to higher frequencies.

4. CONCLUSIONS

In this article I have attempted to show that description of economic time series in the frequency domain is an exceptionally fruitful way of looking at them. Many have been discouraged about the prospects of using spectral techniques in economics because of the very large numbers of observations required and because of the obviously non-stationary character of economic time series. Two important conclusions of this article, however, are, first, that the number of observations available to economists does not prevent satisfactory analysis of the high frequency components, and, second, that much of the supposed non-stationary character of economic time series may simply be treated as high power at low frequencies and allowed for by prewhitening. The method of prewhitening which has been used does not appear to have been discussed in the available literature; it is to take weighted quasi-differences of a series until the estimated spectrum of the series appears relatively flat.

Among the high frequency components of economic time series which may successfully be studied by spectral techniques are those produced by seasonality. It is shown in this article that a slowly changing and stochastic seasonal pattern will reveal itself in the spectrum of an economic time series by a series of peaks at certain frequencies. The breadth of these peaks depends essentially on the regularity of the seasonal pattern, being narrower and more pronounced the more regular the pattern.

A practical problem of considerable significance which may be treated by means of spectral techniques is that of assessing quantitatively the effects of a seasonal adjustment procedure. Seasonal adjustment may produce three types of effects in addition to the elimination of seasonality: (1) elimination of components that are not seasonal; (2) introduction of components that are not seasonal; and (3) distortion of temporal relations among series. Apart from the methods here proposed, there do not seem to be satisfactory methods of measuring these possible effects. Many time series of U.S. employment, unemployment, labor force, and components thereof have been analyzed. The results for total unemployment and its four major age-sex categories are reported here. Two methods of seasonal adjustment have been considered, the one currently in use by the Bureau of Labor Statistics, and the so-called "residual" method. In the analysis of any sort of filtering operation, such as seasonal adjustment, it is found that an estimated frequency response function, derived from estimates of the spectrum of the original series and the cross-spectrum between the original and seasonally adjusted series, provides the most useful method of analysis.

The major substantive conclusion of this article is that both the BLS and residual methods of seasonal adjustment eliminate far more than can properly be called seasonal and produce severe phase shifts at lower frequencies, but neither appears to introduce non-seasonal components. The presence of strong phase shifts at low frequencies casts considerable doubt on the ability of these series as seasonally adjusted by current techniques to serve as sensitive indicators of underlying economic conditions.

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The bulk of the research which this paper reports was carried out under Grant NSF G 16114 from the National Science Foundation to Stanford University. Although the initial writing was completed during the first three months of my tenure as Fellow of the John Simon Guggenheim Memorial Foundation and Fulbright Research Grantee at the Econometric Institute, Rotterdam, certain rewriting was accomplished under NSF GS-142. Additional funds for computations were received from the President's Committee to Appraise Employment and Unemployment Statistics, and some of the results reported here were reported more briefly in the Committee's report [19, pp. 175–75]. I am indebted to the National Science Foundation and to the chairman of the President's Committee, R. A. Gordon, for the funds with which to carry out this research.

Miss Margaret E. Martin of the Bureau of the Budget supplied many unpublished figures and much helpful advice on how to interpret them.

This paper appeared originally as Report 6227 of the Econometric Institute, Netherlands School of Economics (December 4, 1962), and in this connection typing and clerical assistance in the preparation of the charts were supplied and are much appreciated. Since this time, I have learned of a number of other applications of spectral techniques to economic problems: Especially not to be overlooked is the forthcoming volume, *Analysis of Economic Time Series* by C. W. J. Granger and M. Hatanaka, Chapter 3 of which contains a first-rate introduction to spectral theory. An exposition of the theory, with applications to a number of problems including the determination of lead-lag relationships between series, may be found in J. Cunnyngham, "The Spectral Analysis of Economic Time Series," Working Paper No. 14, U. S. Bureau of the Census (Washington, D. C.: 1963). In a paper presented at the meetings of the American Statistical Association, September 4, 1963, in Cleveland, Ohio, "Spectral Analysis and Parametric Methods for Seasonal Adjustment of Economic Time Series," H. Rosenblatt of the Bureau of the Census tests a method of parametric seasonal adjustment due to Hannan (*Australian Journal of Statistics*, 2: 1–15, April, 1960) by methods similar to those employed here. Finally, in an unpublished Master's thesis at the Australian National University, Canberra, N. F. Nettheim has treated the problem of seasonal adjustment analytically by means of spectral theory.

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MINIMUM VARIANCE, LINEAR, UNBIASED SEASONAL ADJUSTMENT OF ECONOMIC TIME SERIES*

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A statistical theory based on the general linear statistical model is developed for seasonal adjustment of economic (and other) time series. A method for seasonal adjustment may be represented as taking place in two steps. The first step is to estimate the unknown parameters of the seasonal component of the series; the second step is to remove the estimated seasonal component from the set of observations. For the unique minimum variance, linear, unbiased method for seasonal adjustment, estimation is carried out through the unique, minimum variance, linear unbiased estimator. Sampling theory for statistical inference about a method for seasonal adjustment may be derived from normal sampling theory for the general linear statistical model. The properties of minimum variance, linearity, and unbiasedness provide a complete basis for the selection of a method for seasonal adjustment.

1. INTRODUCTION

ECONOMIC statisticians are in virtually unanimous agreement that any model for representation of an economic time series must contain both deterministic and random components. In view of this agreement, it is surprising that until recently there has been no rigorous discussion of the problem of seasonal adjustment from the viewpoint of statistical theory. It seems natural to approach the problem of seasonal adjustment as a problem in estimation. The first step in seasonal adjustment is to estimate the seasonal component of a time series. The second step is to remove the estimated seasonal component. Given the random component of an economic time series, it should be possible to characterize any method for seasonal adjustment statistically. The estimators of the seasonal component are random variables. The distributions of these estimators may be deduced from the distribution of the random component of the original series. A seasonally adjusted time series is a well-defined transformation of the original series; its deterministic component may be characterized from the deterministic component of the original series; the distribution of its random component may be deduced from the distribution of the random component of the original series.

In practice, seasonal adjustment is carried out by "estimating" the seasonal component of a time series and removing this component from the original series. But estimates of the seasonal component are regularly presented without standard errors or other means of characterizing the sampling distribution from which the estimates are drawn. Furthermore, the properties of the seasonally adjusted series are rarely characterized from the statistical point of view. Serious confusions about even the most elementary statistical properties of a seasonally adjusted series may be found in the current literature. In the

* The equivalence of the two alternative models for the trend component discussed in Section 2 was first suggested to me by William Kruskal. I would like to thank Professor Kruskal and also Saul H. Hymans and Arthur S. Goldberger for their helpful comments on an earlier version of the paper.

absence of information about the sampling distribution of the estimator of the seasonal component, it is impossible to decide whether a method gives rise to systematic biases in the seasonally adjusted series or whether a given method is efficient. Furthermore, it is impossible to make inferences for problems which arise in selecting a method for seasonal adjustment. Examples of such problems are: whether to adjust at all or if to adjust, what kind of seasonal adjustment to use.

Very recently, several important steps have been taken in the development of a statistical theory for seasonal adjustment of economic (and other) time series. The first problem of seasonal adjustment is estimation of the seasonal component. This problem is discussed by Hannan [14, 15]. Hannan considers estimation of the parameters of a certain model of the seasonal component by "ordinary" least squares after removal of trend by application of a moving average.

Advantages of a moving average transformation for trend removal have been stressed by Kendall [18, p. 372 ff]. One method for choosing a moving average is to fit a "moving polynomial" to the original time series and to subtract the resulting estimate of trend from the series. Kendall shows that this procedure may be represented as a certain moving average transformation of the original series; the coefficients of the moving average are determined by the degree of the moving polynomial and the number of observations to which the polynomial is fitted. As an example, Kendall considers the fitting of a polynomial of degree k to each $k+1$ successive observations. The principal advantage of trend removal by some kind of moving average appears to be that the method is extremely "flexible"; it seems difficult to imagine a model of trend which could not be approximated closely by a "moving polynomial" as described by Kendall.

Very recently, Durbin [4, 5] has raised some doubts about the flexibility of moving average methods for trend removal. He considers removal of trend by a centered moving average followed by estimation of the seasonal component by ordinary least squares, as discussed by Hannan. Durbin shows that in this method each element of the seasonally adjusted series is obtained by subtracting from the original series the deviation of the seasonal mean from the mean of all observations together with a term that depends on only a few observations at each end of the original series. Since seasonal adjustment by least squares with no removal of trend involves subtracting from the original series the deviation of the seasonal mean from the mean of all observations, this result casts serious doubt on the flexibility of the moving average as a method for trend removal.

Aside from the doubts the flexibility of moving average methods raised by Durbin, there is at least one additional difficulty with the method for seasonal adjustment considered by Hannan. If the random component of the original time series is distributed independently over time, the random component of a moving average of this series is not distributed independently. Kendall [18] refers to this alteration in the distribution of the random component as the "Slutsky-Yule effect." To justify the application of an ordinary least squares method for estimation of the seasonal component, in the presence of a random component not distributed independently over time, Hannan appeals to a result of Grenander [11] to the effect that least squares estimates for the parameters of the deterministic component of a time series with a *stationary* random component are asymptotically efficient for a certain class of deterministic components. Under the assumption that a moving average transformation for removal of trend from the original series results in a random component which is stationary, Hannan shows that ordinary least squares estimates for parameters of a certain model of the seasonal component are asymptotically efficient. Hannan also gives asymptotic standard errors for the ordinary least squares estimates as well as exact standard errors.

Two observations must be made about the applicability of Hannan's results. First, when the random component of the original series is distributed independently over time, it is possible to calculate the so-called "Slutsky-Yule effect" exactly from the moving average transformation of the original observations. Given the "Slutsky-Yule effect," small sample standard errors for the ordinary least squares estimator may be obtained, as indicated by Hannan. These standard errors may be employed in the usual way for inference in small samples, so that there is no need to appeal to asymptotic theory as a basis for inference. More significantly, given the "Slutsky-Yule effect" it is possible to improve upon the efficiency of ordinary least squares estimates in small samples; in fact, it is possible to obtain the unique, minimum variance, linear, unbiased estimator of the parameters of the seasonal component. Secondly Hannan's assumption that the moving average transformation of the original series results in a stationary random component is unnecessarily restrictive. For any moving average transformation whatever it is possible to calculate the "Slutsky-Yule effect" and to use this information to obtain an efficient estimator of the seasonal component.

The second problem of seasonal adjustment is to estimate the spectrum of the random component of the original series after both trend and seasonal components have been removed. Hannan discusses this problem where the seasonal component is estimated by "ordinary" least squares applied to a series of observations transformed by a moving average. This problem is also discussed by Nerlove [24, 25], Nettheim [27], and Rosenblatt [31]. We consider the problem of estimating the variance of the random component under the assumption that the random component is stationary and serially independent. Estimation of the spectrum of a serially dependent random component of the original series after both trend and seasonal components have been removed is a straightforward application of standard theory as presented by Grenander [11], Grenander and Rosenblatt [12], and Hannan [16], and will not be considered in this paper.

The purpose of this paper is to complete the development of a statistical theory of seasonal adjustment. This theory is based on a statistical model of an economic time series which satisfies the hypotheses of the general linear statistical model.[1] This statistical model includes that considered by Durbin and Hannan and by many previous investigators of the problem of seasonal adjustment. For this model it is possible to obtain estimates of both trend and

[1] For a discussion of point estimation and statistical inference in the general linear statistical model, a standard reference is Scheffé [32], Chapters 1 and 2.

seasonal components which are minimum variance, linear, and unbiased for samples of any size. This estimator for the seasonal component cannot be obtained by "ordinary" least squares after trend removal by a moving average transformation, so that Hannan's estimator of this component is inefficient in small samples. Secondly, removal of trend may be approached in two alternative and equivalent ways. Trend removal may be represented as a moving average transformation of the original time series (more generally, as an arbitrary linear transformation); alternatively, trend and seasonal components may be estimated simultaneously. These alternative representations of trend removal result in an identical estimator for the seasonal component. Simultaneous estimation of trend and seasonal components requires an explicit representation of the model for the trend component. This model may be examined directly to see just how "flexible" any proposed method for trend removal really is.

The framework provided by the general linear statistical model provides two important points of contact with existing statistical theory. First, properties of the estimators of the trend and seasonal components may be derived from statistical theory for the general linear model. Sampling theory required for statistical inference about a method for seasonal adjustment may be derived from sampling theory for the general linear model. Secondly, methods for estimation of trend and seasonal components may be derived directly from the general theory of regression analysis for stationary time series as developed by Parzen [29]; Parzen's approach is, of course, much more general than that employed in this paper.

The chief disadvantage of a theory of seasonal adjustment based on the general linear statistical model is that this theory does not cover the methods of seasonal adjustment most widely used in practice—ratio-to-moving average methods. An alternative approach to a theory of seasonal adjustment would be to attempt to characterize ratio-to-moving average methods statistically. It would be gratifying, both to statisticians and to practitioners, to be able to present a straightforward statistical theory for seasonal adjustment covering procedures currently in use, simplifying or perhaps justifying these procedures from the statistical point of view, and clearing up the remaining difficulties in application. But it is not easy to argue that there is or could be a straightforward statistical theory for ratio-to-moving average methods. This situation makes practitioners uneasy, since choices among alternative methods for seasonal adjustment, must be made on the basis of criteria which have no basis in statistical inference. As for the statisticians, most of them appear to be unwilling to study seasonal adjustment as it is carried out in practice from the statistical point of view.[2]

The theory of seasonal adjustment presented in this paper is an attempt to a reasonable compromise between the demands of practice and the demands of statistical theory. It is not implied or even suggested that the proposed theory has withstood the test of practical usefulness. Although the theory represents a fairly radical departure from current practice, no attempt has been made to test out the proposed theory of seasonal adjustment in practical situations. Insofar

[2] Undoubtedly, part of the reluctance of statisticians to study ratio-to-moving average methods stems from the difficulty of the statistical problems involved.

as this new approach is consistent with the demands of practice, it should open up an important new line of research—the selection of methods for seasonal adjustment for particular purposes on the basis of standard methods for statistical inference.

2. THE STATISTICAL MODEL

The first step in constructing a theory for seasonal adjustment is to specify the statistical model of an economic time series. Throughout the following discussion, seasonal adjustment is studied for a statistical model which satisfies the hypotheses of the general linear statistical model. The central feature of the model is that an economic time series may be decomposed into deterministic and random components and that the deterministic component may be further decomposed into seasonal and non-seasonal components. Decomposition of an economic time series into seasonal and non-seasonal deterministic components and a random component is standard in the literature of economic statistics. Every discussion of seasonal adjustment presupposes a decomposition of this sort. Further, we assume that the three components of an economic time series are *additive.* Where the ith observation on the time series is denoted y_i, each observation may be written as the sum of its three components:

$$y_i = d_i + s_i + \epsilon_i, \qquad (i = 1 \cdots N), \quad (1)$$

where d_i is the (non-seasonal) deterministic component, s_i the seasonal (deterministic) component, and ϵ_i the random component. Many different methods for seasonal adjustment which are special cases of this model have already been proposed in the literature. Examples are given below.

The assumption that the three components of an economic time series are additive excludes ratio-to-moving-average methods of seasonal adjustment, since the statistical model underlying these methods is additive in the non-seasonal deterministic component and (apparently) multiplicative in the seasonal component. The rationale for this particular model has never been developed in any detail; it might be suggested that the method arose, at least in part, as a computationally convenient approximate method for treating a multiplicative model, that is, a statistical model in which the original series is assumed to be multiplicative in its three components. Let Y_i denote the ith observation and D_i, S_i, and E_i the deterministic, seasonal, and error components. The multiplicative model of an economic time series may be written:

$$Y_i = D_i \cdot S_i \cdot E_i, \qquad (i = 1 \cdots N),$$

where each observation is the product of its three components. Taking logarithms of both sides and writing $y_i = \ln Y_i$, and so on, this model may be rewritten in additive form (1). It is obvious from this example that the statistical model of the original economic time series need not be additive; all that is required is that the model be additive for some transformation of the original series. Obviously, the statistical theory would be the same for any transformation which produces additivity in the statistical model (including no transformation at all as a special case).

At this point it is useful to introduce vector notation for the statistical model; where:

$$y = \begin{bmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ y_N \end{bmatrix}, \quad d = \begin{bmatrix} d_1 \\ d_2 \\ \cdot \\ \cdot \\ \cdot \\ d_N \end{bmatrix}, \quad s = \begin{bmatrix} s_1 \\ s_2 \\ \cdot \\ \cdot \\ \cdot \\ s_N \end{bmatrix}, \quad \epsilon = \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \cdot \\ \cdot \\ \cdot \\ \epsilon_N \end{bmatrix},$$

the model may be re-written in the form:

$$y = d + s + \epsilon. \tag{2}$$

Using vector notation it is possible to describe the specification for the statistical model more concisely than if ordinary scalar notation for the model were used as in (1).

To complete the description of the statistical model of an economic time series, it is necessary to specify each of the three components of the series—deterministic, seasonal, and random. Under the general linear statistical model it is natural to assume that the random component is a vector satisfying the hypotheses:

$$E(\epsilon) = 0 \tag{3}$$
$$V(\epsilon) = \omega I,$$

where ω is a constant parameter representing the variance of each element of the random component and I is the identity matrix of order N. Under hypotheses (3), it is assumed that the expected value of each element of the error vector is zero, that each element has a variance equal to ω, and the covariance of any two distinct components is zero. In the usual way, the last of these assumptions may be relaxed. This point will be taken up again in the following section.

For the seasonal component of an economic time series, it is assumed that this component may be written as a linear combination of certain seasonal variables:

$$s = S\sigma, \tag{4}$$

where S is a matrix with columns representing the seasonal variables. The seasonal component s is expressed as a linear combination of the columns of the matrix S where the vector σ is a set of constant coefficients. Since the seasonal component s is taken to be deterministic, the columns of the matrix S, representing the seasonal variables, may be treated as fixed (non-random). It is assumed further that the matrix S is of *full rank*, that is, that the columns of S are linearly independent. This assumption is required for identification of the unknown parameters σ. We will denote the rank of the matrix S by K_s.

A number of different models for the seasonal component satisfying assumption (4) has been proposed in the literature. To take the most frequently discussed example, it may be assumed that the "effect" associated with each season is constant throughout the sample period. Where K_s is the number of seasons, we will denote the seasonal effects by $\sigma_j (j=1 \cdots K_s)$. Each seasonal effect may be treated as an unknown parameter; associated with each parameter is a certain fixed seasonal variable. Let the vector of observations of the jth seasonal variable be denoted s^j $(j=1 \cdots K_s)$; the elements of the vector s^j, $\{s_{ij}\}$, are unity when i is equal to j plus an integer multiplied by the number of seasons K_s. For the case of four seasons, the seasonal variables s^j $(j=1 \cdots 4)$ have the form:

$$s^1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad s^2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad s^3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad s^4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ \vdots \\ 1 \end{pmatrix};$$

the pattern shown for the first four elements of each vector is repeated for all the remaining elements. The seasonal component has the form:

$$s = S\sigma = \sum_{j=1}^{K_s} \sigma_j s^j;$$

for the example of constant season effects and four seasons, the seasonal component is:

$$\begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \\ \vdots \\ s_N \end{pmatrix} = \sigma_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \sigma_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \sigma_3 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \sigma_4 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \vdots \\ \sigma_4 \end{pmatrix};$$

the pattern shown for the first four elements of the seasonal component is repeated for all the remaining elements.

Many other models of a seasonal component satisfying assumption (4) have been proposed in the literature. To take just one other example, Cowden [3] has proposed that for each season the corresponding seasonal component be represented by a polynomial; this model for the seasonal component has also been discussed by Hannan [14]. Clearly, assumption (4) includes a wide range of seasonal models beyond the examples we have mentioned explicitly. An obvious extension of Cowden's proposed model for the seasonal component is that for each season the corresponding seasonal component be represented by any function whatever, provided only that the function can be written as a linear combination of a set of fixed seasonal variables. But even this extension does not exhaust possible models for the seasonal component. One might wish to include seasonal variables which are non-zero for more than one season; as one example, each seasonal effect may be taken as proportional to the number of working days for the corresponding season. The effects of holidays, the number of weekends in a month, weather, strikes, and so on, may be represented in this way.[3] It would be possible to extend the list of examples of models for the seasonal component of an economic time series almost *ad infinitum*. The important point is that assumption (4) as it stands embodies everything about the seasonal component which is required for a complete statistical theory of seasonal adjustment.

For the non-seasonal deterministic component of an economic time series, two alternative assumptions may be made. First, it may be assumed that this component can be written as a linear combination of certain deterministic variables:

$$d = D\delta \tag{5}$$

where D is a matrix with columns representing the (non-seasonal) deterministic variables and δ is a set of constant coefficients. Obviously, the columns of the matrix D may be treated as fixed (non-random). It is assumed further that the matrix D is of *full rank*, that is, that the columns of D are linearly independent. This assumption is required for identification of the unknown parameters δ. We will denote the rank of the matrix D by K_d.

Alternatively, it may be assumed that the non-seasonal deterministic component is removed by a certain linear transformation of the original series of observations. Such a linear transformation for removal of the deterministic component may be represented in the form:

$$0 = Ld, \tag{6}$$

where L is a linear transformation and d is the non-seasonal deterministic component of the original series of observations. Any method for removal of this component which involves only weighted sums and differences of the elements of the vector of observations y may be represented as such a linear transformation. It is assumed further that the matrix L' is of *full rank*, that is, that the rows of L are linearly independent. We will denote the rank of the matrix L by $N-K_d$. From the assumption that the matrix L' is of full rank, it follows that the linear transformation L takes the vector of observations y with N elements into a vector Ly with $N-K_d$ elements.

The specification of the non-seasonal deterministic component of an economic time series has been the subject of a voluminous literature. The essentials of this literature, much of which ante-dates the development of "classical" statistical theory, are reviewed and summarized by Kendall [18]. Kendall distinguishes between two approaches to the problem of trend-removal. First, one may adopt a particular hypothesis about trend under assumption (5) above. The model of trend may then be estimated by standard methods and removed from the original time series by simply subtracting the estimated trend from the original series. Secondly, one may adopt some particular hypothesis about a transformation of the original time series which will remove trend; Kendall considers only cases in which the transformation is a moving average.

[3] Representation of the seasonal component as a function of certain explanatory variables and estimation of the parameters of this function by regression analysis are discussed by Mendershausen [22, 23] and Eisenpress [8]. An illustration of this approach is provided by a recent publication of the Netherlands Central Bureau of Statistics [26].

Clearly, any moving average of the original observations removing trend is a linear transformation for trend removal as in (6) above.

A number of specific models for the non-seasonal deterministic component satisfying assumptions (5) or (6) has been discussed in the recent literature. In the most familiar example[4] the deterministic component is taken to be a polynomial in time. Such a deterministic component can be written in the form (5) where the columns of the matrix D have elements which are powers of time and the elements of the vector δ are the coefficients of the polynomial. An ordinary regression situation in which the deterministic component is taken to be linear in an arbitrary set of explanatory (predetermined) variables has been considered in detail by Lovell [21]. Perhaps the most familiar example of a method for trend removal (6) is the centered moving average with all seasons weighted equally. Under this method a moving average of the original observations is computed; the moving average is subtracted from the original observations to remove trend. A moving average transformation for removal of trend has been discussed recently by Leong [20]. Many additional examples of models of the non-seasonal deterministic component satisfying hypotheses (5) or (6) could be given.

Apparently, Kendall and others consider the assumptions (5) and (6) to be distinct. For example, in Hannan's recent paper on seasonal adjustment [14], assumption (5) is considered only in the observation: "The use of regression methods to remove trend is a straightforward application of standard theory and will not be discussed here" [14, p. 6]. Of course, it is obvious that under assumption (5) one can always devise a linear transformation which will remove trend; one such linear transformation is the projection of the vector of observations y on the linear subspace of Euclidean N-space orthogonal to any basis for the columns of the matrix D; where D is of full rank (all the columns of D are linearly independent), the matrix representation of this projection is $I-D(D'D)^{-1}D'$. Setting

$$L^* = I - D(D'D)^{-1}D', \tag{7}$$

we apply this linear transformation to the non-seasonal deterministic component under assumption (5):

$$L^*d = L^*D\delta = [I - D(D'D)^{-1}D']D\delta = [D - D]\delta = 0.$$

Although the linear transformation L^* in (7) annihilates the trend component, this transformation is not of full rank. In fact, if the matrix D is of rank K_d, the matrix L^* is of rank $N-K_d$, where the order of L^* is equal to N, the number of observations. To obtain a linear transformation of full rank, we may take any basis of the subspace of Euclidean N-space spanned by the rows of L^*. Let such a basis be denoted by $\{l^1, l^2 \cdots l^{N-K_d}\}$. Let L be a matrix with rows $\{l^1, l^2 \cdots l^{N-K_d}\}$; obviously, L is of full rank. Further:

$$Ld = LD\delta = 0,$$

[4] Representation of the trend component as a polynomial in time has been discussed by Jones [17], Kendall [18], Hald [13], and Hannan [14, 15, 16]. The paper by Jones provides the first example in the literature of estimation of a model of seasonal and trend simultaneously by ordinary least squares. This approach to a statistical model with a seasonal component is also employed by Hald.

since each row of L is orthogonal to each column of D. But L satisfies the assumptions of the model of the deterministic component given by (6). Hence, if a model of the trend component of an economic time series satisfies assumption (5) it must satisfy (6) as well.[5] The fact that assumption (5) implies assumption (6) is, of course, a very familiar one. Even in the older literature fragmentary results of this kind can be found. For example, the usual justification for the variate difference transformation to remove trend is that the trend component can be characterized as a polynomial in time of order, say, p. The corresponding transformation L consists of first differencing each observation p times.

It is somewhat less familiar, perhaps even surprising, that if the model of a trend component of an economic time series satisfies assumption (6) it must also satisfy assumption (5). This fact has two important consequences. First, trend removal by moving average methods (or, more generally, any linear method) may be characterized as a problem in regression. There is no need for separate discussion of moving average and regression methods for trend removal; these methods are equivalent, so that trend removal by any such method may be taken as a straightforward application of standard theory. Secondly, for any linear method of trend removal it is possible to construct the corresponding model of the trend component in the form given by (5). From this form of the model it is possible to find out just how "flexible" a given model for trend really is. For moving average methods of trend removal, construction of the model of the trend component (5) is especially straightforward.

In the discussion that follows, we will first demonstrate the equivalence of assumptions (5) and (6) and then go on to discuss the construction of a model for the trend component of the form given by (5). To demonstrate the equivalence of assumptions (5) and (6) it suffices to show that (6) implies (5), since we have already shown that (5) implies (6). Where the linear transformation L in (6) is of full rank, we may select any basis of the subspace of Euclidean N-space orthogonal to the rows of L, that is, any basis of the subspace of vectors d for which:

$$Ld = 0,$$

as in (6). If L has rank $N-K_d$, this subspace has dimension K_d; denote any basis of the subspace by $\{d^1, d^2 \cdots d^{K_d}\}$. Now, let D be a matrix with columns $\{d^1, d^2 \cdots d^{K_d}\}$; then we may write:

$$d = \sum_{j=1}^{K_d} \delta_j d^j = D\delta, \tag{8}$$

for any vector d satisfying (6) by the definition of a basis; but this is assumption (5), since the matrix D is of full rank. Hence, if a model of the trend component of an economic time series satisfies assumption (5) it must satisfy (6) as well.

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[5] A technique similar to that used here for the analysis of the trend component is used by Lovell [21] for analysis of the seasonal component.

We conclude that the characterizations of the trend component given by assumptions (5) and (6) are equivalent.

Algebraically, there are many ways to choose the basis of the subspace orthogonal to the rows of L in (6) and many ways to choose the basis of the subspace orthogonal to the columns of D in (5). But for the matrix D corresponding to any basis of the subspace orthogonal to the rows of L in (6), the rows of L themselves provide a basis for the subspace orthogonal to the columns of D. It is in this essentially geometric sense that the two representations of the model for the trend component are equivalent. It will be demonstrated that the unique, minimum variance, linear, unbiased estimator of the parameters of the seasonal component is invariant with respect to the choice of a basis for the subspace orthogonal to the rows of L in (6) or to the columns of D in (5); the estimators of the parameters of the seasonal component are *algebraically* equivalent for the two alternative representations of the non-seasonal deterministic component.

We turn next to the construction of a basis for the subspace of Euclidean N-space orthogonal to the rows of the matrix L. In the general situation, this problem reduces to that of solving K_d sets of K_d linear homogeneous equations in N unknowns:

$$Ld^j = 0, \qquad (j = 1 \cdots K_d).$$

This is a standard problem in algebra of little statistical interest and we will not discuss the general situation in detail. We turn instead to the special case in which the linear transformation L is a moving average transformation. In this case a certain moving average annihilates the non-seasonal deterministic component; let the weights of the moving average transformation be represented in scalar form by $\{m_0, m_1 \cdots m_{K_d}\}$ where K_d+1 is the order of the moving average. The coefficients of m may be positive or negative; it is assumed that the coefficients $\{m_0, m_{K_d}\}$ are not zero. Condition (6), that the moving average transformation removes the deterministic component, may be represented in scalar form as:

$$m_0 d_t + m_1 d_{t+1} + \cdots + m_{K_d} d_{t+K_d} = 0, \quad (t = 1 \cdots N - K_d). \tag{9}$$

This condition is, of course, a homogeneous difference equation of order K_d in the elements of the deterministic component. The matrix representation of condition (6) in this case is:

$$Ld = \begin{bmatrix} m_0 & m_1 \cdots m_{Kd} & 0 & \cdots 0 \\ 0 & m_0 \cdots m_{K_{d-1}} & m_{K_d} & \cdots 0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ \cdot \\ \cdot \\ d_N \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{bmatrix} = 0.$$

Each row of the matrix L is obtained by moving each element of the preceding row one column to the right and taking the first element of the row to be the last element of the preceding row. The matrix L has rank $N-K_d$; this matrix is of full rank so that the assumptions under specification (6) are satisfied.

From the characteristic solutions of the difference equation corresponding to condition (9) it is possible to construct a basis for the subspace of Euclidean N-space orthogonal to the rows of the matrix L. First, suppose that the scalar λ is a characteristic root of the polynomial:

$$m_0 + m_1\lambda + \cdots + m_{K_d}\lambda^{K_d} = 0; \qquad (10)$$

then the sequence $\{\lambda^t\}$ is a characteristic solution of the difference equation (9). Secondly, we construct a column vector d with elements equal to the first N elements of the characteristic solution, say $\{\lambda, \lambda^2 \cdots \lambda^N\}$; this column vector is orthogonal to the rows of L from (10):

$$Ld = \begin{bmatrix} m_0 & m_1 \cdots m_{K_d} & 0 \cdots 0 \\ 0 & m_0 \cdots m_{K_{d-1}} & 0 \cdots 0 \\ \cdot & \cdot \quad\quad \cdot & \cdot \quad\quad \cdot \\ \cdot & \cdot \quad\quad \cdot & \cdot \quad\quad \cdot \\ \end{bmatrix} \begin{bmatrix} \lambda \\ \lambda^2 \\ \cdot \\ \cdot \\ \lambda^N \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{bmatrix} = 0.$$

Thirdly, let the set of scalars $\{\lambda_1 \lambda_2 \cdots \lambda_{K_d}\}$ be the set of characteristic roots of the polynomial (10), not necessarily distinct. Then the set of characteristic solutions corresponding to this set of characteristic roots is a linearly independent set, each element of which is orthogonal to the rows of L; but then the set of characteristic solutions spans the subspace of Euclidean N-space orthogonal to the rows of L and is, hence, a basis of this subspace. Of course, some of the roots of the polynomial (10) may be complex; but complex roots occur in conjugate pairs, so that from the set of characteristic solutions corresponding to complex roots it is possible to obtain directly a basis of the subspace orthogonal to the rows of L for which each element in the basis is represented as a vector with real elements. We take the elements of this basis as the columns of the matrix D in (5).

To illustrate the construction of a basis for the subspace orthogonal to the rows of a certain moving average transformation L, we consider an example in which the deterministic component is removed by a centered moving average with as many elements as there are seasons, with all seasons weighted equally. in this example, we suppose that there are four seasons so that $K_d = K_s = 4$. The weights of the moving average may be represented in scalar form by $\{-1/8, -1/4, 3/4, -1/4, -1/8\}$. The condition (6) that the moving average transformation removes the deterministic component may be represented in scalar form as:

$$-1/8\ d_t - 1/4\ d_{t+1} + 3/4\ d_{t+2} - 1/4\ d_{t+3} - 1/8\ d_{t+4} = 0, \qquad (t = 1 \cdots N-4);$$

to remove trend, we subtract an ordinary centered moving average from each observation.

The characteristic roots of the polynomial:

$$-1/8 - 1/4\,\lambda + 3/4\,\lambda^2 - 1/4\,\lambda^3 - 1/8\,\lambda^4 = 0,$$

are $\{1, 1, 2+\sqrt{3}, 2-\sqrt{3}\}$; unity is a repeated root of multiplicity two. The set of characteristic solutions corresponding to these roots is $\{1, t, (2+\sqrt{3})^t, (2-\sqrt{3})^t\}$. The elements of the basis of the subspace of Euclidean N-space orthogonal to the rows of L are:

$$d^1 = \begin{bmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ 1 \end{bmatrix}, \quad d^2 = \begin{bmatrix} 1 \\ 2 \\ \cdot \\ \cdot \\ N \end{bmatrix}, \quad d^3 = \begin{bmatrix} 2+\sqrt{3} \\ (2+\sqrt{3})^2 \\ \cdot \\ \cdot \\ (2+\sqrt{3})^N \end{bmatrix}, \quad d^4 = \begin{bmatrix} 2-\sqrt{3} \\ (2-\sqrt{3})^2 \\ \cdot \\ \cdot \\ (2-\sqrt{3})^N \end{bmatrix};$$

the model for trend may be written in the form (5):

$$d = D\delta = \delta_1 d^1 + \delta_2 d^2 + \delta_3 d^3 + \delta_4 d^4,$$

or more explicitly in the form:

$$\begin{bmatrix} d_1 \\ d_2 \\ \cdot \\ \cdot \\ d_N \end{bmatrix} = \delta_1 \begin{bmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ 1 \end{bmatrix} + \delta_2 \begin{bmatrix} 1 \\ 2 \\ \cdot \\ \cdot \\ N \end{bmatrix} + \delta_3 \begin{bmatrix} 2+\sqrt{3} \\ (2+\sqrt{3})^2 \\ \cdot \\ \cdot \\ (2+\sqrt{3})^N \end{bmatrix} + \delta_4 \begin{bmatrix} 2-\sqrt{3} \\ (2-\sqrt{3})^2 \\ \cdot \\ \cdot \\ (2-\sqrt{3})^N \end{bmatrix}.$$

The model for trend consists of a constant term δ_1, a linear trend corresponding the unknown parameter δ_2, a geometrically increasing component $\{(2+\sqrt{3})^t\}$, which nearly quadruples every quarter, and a geometrically declining component $\{(2-\sqrt{3})^t\}$. Such a model of trend can hardly be called "flexible."

The preceding discussion can be summarized by giving a complete specification of the statistical model for an economic time series. Combining the additivity assumption (2) with the assumptions (4) and (5) that both seasonal and deterministic components are linear, the complete statistical model may be written:

$$y = D\delta + S\sigma + \epsilon, \tag{11}$$

where the random component ϵ satisfies assumptions (3):

$$E(\epsilon) = 0$$
$$V(\epsilon) = \omega I.$$

It is assumed that the matrix of deterministic and seasonal variables, $[DS]$ is of full rank, that is, that the columns of this matrix are linearly independent. This assumption is required for identification of the unknown parameters $\{\delta, \sigma\}$. The assumption implies that D and S are each of full rank. The matrix $[D\ S]$ is taken to be fixed (non-random), since each of the matrices S and D is taken to be fixed. This completes the specification of the statistical model of an economic time series.

As an example of the statistical model of an economic time series (11), we take a model for quarterly time series with constant seasonal effects where the trend component is removed by a centered moving average with all seasons weighted equally. The model may be written:

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ \cdot \\ \cdot \\ y_N \end{bmatrix} = \begin{bmatrix} 1 & 1 & (2+\sqrt{3}) & (2-\sqrt{3}) & 1 & 0 & 0 & 0 \\ 1 & 2 & (2+\sqrt{3})^2 & (2-\sqrt{3})^2 & 0 & 1 & 0 & 0 \\ 1 & 3 & (2+\sqrt{3})^3 & (2-\sqrt{3})^3 & 0 & 0 & 1 & 0 \\ 1 & 4 & (2+\sqrt{3})^4 & (2-\sqrt{3})^4 & 0 & 0 & 0 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & N & (2+\sqrt{3})^N & (2-\sqrt{3})^N & 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \\ \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \end{bmatrix} + \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \cdot \\ \cdot \\ \epsilon_N \end{bmatrix}. \quad (12)$$

This form of the model for a quarterly time series does not satisfy the hypotheses of the statistical model (11) because the sum of the first four columns is equal to the fifth, so that the matrix $[D\ S]$ is not of full rank even though the matrix D and the matrix S, considered separately, are of full rank. This implies that not all of the parameters of the vectors σ and δ may be identified. To reduce the matrix $[D\ S]$ to full rank, it is necessary to eliminate one of the unknown parameters.

There are many ways in which the matrix $[D\ S]$ may be reduced to full rank; in this example, a natural way appears to be to set the sum of the seasonal effects equal to zero:

$$\sum_{j=1}^{4} \sigma_j = 0.$$

Where the statistical model is additive in the original observations, this restriction has an interesting interpretation. It implies that the annual sums of seasonally adjusted and seasonally unadjusted data must be the same. Of course, this interpretation does not carry over to situations in which the statistical model is additive in some transformation of the original set of observations. To eliminate one of the unknown parameters, this restriction may be solved for one of the seasonal effects, say the fourth:

$$\sigma_4 = -\sigma_1 - \sigma_2 - \sigma_3.$$

The model for the seasonal component may be re-written:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{bmatrix}$$

Combining the reduced model for the seasonal component with the model for the non-seasonal deterministic component in (12), we obtain:

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ \vdots \\ \vdots \\ y_N \end{bmatrix} = \begin{bmatrix} 1 & 1 & (2+\sqrt{3}) & (2-\sqrt{3}) & 1 & 0 & 0 \\ 1 & 2 & (2+\sqrt{3})^2 & (2-\sqrt{3})^2 & 0 & 1 & 0 \\ 1 & 3 & (2+\sqrt{3})^3 & (2-\sqrt{3})^3 & 0 & 0 & 1 \\ 1 & 4 & (2+\sqrt{3})^4 & (2-\sqrt{3})^4 & -1 & -1 & -1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & N & (2+\sqrt{3})^N & (2-\sqrt{3})^N & -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \\ \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{bmatrix} \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \cdot \\ \cdot \\ \epsilon_N \end{bmatrix}.$$

For this form of the model, the matrix $[DS]$ is of full rank.

3. MINIMUM VARIANCE, LINEAR, UNBIASED SEASONAL ADJUSTMENT

The next step in constructing a statistical theory for seasonal adjustment will be to derive the unique, minimum variance, linear, unbiased estimator of the parameters of the seasonal component. For the statistical model of an economic time series described in the previous section, this derivation is a straightforward application of the theory of estimation for the general linear statistical model. The problem of estimation may be approached in two alternative ways. First, we consider estimation of the parameters of the seasonal component where the non-seasonal deterministic or trend component has been removed by a linear transformation of the original set of observations, for example, a moving average transformation of the observations. In matrix notation such a linear transformation may be represented as in (6):

$$Ly = L(D\delta + S\sigma + \epsilon) = LS\sigma + L\epsilon. \tag{13}$$

The matrix L representing the transformation of the original set of observations to remove the deterministic component is of full rank. The matrix D has as columns the elements of a basis of the sub-space of Euclidean N-space orthogonal to the rows of L. It is assumed, as in (11), that the matrix $[DS]$ is of full rank.

The statistical model of the transformed series of observations (13) satisfies the hypotheses of the Gauss-Markov Theorem on estimation in the general linear statistical model as generalized by Aitken [1]. To demonstrate this we observe first that if the matrix $[DS]$ in the model of an economic time series (11) is of full rank, then the matrix LS is of full rank; furthermore, the variance-covariance matrix of the random component for the transformed series is a known positive definite matrix:

$$V(L\epsilon) = \omega LL',$$

where LL' is a positive definite since the matrix L is of full rank. The matrix LL' represents the "Slutsky-Yule effect" as discussed by Kendall [18, pp. 378–87] and others. Obviously the "Slutsky-Yule effect" can be calculated directly from the linear transformation L, whether or not this transformation results in a random component for the transformed series $L\epsilon$ which is stationary.

Somewhat surprisingly, in previous discussions of the statistical model for a transformed series of observations (13) by Durbin [4, 5], Hannan [14, 15], and Nettheim [27], the problem of estimation of the parameters of the seasonal component σ has been approached through application of "ordinary" least squares to the model of the transformed series. The ordinary least squares estimator for the parameters σ, say $\hat{\hat{\sigma}}$, is simply:

$$\hat{\hat{\sigma}} = (S'L'LS)^{-1}S'L'Ly,$$

which has variance-covariance matrix:

$$V(\hat{\hat{\sigma}}) = \omega(S'L'LS)^{-1}S'L'(LL')^{-1}LS(S'L'LS)^{-1}.$$

On the assumption that the random component of the transformed series $L\epsilon$ is stationary and under certain restrictions on the matrix LS, Hannan has demonstrated that the asymptotic variance-covariance matrix of the ordinary least squares estimator is identical to the asymptotic variance-covariance matrix of the unique minimum variance, linear, unbiased estimator.

It would appear that a more natural approach to the estimation of parameters of the seasonal component is through the unique, minimum variance, linear, unbiased estimator itself. In addition to having the same asymptotic efficiency as the least squares estimator under the conditions stated by Hannan, the minimum variance estimator is efficient (in the class of linear unbiased estimators) for samples of any size without any restrictions beyond those given in the specification of the statistical model (11) for an economic time series. Furthermore, this estimator gives rise to statistics for inference which have the usual optimality properties for tests of hypotheses or confidence interval estimation. It will be shown below that the computational difficulty associated with calculation of the unique minimum variance estimates is no greater than that associated with calculation of the ordinary least squares estimates.

The unique minimum variance, linear, unbiased estimator, say σ, of the parameters of the seasonal model σ is:

$$\hat{\sigma} = [S'L'(LL')^{-1}LS]^{-1}S'L'(LL')^{-1}Ly, \tag{14}$$

which has variance-covariance matrix:

$$V(\hat{\sigma}) = \omega[S'L'(LL')^{-1}LS]^{-1} \tag{15}$$

The properties of unbiasedness and efficiency *in finite samples* for this estimator follow directly from the Gauss-Markov Theorem on estimation of the general linear statistical model as generalized by Aitken [1].

To complete the theory of estimation for the statistical model of the transformed series of observations (13), it is necessary to give the usual unbiased estimators for the parameter ω under "ordinary" least squares and minimum variance estimation. For the "ordinary" least squares estimator of the parameters of the seasonal component, the residuals from the fitted regression, say $\hat{\hat{\epsilon}}$, are defined as follows:

$$\hat{\hat{\epsilon}} = Ly - LS\hat{\hat{\sigma}};$$

an unbiased estimator of the parameter ω, say $\hat{\hat{\omega}}$, is:

$$\hat{\hat{\omega}} = \frac{\hat{\hat{e}}'(LL')^{-1}\hat{\hat{e}}}{N - (K_d + K_s)} .$$

For the minimum variance estimator of the parameters of the seasonal component, residuals from the fitted regression, say $\hat{e}$, are defined as follows:

$$\hat{e} = Ly - LS\hat{\sigma};$$

an unbiased estimator of the parameter ω, say $\hat{\omega}$, is:

$$\hat{\omega} = \frac{\hat{e}'(LL')^{-1}\hat{e}}{N - (K_d + K_s)} . \tag{16}$$

Where the random component of the original series of observations is normally distributed, the estimator (16) of the parameter ω, is the unique, minimum variance, unbiased estimator of ω.[6] In this case the estimator (14) is the unique, minimum variance estimator of the parameters σ in the class of unbiased estimators; restriction to the class of linear estimators is no longer required.[7]

Secondly, we consider estimation of the parameters of the seasonal component where the non-seasonal deterministic or trend component is represented by a certain linear combination of fixed non-seasonal deterministic variables as in (5); in matrix notation, the statistical model of an economic time series may be written:

$$y = D\delta + S\sigma + \epsilon$$

as in (11). This model of an economic time series satisfies the hypotheses of the general linear statistical model; applying the Gauss-Markov Theorem, the unique, minimum variance, linear, unbiased estimator of the parameters $\{\delta, \sigma\}$, say $\{\hat{\delta}, \hat{\sigma}\}$, is the ordinary least squares estimator:

$$\begin{bmatrix} \hat{\delta} \\ \hat{\sigma} \end{bmatrix} \begin{bmatrix} D'D & D'S \\ S'D & S'S \end{bmatrix}^{-1} \begin{bmatrix} D'y \\ S'y \end{bmatrix},$$

which may be written as:

$$\begin{aligned} \hat{\delta} &= (D'D)^{-1}D'(I - S\{S'[I - D(D'D)^{-1}D']S\}^{-1}S'[I - D(D'D)^{-1}D'])y, \\ \hat{\sigma} &= \{S'[I - D(D'D)^{-1}D']S\}^{-1}S'[I - D(D'D)^{-1}D']y. \end{aligned} \tag{17}$$

The estimator, $\hat{\sigma}$, of the parameters of the seasonal component of the statistical model of an economic time series (11) is identical to the estimator derived from the statistical model for the transformed observations (13). To demonstrate this, it is necessary to show that:

$$I - D(D'D)^{-1}D' = L'(LL')^{-1}L.$$

[6] See, for example, Graybill [10, p. 113].
[7] See Graybill [10, p. 113].

We observe first that the matrix $[L'D]$ is N by N and of full rank, since if:

$$L'\alpha_1 + D\alpha_2 = 0,$$

for $\{\alpha_1', \alpha_2\}$ with elements not all zero,

$$L(L'\alpha_1 + D\alpha_2) = LL'\alpha_2 = 0,$$
$$D'(L'\alpha_1 + D\alpha_2) = D'D\alpha_1 = 0;$$

but the matrices LL' and $D'D$ are positive definite, hence all elements of α_1 and α_2 are zero, a contradiction. Since the matrix $[L'D]$ is of full rank, the orthogonal projection on the subspace of Euclidean N-space spanned by the columns of this matrix (that is, the whole space) is the identity. Hence,

$$I = [L'D]\begin{bmatrix} LL' & 0 \\ 0 & D'D \end{bmatrix}^{-1} [L'D]',$$
$$= L'(LL')^{-1}L + D(D'D)^{-1}D',$$

as was to be demonstrated.[8] We conclude that the estimator of the parameters of the seasonal component derived from the statistical model (11) is identical to the estimator derived from the statistical model for the transformed series (13).

The variance-covariance matrix of the least squares estimator (17) is:

$$\begin{bmatrix} \hat{\delta} \\ \hat{\sigma} \end{bmatrix} = \omega \begin{bmatrix} D'D & D'S \\ S'D & S'S \end{bmatrix}^{-1}; \tag{18}$$

the variance-covariance matrix (15) for the estimator of the parameters of the seasonal model (14) may be derived from this expression. To complete the theory of estimation for the statistical model of an economic time series (11), it is necessary to give the unbiased estimator for the parameter ω corresponding to the minimum variance estimators, $\{\hat{\delta}, \hat{\sigma}\}$. The residuals from the fitted regression, say e, are defined as follows:

$$e = y - D\hat{\delta} - S\hat{\sigma};$$

an unbiased estimator of the parameter ω, say $\hat{\omega}$, is:

$$\hat{\omega} = \frac{e'e}{N - (K_d + K_s)}, \tag{19}$$

which is identical to the estimator (16) of ω. To show this, it suffices to show that:

$$e'e = \hat{e}'(LL')^{-1}\hat{e},$$

from (16) and (19). The sum of squared residuals from (19) may be written in the form $\epsilon' M_{DS}\epsilon$, where M_{DS} is the idempotent matrix:

$$M_{DS} = I - [DS]\begin{bmatrix} D'D & D'S \\ S'D & S'S \end{bmatrix}^{-1} [DS]';$$

[8] This proof was suggested to me by William Kruskal.

letting M_D be the idempotent matrix:

$$M_D = I - D(D'D)^{-1}D',$$

the matrix M_{DS} may be expressed as:

$$M_{DS} = M_D - M_DS\{S'M_DS\}^{-1}S'M_D.$$

Secondly, from the fact that:

$$\hat{e} = Ly - LS\hat{\sigma} = L[I - D\hat{\delta} - S\hat{\sigma}] = LM_{DS}\epsilon,$$

the sum of squared residuals from (16) may be written:

$$\hat{e}'(LL')^{-1}\hat{e} = \epsilon'M_{DS}L'(LL')^{-1}LM_{DS}\epsilon = \epsilon'M_{DS}\epsilon = e'e,$$

as was to be demonstrated.

To summarize, we have shown that the unique minimum variance estimators (14) and (17) of the parameters σ of the seasonal component and the unbiased estimators (16) and (19) of the variance of the random component are identical. These results are independent of the choice of a basis for the sub-space of Euclidean N-space orthogonal to the rows of the linear transformation L in proceeding from the representation (6) of the non-seasonal deterministic component to the alternative representation (5) or the choice of a basis for the sub-space of Euclidean N-space orthogonal to the columns of D is proceeding from (5) to (6). For minimum variance, linear, unbiased estimation of the parameters of the seasonal component of an economic time series and for unbiased estimation of the variance of the random component, the two alternative representations of the trend component (5) and (6) are equivalent. Where the random component of the original time series is normally distributed, the estimator (19) is, as indicated previously, the unique, minimum variance, unbiased estimator of the variance of the random component; in this case the estimator (17) of the parameters of the seasonal component is the unique, minimum variance, unbiased estimator without restriction to the class of linear estimators.

Representation of the statistical model for an economic time series in the form (11) as opposed to (13) has the advantage that the form of the model for the trend component is given explicitly. The particular form for the trend component depends, of course, on the choice of a basis for the sub-space of Euclidean N-space orthogonal to the rows of the matrix L in (13). However, any particular form for the trend component provides the information necessary to decide just how "flexible" a given model for the trend component given as a linear transformation of the original observations as in (13) really is. Representation of the statistical model in the form (11) has the additional advantage that an explicit estimator of the parameters of the trend component $\hat{\delta}$ may be obtained as in (17); this estimator may be used as the basis for inference about the trend component. Such an estimator is essential for inference about trend and seasonal components simultaneously; we will return to this point in the following section. Of course, for inference about the seasonal component alone, an explicit estimator of the parameters of the trend component is not required. Finally, the unique, minimum variance, linear, unbiased estimator (17) for the parameters of both seasonal and trend components is an ordinary least squares estimator so that the computation of estimates for any given sample is a routine least squares computation. Computation of the corresponding estimates for parameters of the seasonal component from the transformed data using the formulas (14) would require a different type of calculation.[9]

To complete the statistical theory of seasonal adjustment for an economic time series, it is necessary to characterize the process of seasonal adjustment. If the seasonal component of an economic time series were known, that is, if the parameters σ of the seasonal component were known, the seasonal component could be removed from the original series by simple subtraction; where the original series of observations is denoted y, let y_S denote the seasonally adjusted series:

$$y_S = y - S\sigma = D\delta + \epsilon; \tag{20}$$

like the original series, the seasonally adjusted series has both deterministic and random components. Of course, in any practical problem the seasonal component of an economic time series is unknown. Where the seasonal component is unknown, it is customary to speak of "removing" the seasonal in a rather loose fashion. It is frequently alleged, for example, that this or that method does not remove the seasonal component or "under-adjusts" or, alternatively, that a method removes more than the seasonal component or "over-adjusts." In general, the only definition of the term *method for seasonal adjustment* which appears to be appropriate is simply that a method for seasonal adjustment is a transformation of the original set of observations which takes the observations into a "seasonally adjusted" set of observations. To stipulate further that such a transformation must "remove the seasonal component" would rule out "under-adjustment" and "over-adjustment" by definition.[10]

We limit further consideration to linear methods of seasonal adjustment; a method of seasonal adjustment is defined as *linear* if and only if the seasonally adjusted series may be obtained from the original series of observations by a linear transformation. For an arbitrary linear method of seasonal adjustment let B represent the linear transformation of the original series by which the seasonally adjusted series is obtained; let y_S^B represent the corresponding seasonally adjusted series. The method of seasonal adjustment may be written in the form:

$$y_S^B = By.$$

[9] The calculation required involves inversion of the matrix $(LL')^{-1}$ prior to computation of the "weighted" moment matrices $S'L'(LL')^{-1}LS$ and $S'L'(LL')^{-1}Ly$; once the moment matrices are computed, the computation is essentially an ordinary least squares computation. This computation together with the transformation of the original data is equivalent in difficulty to direct computation of the estimator $\hat{\sigma}$ from the formulas in (17). An additional difficulty in the use of transformed observations as in (14) is that special computer programs must be written first for transformation of the observations and then for computation of the "weighted" moment matrices. Many computer programs are already available for the ordinary least squares computation employed in (17).

[10] It does not appear to be useful to define "seasonally adjusted" observations except as some transformation of the original observations. This problem of definition is analogous to that of defining the term "estimator," which is usually taken to mean any function of the set of observations. Further restrictions on the function are then indicated explicitly as in "linear estimator."

As an example of a linear method for seasonal adjustment, let $\hat{y}_S$ denote a seasonally adjusted series obtained by replacing the parameters σ in (20) by the unique, minimum variance, linear, unbiased estimator of these parameters, $\hat{\sigma}$:

$$\hat{y}_S = y - S\hat{\sigma} = [I - S\{S'M_DS\}^{-1}S'M_D]y; \tag{21}$$

$$B = [I - S\{S'M_DS\}^{-1}S'M_D]. \tag{22}$$

Any method of seasonal adjustment based on a linear estimator of the parameters of the seasonal component σ is a linear method; the "ordinary" least squares estimator $\hat{\hat{\sigma}}$ considered by Hannan [14, 15] and Durbin [4, 5] is a linear estimator so that the corresponding method of seasonal adjustment is a linear method. An obvious property of linear methods of seasonal adjustment is that the seasonally adjusted series is additive in its deterministic and random components:

$$\overset{B}{y_S} = By = By = B(D\delta + S\sigma + \epsilon) = BD\delta + BS\sigma + B\epsilon.$$

Any method for seasonal adjustment may be characterized by the distribution of the difference between the seasonally adjusted series obtained by that method and the seasonally adjusted series which would be obtained if the parameters of the seasonal component were known. For an arbitrary linear method for seasonal adjustment, this difference may be represented in the form:

$$\overset{B}{y_S} - y_S = [B - I]D\delta + BS\sigma + [B - I]\epsilon. \tag{23}$$

An unbiased linear method for seasonal adjustment may be defined as a linear method for which the expected value of the difference (23) is zero; for an unbiased method:

$$E(\overset{B}{y_S} - y_S) = [B - I]D\delta + BS\sigma = 0,$$

so that:

$$\begin{aligned} [B - I]D &= 0, \\ BS &= 0. \end{aligned} \tag{24}$$

An unbiased linear method for seasonal adjustment leaves the trend component unchanged and annihilates the seasonal component. The method for seasonal adjustment based on the estimator $\hat{\sigma}$ as in (21) is unbiased since:

$$- S\{S'M_DS\}^{-1}S'M_DD = 0,$$

$$[I - S\{S'M_DS\}^{-1}S'M_D]S = 0.$$

It is easily verified that the method based on the "ordinary" least squares estimator $\hat{\hat{\sigma}}$ is also unbiased.

Finally, a minimum variance, linear, unbiased method for seasonal adjustment may be defined as a linear unbiased method for which the variance of an arbitrary linear combination of the differences (23) is a minimum. Letting a' denote an arbitrary vector of fixed coefficients, such a linear combination may be represented in the form $a'(y_S^B - y_S)$. For any linear method of seasonal adjustment, the variance of such a linear combination is:

$$V[a'(y_S^B - y_S)] = V[a'(B - I)\epsilon] = \omega a'[B - I][B - I]'a.$$

For a linear unbiased method of seasonal adjustment, this variance may be written in the form:

$$\begin{aligned} V[a'(y_S^B &- y_S)] \\ &= \omega a'[S\{S'M_DS\}^{-1}S' \\ &\quad + (I - S\{S'M_DS\}^{-1}S'M_D - B)(I - S\{S'M_DS\}^{-1}S'M_D - B)']a, \\ &= \omega a'[S\{S'M_DS\}^{-1}S' + S\{S'M_DS\}^{-1}S' \\ &\quad + (B - I)M_DS\{S'M_DS\}^{-1}S' + S\{S'M_DS\}^{-1}S'M_D(B - I)' \\ &\quad + (B - I)(B - I)']a, \end{aligned} \tag{25}$$

where:

$$(B - I)M_DS\{S'M_DS\}^{-1}S' = -S\{S'M_DS\}^{-1}S',$$

since:

$$\begin{aligned} (B - I)M_D &= (B - I)(I - D(D'D)^{-1}D') = B - I, \\ (B - I)S &= -S, \end{aligned}$$

from the conditions for unbiasedness of a linear method for seasonal adjustment (24). The expression (25) for the variance of an arbitrary linear combination of the differences (23) is the sum of two positive semi-definite quadratic forms, $\omega a'S\{S'M_DS\}^{-1}S'a$ and

$$\omega a'(I - S\{S'M_DS\}^{-1}S'M_D - B)(I - S\{S'M_DS\}^{-1}S'M_D - B)'a.$$

The first of these terms does not depend on the linear transformation B used to obtain the seasonally adjusted series from the original series of observations. This term represents a lower bound for the variance of the linear combination of differences (25). The bound is attained by taking the linear transformation B to be equal to the transformation associated with the estimator $\hat{\sigma}$ of the parameters of the seasonal component as in (22). Clearly, this is the only choice of the linear transformation B for which the lower bound for the variance (25) is attained for every possible choice of the vector of fixed coefficients a'.

We can summarize the statistical theory of seasonal adjustment by saying that the method for seasonal adjustment associated with the unique, minimum variance, linear, unbiased estimator of the parameters of the seasonal component $\hat{\sigma}$ as in (21) in the unique, minimum variance, linear, unbiased method for seasonal adjustment of an economic time series (11). Any other linear unbiased method for seasonal adjustment has larger variance for any finite sample size. In particular, the method for seasonal adjustment based on the "ordinary" least squares estimator $\hat{\hat{\sigma}}$ has larger variance for any finite sample size.

It may be observed that the definitions of properties of linearity, unbiasedness, and minimum variance for an arbitrary method for seasonal adjustment do not depend in any way on the use of an estimator for the unknown parameters σ of the seasonal component of an economic time series. We have seen that implicit in the minimum variance, linear, unbiased method for seasonal adjustment there is an estimator for the parameters σ which is itself a minimum variance, linear, unbiased estimator of these parameters. It can be shown that implicit in any linear method there is a linear estimator for the parameters σ; let such a linear estimator, say σ^C, be represented by:

$$\sigma^C = Cy;$$

then, from the definition of a linear method for seasonal adjustment we may write:

$$y_S^B = By = [I - SC]y,$$

where:

$$C = -S_L^{-1}[B - I], \tag{26}$$

with S_L^{-1} representing a left-inverse of the matrix S. Finally, a linear estimator obtained from a linear method for seasonal adjustment is unbiased if and only if the method for seasonal adjustment is unbiased, since the conditions for unbiasedness of a linear estimator σ^C are easily shown to be:

$$CD = 0,$$
$$CS = I;$$

from the expression (26) it may be verified that these conditions are equivalent to the conditions (24) for unbiasedness of a method for seasonal adjustment.

We conclude that any linear method for seasonal adjustment may be represented as taking place in two steps. The first step is to estimate the unknown parameters of the seasonal component of the series; the second step is to remove the estimated seasonal component from the set of original observations. For a linear method of seasonal adjustment, estimation of the unknown parameters of the seasonal component is carried out by means of a linear estimator. For a linear unbiased method, estimation of the unknown parameters is carried out through a linear unbiased estimator. Finally, for the unique, minimum variance, linear, unbiased method of seasonal adjustment, estimation of the unknown parameters of the seasonal component is carried out by means of the unique, minimum variance, linear, unbiased estimator. This completes the statistical characterization of linear methods for seasonal adjustment.

The final step in the construction of a statistical theory for seasonal adjustment is to derive predictors for the original time series and for the seasonally adjusted time series. For prediction it is assumed that a set of N observations on an economic time series is known; these observations are the components of the vector y in the statistical model (11). Further, a set of $N+n$ observations on the set of seasonal and non-seasonal deterministic variables is known. It is customary to identify the first N observations on these variables with the N known observations on the economic time series, y; the problem of prediction is to "forecast" or extrapolate forward in time n additional observations on the economic time series corresponding to the n additional known observations on the set of seasonal and non-seasonal deterministic variables. This problem is formally identical with interpolation or backward extrapolation; each of these problems is a special instance of the general problem of "missing observations" as discussed, for example, by Kruskal [19]. Although we continue to speak of "prediction," the n observations to be "predicted" may occur in time before or after the N known observations or they may be interspersed in time among the known observations.

We denote the N known observations on an economic time series by the vector y; similarly, we denote the n observations to be predicted by the vector y_0. Secondly, we denote the N observations on the non-seasonal and seasonal deterministic variables corresponding to the known observations y by the matrix $[DS]$; we denote the n observations on these variables corresponding to the n observations to be predicted by the matrix $[D_0S_0]$. Similarly, the random components of the observed and predicted observations of the time series are denoted by ϵ and ϵ_0, respectively. From the statistical model for an economic time series (11), the observations to be predicted may be expressed as:

$$y_0 = D_0\delta + S_0\sigma + \epsilon_0. \tag{27}$$

We assume that the random component satisfies:

$$E(\epsilon_0) = 0,$$
$$V\begin{bmatrix} \epsilon \\ \epsilon_0 \end{bmatrix} = \omega\begin{bmatrix} I_N & 0 \\ 0 & I_n \end{bmatrix}, \tag{28}$$

where I_N is an identity matrix of order N and I_n is an identity matrix of order n. The second of the relationships (28) implies the corresponding relationship for the random component of the original series of observations ϵ, as given in assumption (3).

We consider only linear predictors of the vector of unknown observations y_0; a predictor is *linear* if and only if the predictor is a linear transformation of the series of known observations y. For an arbitrary linear predictor let P represent the linear transformation of the original series and let y_0^P represent the predictor itself. Any linear predictor may be written in the form:

$$y_0^P = Py.$$

As an example of a linear predictor, let $\hat{y}_0$ denote the predictor obtained by replacing the parameters $\{\delta, \sigma\}$ in (27) by the unique, minimum variance, linear, unbiased estimator of these parameters $\{\hat{\delta}, \hat{\sigma}\}$ and by setting the random component ϵ_0 equal to expected value, zero:

$$\hat{y}_0 = D\hat{\delta} + S\hat{\sigma}; \tag{29}$$

for this predictor:

$$P = D_0(D'D)^{-1}D'(I - S\{S'M_DS\}^{-1}S'M_D) + S_0\{S'M_DS\}^{-1}S'M_D, \tag{30}$$

from (17). Clearly, any predictor based on a linear estimator of the parameters of the non-seasonal and seasonal deterministic components is a linear predictor.

Any predictor may be characterized by the distribution of the difference between predicted and actual values of the observations to be predicted; for an arbitrary linear predictor this difference may be represented in the form:

$$\begin{aligned} \overset{P}{y_0} - y_0 &= P(D\delta + S\sigma + \epsilon) - (D_0\delta + S_0\sigma + \epsilon_0) \\ &= (PD - D_0)\delta + (PS - S_0)\sigma + P\epsilon - \epsilon_0. \end{aligned} \tag{31}$$

An unbiased linear predictor may be defined as a linear predictor for which the expected value of the difference (31) is zero; for an unbiased predictor:

$$E(\overset{P}{y_0} - y_0) = (PD - D_0)\delta + (PS - S_0)\sigma,$$

so that:

$$\begin{aligned} PD - D_0 &= 0, \\ PS - S_0 &= 0. \end{aligned} \tag{32}$$

A minimum variance, linear, unbiased predictor may be defined as a linear, unbiased predictor for which the variance of an arbitrary linear combination of the differences (31) is a minimum. Letting a' denote an arbitrary vector of fixed coefficients, such a linear combination may be represented in the form $a'(\overset{P}{y_0} - y_0)$. For any linear predictor the variance of such a linear combination is:

$$V[a'(\overset{P}{y_0} - y_0)] = V[a'(P\epsilon - \epsilon_0)] = \omega a'[PP' + I_n]a.$$

It is well known that predictor (29) is the unique, minimum variance, linear, unbiased predictor of the vector of unknown observations, y_0. Unbiasedness follows directly from (30) and (32). The proof of minimum variance is analogous to the proof for the minimum variance property of a method of seasonal adjustment in (25) and the following discussion.

A second problem is to "predict" the n seasonally adjusted observations on the economic time series. We denote the n observations to be predicted by the vector y_{0S}. From the statistical model (20) for the seasonally adjusted series:

$$y_{0S} = y_0 - S_0\sigma = D_0\delta + \epsilon_0. \tag{33}$$

As before, we consider only linear predictors; let P_S represent the linear transformation of the original series (not adjusted for seasonal variation) by which the predictor is obtained; let $\overset{P}{y}_{0S}$ represent the predictor itself. Any linear predictor may be written in the form:

$$\overset{P}{y}_{0S} = P_S y.$$

Let $\hat{y}_{0S}$ denote the predictor obtained by replacing the parameters δ in (33) by the unique, minimum variance, linear, unbiased estimator of these parameters $\hat{\delta}$ and by setting the random component ϵ_0 equal to zero:

$$\hat{y}_{0S} = D_0\hat{\delta}; \tag{34}$$

obviously, $\hat{y}_{0S}$ is a linear predictor of the seasonally adjusted observations y_{0S}. For an arbitrary linear predictor the difference between the predicted and actual values of the observations to be predicted is:

$$\overset{P}{y}_{0S} - y_{0S} = (P_S D - D_0)\delta + P_S S\sigma + P_S\epsilon - \epsilon_0, \tag{35}$$

so that the conditions for unbiased prediction are:

$$\begin{aligned} P_S D - D_0 &= 0, \\ P_S S &= 0. \end{aligned} \tag{36}$$

A minimum variance predictor must minimize the expression:

$$V[a'(\overset{P}{y}_{0S} - y_{0S})] = V[a'(P_S\epsilon - \epsilon_0)] = \omega a'[P_S P'_S + I_n]a,$$

for an arbitrary vector a' of fixed coefficients. For a linear unbiased predictor of the seasonally adjusted observations, this variance may be written in the form:

$$\begin{aligned} V[a'(\overset{P}{y}_{0S} - y_{0S})] = \omega a'[&D_0(D'D)^{-1}D_0' + I_n \\ &+ (D_0(D'D)^{-1}D'[I - S\{S'M_D S\}^{-1}S'M_D] - P_S) \\ &(D_0(D'D)^{-1}D'[I - S\{S'M_D S\}^{-1}S'M_D] - P_S)']a \end{aligned} \tag{37}$$

using (36). To minimize this expression we take the linear transformation P_S to be:

$$P_S = D_0(D'D)^{-1}D'[I - S\{S'M_D S\}^{-1}S'M_D];$$

the corresponding predictor is

$$P_S y = D_0(D'D)^{-1}D'[I - S\{S'M_D S\}^{-1}S'M_D]y = D_0\hat{\delta} = \hat{y}_{0S}.$$

The unique, minimum variance, linear, unbiased predictor of the vector of unknown seasonally adjusted observations y_{0S} is obtained by replacing δ in (33) by the unique, minimum variance, linear unbiased estimator $\hat{\delta}$ and setting the random component ϵ_0 equal to its expected value, zero.

For prediction of n unknown observations of an economic time series given N known observations it is necessary to calculate the ordinary least squares estimator $\{\hat{\delta}, \hat{\sigma}\}$ of the parameters $\{\delta, \sigma\}$; for prediction of n unknown observations adjusted for seasonal variation, only the ordinary least squares estimator $\hat{\delta}$ of the parameters δ is required. For either type of prediction it is impossible to use the estimator $\hat{\sigma}$ of the parameters σ alone. For prediction it is essential to employ a model of trend in the form given by (5) rather than the form given by (6); the estimator $\hat{\sigma}$ in the form derived from generalized least squares as in (14) by itself is useless for prediction. We have already pointed out that the computations required to obtain the ordinary least squares estimator $\{\delta, \hat{\sigma}\}$ are quite routine, while the computations required for $\hat{\sigma}$ in the form given by (14) require a different type of calculation. An additional advantage of the model of trend (5) as opposed to (6) is that an explicit model for the trend component as in (5) provides useful information about the statistical model for an economic time series not easily obtained from the implicit model for trend as in (6).

Up to this point we have proceeded under the assumption that the random component of an economic time series is stationary and that the elements of this component are independently distributed. This assumption can be relaxed in the usual way; we first replace assumption (28) by the following assumption:

$$V\begin{bmatrix} \epsilon \\ \epsilon_0 \end{bmatrix} = \omega \begin{bmatrix} \Omega_{NN} & \Omega_{Nn} \\ \Omega_{nN} & \Omega_{nn} \end{bmatrix} = \omega\Omega \tag{38}$$

where Ω is a known, positive definite matrix and Ω_{NN}, Ω_{nn} are square matrices of order N and n, respectively. For this situation the unique, minimum variance, linear, unbiased estimator of the parameters $\{\delta, \sigma\}$ may be written:

$$\begin{bmatrix} \hat{\delta} \\ \hat{\sigma} \end{bmatrix} = \begin{bmatrix} D'\overset{-1}{\Omega}_{NN}D & D'\overset{-1}{\Omega}_{NN}S \\ S'\overset{-1}{\Omega}_{NN}D & S'\overset{-1}{\Omega}_{NN}S \end{bmatrix}^{-1} \begin{bmatrix} D'\overset{-1}{\Omega}_{NN}y \\ S'\overset{-1}{\Omega}_{NN}y \end{bmatrix}; \tag{39}$$

this estimator has variance-covariance matrix,

$$V\begin{bmatrix} \hat{\delta} \\ \hat{\sigma} \end{bmatrix} = \begin{bmatrix} D'\overset{-1}{\Omega}_{NN}D & D'\overset{-1}{\Omega}_{NN}S \\ S'\overset{-1}{\Omega}_{NN}D & S'\overset{-1}{\Omega}_{NN}S \end{bmatrix}^{-1}; \tag{40}$$

an unbiased estimator of the parameter ω, say $\hat{\omega}$, is:

$$\hat{\omega} = \frac{e'\overset{-1}{\Omega}_{NN}e}{N - K_d - K_s}. \tag{41}$$

The unique, minimum variance, linear, unbiased method for seasonal adjustment may be represented in the form:

$$\hat{y}_S = y - S\hat{\sigma}. \tag{42}$$

Finally, the unique, minimum variance, linear, unbiased predictors for the unknown observations unadjusted and adjusted for seasonal variation as in (27) and (33), respectively, are:

$$\begin{aligned} \hat{y}_0 &= D_0\hat{\delta} + S_0\hat{\sigma} + \Omega_{nN}\overset{-1}{\Omega}_{NN}e, \\ \hat{y}_{0S} &= D_0\hat{\delta} + \Omega_{nN}\overset{-1}{\Omega}_{NN}e. \end{aligned} \tag{43}$$

The predictor $\hat{y}_0$ is discussed by Goldberger [9].

Where the matrix Ω in (38) is unknown, the ordinary least squares estimator of the unknown parameters $\{\delta, \sigma\}$ is unbiased. Furthermore, under certain restrictions on the matrix of non-seasonal deterministic and seasonal variables $\{DS\}$ and the variance-covariance matrix of the random component, Ω_{NN}, the least squares estimator is asymptotically efficient.[11] When the appropriate restrictions on the variance-covariance matrix Ω_{NN} are satisfied and where each variable in the matrix $[DS]$ may be represented as a trigonometric or polynomial function, it seems appropriate to use the ordinary least squares estimator. Otherwise, an asymptotically efficient estimator may be obtained if it is possible to estimate the variance-covariance matrix consistently from the

[11] See Grenander [11], Grenander and Rosenblatt [12, pp. 231–55], and Hannan [16, pp. 122–8].

residuals to an ordinary least squares regression. The first step is to apply the least squares estimator; secondly, an estimate of the variance-covariance matrix is obtained from the residuals; finally, the variance-covariance matrix Ω_{NN} in (39), (40), and (41) is replaced by its estimated value. From an estimator of the unknown parameters $\{\delta, \sigma\}$ which is asymptotically unbiased and efficient, it is possible to obtain a method of seasonal adjustment analogous to (42) and predictors analogous to (43) which are asymptotically unbiased and efficient, provided that a consistent estimator of the matrix Ω_{NN}^{-1} is available.

In a recent paper Lovell [21] has provided a non-statistical axiomatization for linear methods of seasonal adjustment. Lovell postulates that a linear method of seasonal adjustment should have the following properties:

(1) *orthogonality* of the estimated seasonal component and the seasonally adjusted series:

$$(y - \overset{B}{y_S})'\overset{B}{y_S} = y'[B - B'B]y;$$

(2) *idempotence* of the method for seasonal adjustment:

$$B\overset{B}{y_S} = BBy = By = \overset{B}{y_S};$$

(3) *symmetry* of the method for seasonal adjustment:

$$By = B'y.$$

As Lovell points out, any two of these axioms implies the third. We observe from (22) that the unique, minimum variance, linear, unbiased method for seasonal adjustment is symmetric, since the matrix, $[I - S\{S'M_DS\}^{-1}S'M_D]$, is symmetric. It may be verified directly that this matrix is also idempotent, so that the method for seasonal adjustment is idempotent. By implication the estimated seasonal component and seasonally adjusted series are orthogonal.

The properties of linear methods for seasonal adjustment embodied in Lovell's set of axioms are undoubtedly highly desirable ones. Idempotence, for example, implies that "seasonal adjustment" of a series which has already been adjusted for seasonal variation will leave the series unchanged. The chief difference between Lovell's viewpoint and that adopted in this paper is that in this paper seasonal adjustment is approached as a problem which is essentially statistical in nature. The definitions of minimum variance, linearity, and unbiasedness for methods of seasonal adjustment may be taken as an alternative axiomatization of seasonal adjustment procedures. We have shown that Lovell's non-statistical axioms of orthogonality, idempotence, and symmetry are implied in the statistical axioms of minimum variance and unbiasedness; both sets of axioms are developed only for linear methods of seasonal adjustment. The question that remains is: are the statistical axioms of minimum variance and unbiasedness implied in whole or in part by Lovell's non-statistical axioms?

On the basis of linearity and the axioms of orthogonality, idempotence, and symmetry, Lovell goes on to show that any method for seasonal adjustment satisfying these axioms may be expressed in the form:

$$\overset{B}{y_S} = By = [I - X(X'X)^{-1}X']y \tag{44}$$

for some fixed matrix X, and, conversely, that any method which may be expressed in the form (44) satisfies the axioms. Among many other examples, seasonal adjustment by estimation of the unknown parameters of the seasonal component by least squares regression on the matrix of seasonal variables alone satisfies Lovell's axioms:

$$\overset{B}{y_S} = By = [I - S(S'S)^{-1}S']y.$$

This method for seasonal adjustment is unbiased only if the rows of the matrix D' are orthogonal to the columns of the matrix S, that is:

$$D'S = 0;$$

in this case, the "least squares" method for seasonal adjustment is also minimum variance. We conclude that Lovell's axioms do not imply the statistical axiom of unbiasedness. As a second example, it is easy to verify the fact that the "ordinary" least squares method for seasonal adjustment considered by Hannan [14, 15] and others satisfies Lovell's axioms, so that these axioms, even taken together with the statistical axiom of unbiasedness, do not imply the statistical axiom of minimum variance. Obviously, there are any number of linear methods for "seasonal adjustment" which satisfy Lovell's axioms, some of which have no relation whatever to the problem of seasonal adjustment. An example of this type would be "seasonal adjustment" by regression on the non-seasonal deterministic variables alone:

$$\overset{B}{y_S} = By = [I - D(D'D)^{-1}D']y.$$

It is a straightforward matter to verify the fact that this "method for seasonal adjustment" satisfies Lovell's axioms.

To summarize, the statistical axioms of minimum variance and unbiasedness are not implied by the non-statistical axioms of orthogonality, idempotence, and symmetry. Given a statistical model, the properties of minimum variance and unbiasedness seem to be much more appealing than the non-statistical axioms of orthogonality, idempotence, and symmetry. Fortunately, there is no need to set aside these desirable non-statistical properties of linear methods for seasonal adjustment in order to obtain a method for seasonal adjustment with desirable statistical properties. From the point of view of statistical theory, Lovell's non-statistical axioms may be viewed as theorems pertaining to the minimum variance, linear, unbiased method for seasonal adjustment. From a purely formal point of view an additional advantage of the statistical axioms of minimum variance and unbiasedness is that these axioms provide a complete basis for the choice of a method for seasonal adjustment, since the minimum variance, linear, unbiased method is unique.

Finally, Lovell goes on to consider the problem of regression with seasonally adjusted data where the method for seasonal adjustment satisfies the axiom of linearity and the three non-statistical axioms given above. All of Lovell's results carry over directly to the unique, minimum variance, linear, unbiased method for seasonal adjustment, since this method satisfies the three non-statistical axioms.

4. STATISTICAL INFERENCE ABOUT METHODS FOR SEASONAL ADJUSTMENT

Sampling theory for statistical inference about a method for seasonal adjustment may be derived from sampling theory for the general linear statistical model. In addition to assumptions (2), (3), (4), and (5) for the statistical model of an economic time series (11) and the assumption of full rank for the matrix of deterministic and seasonal variables $[DS]$, it is assumed that the random component of the series of observations is normally distributed:

$$\epsilon \text{ is } N(O, \omega I). \tag{45}$$

On the basis of this assumption the estimator $\{\hat{\delta}, \hat{\sigma}\}$ is normally distributed:

$$\begin{bmatrix} \hat{\delta} \\ \hat{\sigma} \end{bmatrix} \text{ is } N\left(\begin{bmatrix} \delta \\ \sigma \end{bmatrix}, \quad \omega \begin{bmatrix} D'D & D'S \\ S'D & S'S \end{bmatrix}^{-1}\right) \tag{46}$$

or, considering $\hat{\delta}$ and $\hat{\sigma}$ separately:

$$\begin{aligned} &\hat{\delta} \text{ is } N(\delta, \omega\{D'[I - S(S'S)^{-1}S']D\}^{-1}), \\ &\hat{\sigma} \text{ is } N(\sigma, \omega\{S'[I - D(D'D)^{-1}D']S\}^{-1}). \end{aligned} \tag{47}$$

For statistical inference about a method for seasonal adjustment we consider the distribution of an arbitrary set of linear functions of the estimator $\{\hat{\delta}, \hat{\sigma}\}$. Let A' be a matrix of full rank with elements which are known constants; the columns of this matrix are linearly independent. We will denote the rank of the matrix A' by K_a. Obviously, the rank of the matrix A' is less than or equal to the order of the vector of elements of the estimator $\{\hat{\delta}, \hat{\sigma}\}$, that is, $K_a \leqq K_d + K_s$. A set of linear functions of the estimator $\{\hat{\delta}, \hat{\sigma}\}$ may be represented in the form:

$$z = A\begin{bmatrix} \hat{\delta} \\ \hat{\sigma} \end{bmatrix}. \tag{48}$$

On the basis of assumption (45), z is normally distributed:

$$z \text{ is } N\left(A\begin{bmatrix} \delta \\ \sigma \end{bmatrix}, \omega A\begin{bmatrix} D'D & D'S \\ S'D & S'S \end{bmatrix}^{-1} A'\right). \tag{49}$$

Further, the quadratic form,

$$u_a = [z - E(z)]'V(z)^{-1}[z - E(z)], \tag{50}$$

is distributed as χ^2 with K_a degrees of freedom:

$$u_a \text{ is } \chi^2_{K_a}. \tag{51}$$

Finally, the sum of squared residuals $e'e$ divided by the parameter ω, say u_e, is distributed as χ^2 with $N-K_d-K_s$ degrees of freedom:

$$u_e \text{ is } \chi^2_{N-K_d-K_s}; \tag{52}$$

the statistics u_a and u_e are distributed independently so that the ratio,

$$v = \frac{u_a/K_a}{u_e/N - K_d - K_s}, \tag{53}$$

is distributed as F with K_a and $N-K_d-K_s$ degrees of freedom:

$$v \text{ is } F_{K_a, N-K_d-K_s}. \tag{54}$$

For applications it is useful to consider the special case of sampling theory in which each of the linear functions of the estimator $\{\hat{\delta}, \hat{\sigma}\}$ includes only elements of $\hat{\delta}$ or only elements of $\hat{\sigma}$ as arguments. Partitioning the vector z and the matrix A conformably with the vector of elements of the estimator $\{\hat{\delta}, \hat{\sigma}\}$:

$$\begin{bmatrix} z_d \\ z_s \end{bmatrix} = \begin{bmatrix} A_d & 0 \\ 0 & A_s \end{bmatrix} \begin{bmatrix} \hat{\delta} \\ \hat{\sigma} \end{bmatrix}. \tag{55}$$

On the basis of assumption (55) the sets of linear functions z_d and z_s are distributed normally:

$$\begin{aligned} z_d &\text{ is } N(A_d\delta, \omega A_d\{D'[I - S(S'S)^{-1}S']D\}^{-1}A_d'), \\ z_s &\text{ is } N(A_s\sigma, \omega A_s\{S'[I - D(D'D)^{-1}D']S\}^{-1}A_s'). \end{aligned} \tag{56}$$

If, further, the rows of the matrix D' are orthogonal to the columns of the matrix S, then $\hat{\delta}$ and $\hat{\sigma}$ are distributed independently and:

$$\begin{aligned} z_d &\text{ is } N(A_d\delta, \omega A_d[D'D]^{-1}A_d'), \\ z_s &\text{ is } N(A_s\sigma, \omega A_s[S'S]^{-1}A_s'). \end{aligned} \tag{57}$$

Finally, the statistic u_a in (50) may be written:

$$u_a = u_d + u_s,$$

where:

$$\begin{aligned} u_d &= [z_d - E(z_d)]'V(z_d)^{-1}[z_d - E(z_d)], \\ u_s &= [z_s - E(z_s)]'V(z_s)^{-1}[z_s - E(z_s)]; \end{aligned} \tag{58}$$

the statistic u_a is the sum of two independently distributed χ^2 variables. This completes the discussion of sampling theory for statistical inference about a method for seasonal adjustment. We will now consider applications of this theory to certain specific problems.

The first problem for statistical inference is to derive a confidence interval for a seasonally adjusted observation based on the unique, minimum variance, linear, unbiased method for seasonal adjustment. The difference between the seasonally adjusted series, $\hat{y}_s$ in (21), and the "true" seasonally adjusted series, y_s in (20), is:

$$\hat{y}_s - y_s = -S(\hat{\sigma} - \sigma).$$

Considering the ith element of this difference, we may write:

$$z_{Si} = \hat{y}_{Si} - y_{Si} = -\sum_{j=1}^{K_s} s_{ij}\hat{\sigma}_j + \sum_{j=1}^{K_s} s_{ij}\sigma_j, \quad (i = 1 \cdots N). \tag{59}$$

The first term from the right-hand side of (61) is a linear function of the estimator $\hat{\sigma}$ with known constant coefficients; the second term is a constant. Where s^i is taken to be the ith row of the matrix S, the scalar $s^{i\prime}\hat{\sigma}$, $(i = \cdots N)$, is normally distributed; hence the statistic z_{Si} is normally distributed:

$$z_{Si} \text{ is } N(O, \omega s^{i\prime}S'[I - D(D'D)^{-1}D']Ss^i).$$

Further, the statistic,

$$v_{Si} = t_{Si}^2 = \frac{(\hat{y}_{Si} - y_{Si})^2}{e'e \cdot s^{i\prime}S'[I - D(D'D)^{-1}D']Ss^i/N - K_d - K_s}, \tag{60}$$

$$= \frac{(\hat{y}_{Si} - y_{Si})^2}{\hat{\omega}_{Si}},$$

where $\hat{\omega}_{Si}$ is the standard error of estimate of $\hat{y}_{Si}$, is distributed as F with 1 and $N-K_d-K_s$ degrees of freedom from (53). Of course, the statistic t_{Si} is distributed as Student's t-distribution with $N-K_d-K_s$ degrees of freedom.

From the statistic v_{Si} in (60) the usual confidence interval estimate of y_{Si} may be derived. As an illustration, suppose that S is as given in the example of Section 3 and that the rows of the matrix D' are orthogonal to the columns of S; then the standard error of estimate of $\hat{y}_{Si}$ reduces to:

$$\hat{\omega}_{Si} = e'e \cdot s^{i\prime}S'Ss^i/N - K_d - K_s = 3e'e/4N(N - K_d - K_s),$$

so that a confidence interval estimate of the seasonally adjusted observation y_{Si} is:

$$\hat{y}_{Si} + \frac{3e'e}{4N(N - K_d - K_s)} t_{\frac{1}{2}\alpha} \leqq y_{Si} \leqq \hat{y}_{Si} + \frac{3e'e}{4N(N - K_d - K_s)} t_{1-\frac{1}{2}\alpha},$$

where $1-\alpha$ is the level of confidence. A simultaneous confidence ellipsoid for several seasonally adjusted observations may be constructed from the distribution of the differences (59) by analogy with the confidence interval for a single seasonally adjusted observation. In no case can such an ellipsoid be constructed for more than K_s observations; particular attention must be paid to the assumption that the matrix A' of (48) is of full rank.

A second problem for statistical inference is to derive tests for hypotheses such as whether to adjust at all or if to adjust, what kind of seasonal adjustment to use. More formally, we consider the hypothesis:

$$\sigma = 0, \tag{61}$$

that is, that all parameters of the seasonal component are zero. If this hypothesis were true, there would be no reason to adjust the series of observations for seasonal variation. A test statistic for this hypothesis is obtained by choosing the vector z_s in (56) to be:

$$z_s = A_s\hat{\sigma} = \hat{\sigma},$$

so that in (55), the matrix A_s is taken to be the identity matrix of order K_s. From (53) we obtain the statistic:

$$v_{\hat{\sigma}} = \frac{\hat{\sigma}'S'[I - D(D'D)^{-1}D']S\hat{\sigma}/K_s}{e'e/N - K_d - K_s}, \tag{62}$$

which is distributed as F with K_s and $N-K_d-K_s$ degrees of freedom. If the rows of the matrix D' are orthogonal to the columns of S, this statistic reduces to:

$$v_{\hat{\sigma}} = \frac{\hat{\sigma}'S'S\hat{\sigma}/K_s}{e'e/N - K_d - K_s}.$$

A critical region for testing the hypothesis (61) is given by:

$$v_{\hat{\sigma}} \geqq F_{1-\alpha;K_s,N-K_d-K_s},$$

where α is the level of significance of the test. Acceptance of the null hypothesis for this test may be interpreted as acceptance of the hypothesis that no seasonal adjustment of the original series of observations is required. We have given an expression for the statistic $\hat{\sigma}$ in (47) based on an explicit model for the trend component as in (5). It is obvious that this statistic may be computed in an equally straightforward manner from either of the two forms (14) and (17) for the estimator $\hat{\sigma}$ and either of the two forms (16) and (19) for the estimator $\hat{\delta}$. If the hypothesis under consideration is whether to adjust for seasonal variation as in (61), the resulting test statistic is identical for the two forms (5) and (6) of the model for the trend component. As we have already pointed out, there is no computational advantage in using the implicit form of the model for the trend component as in (6).

Another hypothesis which may be of interest is the following: Suppose that a decision to adjust the original series of observations for seasonal variation has been made on *a priori* grounds; the decision to be made on the basis of statistical inference is the type of seasonal adjustment to use. On the basis of standard theory for the general linear statistical model, it is not possible to choose one method for seasonal adjustment among an arbitrary set of alternative methods by means of statistical inference. It is only possible to choose between a pair of alternatives in which all the variables included in the seasonal and non-seasonal deterministic components for one of the alternatives are included for the other. Even within this restricted framework the problem of choosing a method for seasonal adjustment may be interpreted in at least three different ways: first, methods may differ in the model for trend; secondly, methods may differ in the model for seasonal; thirdly, methods may differ in models for both components. Test statistics for choosing between methods for seasonal adjustment which differ only in the model for trend or only in the model for seasonal may be derived as a special case of a test statistic for choosing between methods which differ in models for trend and for seasonal. We will discuss a test statistic only for the general case.

Proceeding formally, the statistical hypothesis to be tested is: for variables excluded from the model for the first method but included in the model for the second, the corresponding parameters are equal to zero. Without loss of generality we may assume that the excluded variables for trend are the first K_d^E variables of the trend component; similarly, we may assume that the excluded variables for the seasonal component are the first K_s^E variables. We let A_d in (55) be a matrix having as its first K_d^E columns an identity matrix of order K_d^E with zeros elsewhere; similarly, we let A_s in (55) be a matrix with its first K_s^E columns an identity matrix of order K_s^E and zeros elsewhere. From (53) we obtain the statistic:

$$v_{\hat{\delta},\hat{\sigma}} = \frac{\begin{bmatrix} \hat{\delta} \\ \hat{\sigma} \end{bmatrix}' A'A \begin{bmatrix} D'D & D'S \\ S'D & S'S \end{bmatrix}^{-1} A'A \begin{bmatrix} \hat{\delta} \\ \hat{\sigma} \end{bmatrix} / K_d^E + K_s^E}{e'e/N - K_d - K_s} \tag{63}$$

which is distributed as F with $K_d^E + K_s^E$ and $N - K_d - K_s$ degrees of freedom. An appropriate critical region may be derived from (63) by a straightforward calculation analogous to that for v given in (62) and the following discussion. To choose between pairs of alternative methods which differ in the model for trend, the estimator $\hat{\delta}$ in (17) must be computed. For this problem it is essential to use an explicit model for the trend component as in (5).

A third problem for statistical inference is to derive statistics for confidence interval predictions for both seasonally unadjusted and seasonally adjusted observations based on the unique, minimum variance, linear, unbiased predictors. For the original series of observations (11) and the observations to be predicted (27), we assume that the random component is normally distributed:

$$\begin{bmatrix} \epsilon \\ \epsilon_0 \end{bmatrix} = N\left(0, \begin{bmatrix} I_N & 0 \\ 0 & I_n \end{bmatrix}\right); \tag{64}$$

assumption (65) implies assumptions (28) and (45).

For unadjusted observations the difference between the predictor, $\hat{y}_0$ in (29), and the actual value of the observations to be predicted, y_0 in (27), is:

$$\hat{y}_0 - y_0 = [D_0\hat{\delta} + S_0\hat{\sigma}] - [D_0\delta + S_0\sigma] - \epsilon_0. \tag{65}$$

The first term from the right-hand side of (66) is a set of linear functions of the estimator $\{\hat{\delta}, \hat{\sigma}\}$. From (49) we see that this set of functions is normally distributed. The last is normally distributed from (64). Hence the difference (65) is normally distributed; where:

$$z_0 = \hat{y}_0 - y_0,$$

we may write:

$$z_0 \text{ is } N\left(0, \omega\left\{[D_0S_0]\begin{bmatrix} D'D & D'S \\ S'D & S'S \end{bmatrix}^{-1}[D_0S_0]' + I_n\right\}\right).$$

The statistic:

$$v_0 = \frac{(\hat{y}_0 - y_0)'\left([D_0S_0]\begin{bmatrix} D'D & D'S \\ S'D & S'S \end{bmatrix}^{-1}[D_0S_0]' + I_n\right)^{-1}(\hat{y}_0 - y_0)/n}{e'e/N - K_d - K_s}, \tag{66}$$

is distributed as F with n and $N-K_d-K_s$ degrees of freedom from (53). The restriction $n=K_a \leqq K_d+K_s$ in (53) need not be satisfied for the statistic v_0 in (66).

From the statistic v_0 the usual confidence ellipsoid for observations to be predicted, y_0 in (27) may be derived. For confidence interval predictions it is necessary to calculate the ordinary least squares estimator $\{\hat{\delta}, \hat{\sigma}\}$; it is impossible to use the estimator $\hat{\sigma}$ alone.

For seasonally adjusted observations the difference between the predictor, $\hat{y}_{0S}$ in (34), and the actual value of the observations to be predicted, y_{0S} in (33), is:

$$\hat{y}_{0S} - y_{0S} = D_0\hat{\delta} - D_0\delta - \epsilon_0. \tag{67}$$

This difference is normally distributed from (56) and (64); where:

$$z_{0S} = \hat{y}_{0S} - y_{0S},$$

we may write:

$$z_{0S} \text{ is } N(0, \omega[D_0\{D'[I - S(S'S)^{-1}S']D\}^{-1}D_0' + I_n]).$$

The statistic:

$$v_{0S} = \frac{(\hat{y}_{0S} - y_{0S})'(D_0\{D'[I - S(S'S)^{-1}S']D\}^{-1}D_0' + I_n)^{-1}(\hat{y}_{0S} - y_{0S})/n}{e'e/N - K_d - K_s}, \tag{68}$$

is distributed as F with n and $N-K_d-K_s$ degrees of freedom from (53). As for the statistic v_0 in (66), the restriction $n=K_a \leqq K_d+K_s$ in (53) need not be satisfied for the statistic v_{0S} in (68). From the statistic v_{0S} in (68) the usual confidence ellipsoid for seasonally adjusted observations to be predicted, y_{0S} in (33), may be derived. For confidence interval predictions of the seasonally adjusted observations it is necessary to calculate the ordinary least squares estimator $\hat{\delta}$; for this purpose an explicit model for trend is required as in (5).

Up to this point we have proceeded under the assumption that the random component of an economic time series is stationary and that the elements of this component are independently distributed. This assumption can be relaxed by replacing assumption (64) by the assumption:

$$\begin{bmatrix} \epsilon \\ \epsilon_0 \end{bmatrix} \text{ is } N\left(0, \omega\begin{bmatrix} \Omega_{NN} & \Omega_{Nn} \\ \Omega_{nN} & \Omega_{nn} \end{bmatrix}\right); \tag{69}$$

this assumption implies assumption (38). Normal sampling theory for statistical inference about a method for seasonal adjustment based on the minimum variance, linear, unbiased estimator, $\{\hat{\delta}, \hat{\sigma}\}$ in (39), and the corresponding predictors, $\hat{y}_0$ and $\hat{y}_{0S}$ in (43), may be derived by direct analogy with the derivation for these statistics given explicitly in the foregoing discussion under assumption (64). Where the variance-covariance matrix in (69) is unknown but a consistent estimator for this matrix is available, sampling theory under assumption (69) is valid asymptotically for the estimator and predictors obtained by replacing the variance-covariance matrix which occurs in (39) and

(43) by a consistent estimator for this matrix. This completes the discussion of sampling theory for statistical inference about a method for seasonal adjustment based on normal sampling theory for the minimum variance, linear, unbiased estimator $\{\hat{\delta}, \hat{\sigma}\}$ of the parameters $\{\delta, \sigma\}$.

In the analysis of economic (and other) time series it is frequently of interest to test the hypothesis that elements of the random component are distributed independently as assumed in (3). Many different tests have been proposed for this purpose; perhaps the most common test is derived from sampling theory for the ratio of mean squared successive difference to mean square error[12] for the residuals from the fitted regression. This test is essentially equivalent to a test of the hypothesis that the first-order autocorrelation coefficient of the elements of the random component is zero. For the statistical model of the transformed series of observations (13), a test of the hypothesis of independence for the elements of the random component has been proposed in the report of the President's Committee to Appraise Employment and Unemployment Statistics [30]. Since the published literature on this aspect of the problem of seasonal adjustment is very sparse, the approach of the President's Committee will be reviewed in detail.

First, given the "Slutsky-Yule effect":

$$V(L_\epsilon) = \omega LL',$$

there exists a matrix T such that:

$$TLL'T' = I, \qquad T'T = (LL')^{-1}; \tag{70}$$

the statistical model (13) may be further transformed as follows:

$$TLy = TLS\sigma + TL\epsilon. \tag{71}$$

For the transformed model (71), the elements of the random component are distributed independently:

$$V(TL\epsilon) = TLL'T' = \omega I.$$

The unique, minimum variance, linear, unbiased estimator $\hat{\sigma}$ of the parameters σ is the ordinary least squares estimator for model (71):

$$\hat{\sigma} = (S'L'T'TLS)^{-1}S'L'T'TLy;$$

using (70), this estimator may be written as in (14):

$$\hat{\sigma} = (S'L'(LL')^{-1}LS)^{-1}S'L'(LL')^{-1}Ly.$$

Any of the standard tests for independence of the elements of the random component may be applied to the transformed residuals:

$$T\hat{e} = TLy - TLS\hat{\sigma}. \tag{72}$$

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[12] This test is discussed by Durbin and Watson [6, 7], by Hannan [16, pp. 128-33], and by Theil and Nagar [37].

The approach of the President's Committee is:

> . . . to inspect the unexplained residuals—the discrepancies between the adjusted and the trend-cycle series—to determine whether they behave in a random manner, as residuals should [30, p. 169].

To evaluate this approach it is necessary to determine just how discrepancies between the seasonally adjusted and trend-cycle series "should" behave. For any fitted regression the residuals may be treated as distributed independently only in certain approximations for large samples; presumably, discrepancies between adjusted and trend-cycle series "behave in a random manner" if they behave like residuals from a fitted regression in which elements of the underlying random component are distributed independently. Examples of such residuals would be the transformed residuals in (72) or the residuals from the fitted regression which enter the definition of $\hat{\omega}$ in (19). We begin with the assumption that the original data may be transformed so as to remove trend as in (13). The estimated trend component implicit in this transformation is obtained by subtracting the transformed series from a truncation of the original series. Truncation of the original series may be represented as a linear transformation of the series; such a linear transformation may be represented in the form:

$$Ry = \bar{y}, \tag{73}$$

where R is a matrix with first and last rows composed entirely of zeros; these rows correspond to the elements of the original series which are deleted in the truncation; the remaining rows of R comprise an identity matrix. The vector $\bar{y}$ is a truncation of the original series of observations. The estimated trend component is $(R-L)y$, the difference between a truncation of the original series and the transformed series. An identical truncation of the seasonally adjusted series is the following:

$$R\hat{y}_S = Ry - RS\hat{\sigma} = R[I - S\{S'M_DS\}^{-1}S'M_D]y, \tag{74}$$

as in (21). The difference between the adjusted and trend-cycle series is:

$$\begin{aligned} R\hat{y}_S + [R - L]y &= (R[I - S\{S'M_DS\}^{-1}S'M_D] - [R - L])y, \\ &= [L - RS\{S'M_DS\}^{-1}S'M_D]y, \\ &= [L - R]S\sigma + [L - RS\{S'M_DS\}^{-1}S'M_D]\epsilon. \end{aligned} \tag{75}$$

From (75) we may observe that unlike residuals from a fitted regression, the difference between adjusted and trend-cycle series has a non-zero expectation:

$$E(\hat{y}_S - [R - L]y) = [L - R]S\sigma.$$

Obviously, this difference cannot behave like residuals from a fitted regression unless:

$$LS\sigma = RS\sigma,$$

that is, unless the transformation to remove trend leaves the seasonal component unchanged except for truncation; this condition is satisfied, for example, by the moving average transformation for removal of trend and the model for the seasonal component considered as an illustration in Section 2. If this condition is satisfied, the difference (75) may be written:

$$R\hat{y}_S - [R - L]y = L[I - S\{S'M_DS\}^{-1}S'M_D]\epsilon. \tag{76}$$

But this expression is identical to the untransformed residuals $\hat{e}$ which enter the definition of $\hat{\omega}$ in (16) and occur in (72). To demonstrate this fact, we may write:

$$\begin{aligned} \hat{e} &= Ly - LS\hat{\sigma}, \\ &= L[I - S\{S'M_DS\}^{-1}S'M_D]y, \\ &= L[I - S\{S'M_DS\}^{-1}S'M_D]\epsilon, \end{aligned}$$

as in (76). Hence, even in the special case for which the transformation to remove trend leaves the seasonal unchanged, except for truncation, the difference between adjusted and trend-cycle series behaves like untransformed residuals from a fitted regression in which the elements of the underlying random component are *not* distributed independently. If the standard tests for independence are applied to these residuals, the null hypothesis of independence *should be rejected* if the elements of the random component underlying the untransformed model (11) are distributed independently.

From this point forward the analysis is meant to be suggestive rather than conclusive. In particular we will suggest that the approach of the President's Committee is invalid, not only for the minimum variance, linear, unbiased method for seasonal adjustment, but also for methods of seasonal adjustment currently used in practice. First, taking a centered twelve-month moving average transformation for removal of trend as an approximation to the non-linear transformation used in practice, we observe that the moving average transformation leaves a set of seasonal variables analogous to those employed in Section 2 invariant except for truncation. The first-order autocorrelation coefficient for discrepancies between the adjusted and the trend-cycle series, say ρ_1 is:

$$\rho_1 = -\frac{13}{144};$$

the first-order autocorrelation coefficient is small and negative. The first test applied to the discrepancies between the adjusted and trend-cycle series in the report of the President's Committee is based on a runs statistic. The null hypothesis is that the elements of the random component of the transformed series are distributed independently. This hypothesis is rejected; the runs statistic suggests *small negative* first-order autocorrelation. But this result is precisely what should be expected if the elements of the random component of the untransformed observations are distributed independently. The Committee's conclusion is the following:

> This raises the suspicion of an overly sensitive adjustment procedure—one that, as soon as it notices a discrepancy in one direction, tends to overcompensate and create a discrepancy in the opposite direction in the following month [30, p. 170].

If the elements of the random component of the untransformed observations are distributed independently, small negative first-order autocorrelation of the elements of the random component of the transformed series should be expected. There is no evidence in the results of the test reported for the type of specification error suggested in the Committee's conclusion. A second test of the hypothesis of independence based on the analysis of variance is reported by the Committee. The results are similar to those for the test based on the runs statistic. The Committee draws the same conclusion from the results of the analysis of variance test as that cited above for the results of the runs test. On the basis of our analysis of the discrepancies between adjusted and trend-cycle series, it may be suggested that the Committee's conclusion is just the opposite of the correct conclusion.

5. SUMMARY AND CONCLUSION

In this paper we have attempted to complete the development of a statistical theory for seasonal adjustment. This theory is based on a statistical model for an economic time series which satisfies the hypotheses of the general linear statistical model. The complete statistical model may be written as in (11):

$$y = D\delta + S\sigma + \epsilon,$$

where y is the vector of original observations, $D\delta$ is the non-seasonal deterministic component, $S\sigma$ is the seasonal deterministic component, and ϵ is the random component of the series. The matrix of deterministic and seasonal variables $[DS]$ is assumed to be of full rank; this matrix is taken to be fixed (non-random). The random component is assumed to satisfy assumptions (3):

$$E(\epsilon) = 0,$$
$$V(\epsilon) = \omega I.$$

Under these assumptions there exist unique, minimum variance, linear unbiased estimators $\{\hat{\delta}, \hat{\sigma}\}$ of the unknown parameters $\{\delta, \sigma\}$ and an unbiased estimator $\hat{\omega}$ of the unknown parameter ω. These estimators may be written as in (17) and (19):

$$\hat{\delta} = (D'D)^{-1}D'(I - S\{S'[I - D(D'D)^{-1}D']S\}^{-1}S'[I - D(D'D)^{-1}D'])y,$$
$$\hat{\sigma} = \{S'[I - D(D'D)^{-1}D']S\}^{-1}S'[I - D(D'D)^{-1}D']y;$$

letting e represent the vector of residuals from the fitted regression for the statistical model (11):

$$\hat{\omega} = \frac{e'e}{N - (K_d + K_s)}.$$

Where the random component is normally distributed, the estimators (17) are the unique, minimum variance, unbiased estimators of the parameters $\{\delta, \sigma\}$ without restriction to the class of linear estimators. Further, the estimator (19) is the unique, minimum variance, unbiased estimator of the parameter ω.

Alternatively, the statistical model for an economic time series may be written as in (13):

$$Ly = LS\sigma + L\epsilon,$$

where L is a linear transformation which removes the non-seasonal deterministic component. For this form of the model, there exists a unique, minimum variance, linear, unbiased estimator $\hat{\sigma}$ of the parameters σ which may be written as in (14):

$$\hat{\sigma} = \{S'L'(LL')^{-1}LS\}^{-1}S'L'(LL')^{-1}Ly;$$

corresponding to this estimator, there exists an unbiased estimator $\hat{\omega}$ of the parameter ω which may be written as in (16):

$$\hat{\omega} = \frac{\hat{e}'(LL')^{-1}\hat{e}}{N - (K_d + K_s)},$$

where $\hat{e}$ represents the vector of residuals from the fitted regression for the statistical model (13). We have shown that any model for economic time series which can be represented in the form (11) may also be represented in the form (13). We have also shown that the estimators $\hat{\sigma}$ and $\hat{\omega}$ are identical for the two alternative forms of the model of an economic time series (11) and (13).

Any linear method for seasonal adjustment may be represented as taking place in two steps. The first step is to estimate the unknown parameters of the seasonal component of the series; the second step is to remove the estimated seasonal component from the set of original observations. For a linear method of seasonal adjustment, estimation of the unknown parameters of the seasonal component is carried out by means of a linear estimator. For a linear unbiased method, estimation is carried out through a linear unbiased estimator. Finally, for the unique, minimum variance, linear, unbiased method, estimation is carried out by means of the unique, minimum variance, linear, unbiased estimator. The unique, minimum variance, linear, unbiased method for seasonal adjustment can be shown to have the properties of orthogonality of the estimated seasonal component and the seasonally adjusted series, and of idempotence and symmetry of the method for seasonal adjustment.

The final step in the construction of a statistical theory for seasonal adjustment is to derive predictors for the original time series and for the seasonally adjusted time series. The predictor obtained by replacing the unknown parameters $\{\delta, \sigma\}$ in the statistical model for unknown observations to be predicted (27) by the estimators $\{\hat{\delta}, \hat{\sigma}\}$ and by setting the random component equal to zero is the unique, minimum variance, linear, unbiased predictor of the unknown observations. A predictor obtained by replacing the unknown parameter δ in the statistical model for unknown observations of the seasonally adjusted series (33) by the estimator $\hat{\delta}$ and by setting the random component equal to zero is the unique, minimum variance, linear, unbiased predictor of the unknown observations for the seasonally adjusted series.

Sampling theory for statistical inference about a method for seasonal adjustment may be derived by assuming that the random component of the statistical model for an economic time series is normally distributed as in (45):

$$\epsilon \text{ is } N(0, \omega I).$$

This assumption implies assumption (3). Normal sampling theory may be utilized for statistical inference in the following problems: to derive a confidence interval for a seasonally adjusted observation based on the unique, minimum variance, linear, unbiased method for seasonal adjustment; to derive tests for hypotheses such as whether to adjust at all or if to adjust, what kind of seasonal adjustment to use; to derive confidence interval predictions for both unadjusted and seasonally adjusted observations based on the unique, minimum variance, linear, unbiased predictors of these observations.

All of the results summarized up to this point may be generalized by replacing assumptions (3) and (45) and the corresponding assumptions (28) and (65) by assumptions (38) and (69), respectively.

Representation of the statistical model for economic time series in the form (11) as opposed to (13) has the advantage that the model for the trend component is given explicitly. Such a model provides the information necessary to decide just how "flexible" a given model for trend really is. Representation of the statistical model in the form (11) has the additional advantage that an estimator of the parameters of the trend component may be obtained as in (17); such an estimator is essential for inference about the trend component. It is also essential for prediction of unknown observations, whether these observations are unadjusted or adjusted for seasonal variation. Finally, an explicit estimator of the parameters of the trend component is required for confidence interval prediction. The unique, minimum variance, linear, unbiased estimator (17) for parameters of seasonal and trend components is an ordinary least squares estimator so that computation of estimates for any given sample is a routine least squares computation. Computation of the corresponding estimates for parameters of the seasonal component (14) requires a different type of calculation.

The theory of seasonal adjustment presented in this paper represents a fairly radical departure from current statistical practice. The primary objective of recent research on seasonal adjustment has been to mechanize the laborious process of seasonal adjustment by hand through the use of electronic computers. A summary of research on seasonal adjustment by electronic computer is given in a series of papers by Shiskin [33, 34, 35], by Shiskin and Eisenpress [36], and in a recent OECD publication [28][13]. This research has been largely confined to ratio-to-moving average methods for seasonal adjustment. As we have already pointed out, it is difficult if not impossible to obtain a straightforward statistical theory for ratio-to-moving average methods. The theory of seasonal adjustment presented here is an attempt to find a reasonable compromise between the demands of practice and the demands of statistical theory. Insofar as this approach is consistent with the demands of practice, it should open up an important new line of research on methods for seasonal adjustment.

The first objective of this new line of research should be to obtain a method

[13] An illustration of applications of "Census" methods for seasonal adjustment is provided by a recent publication of the Bureau of Labor Statistics [2].

for seasonal adjustment based on the general linear statistical model which is as satisfactory as ratio-to-moving average methods from the practical point of view. An important obstacle to this research arises from the fact that selection among alternative ratio-to-moving average methods in the past has been made primarily on the basis of informal inspection of the seasonally adjusted series. Any attempt to formalize the criteria which have been employed in the past is likely to reveal that these criteria are not entirely appropriate or that they do not provide a complete basis for selection among alternative methods for seasonal adjustment. Nevertheless, some attempt must be made to find an orderly basis for selection among alternative methods which will command the support of practitioners. The first principle in selection among alternative methods must be to base the selection on statistical properties of the alternative methods. This principle may be justified by the virtually unanimous agreement among economic statisticians that any model for representation of an economic time series must contain both deterministic and random components. This principle would rule out choice on the basis of criteria without a direct basis in statistical theory such as those proposed by Lovell. Among the criteria which have a direct basis in statistical inference, the selection of any particular criterion must be justified by careful statistical analysis of tests based on the criterion. Examination of the recent Report of the President's Committee to Appraise Employment and Unemployment Statistics reveals that criteria ostensibly based on statistical inference may be highly unsatisfactory.

The second objective of research on methods for seasonal adjustment should be to obtain a method for seasonal adjustment based on a specification of the model for an economic time series which is satisfactory from the point of view of statistical inference. Since it will be necessary to choose a specification by experimentation with alternative models, primary reliance must be placed on predictive tests of the validity of a particular specification. It is very important to note that such tests are not possible for ratio-to-moving average methods or for methods based on a statistical model for an economic time series in which the model for trend is given only implicitly as in (13). Failure of a method in which prediction is possible to pass predictive tests cannot provide support for an alternative method in which prediction is impossible.

The final objective of research should be to obtain a method for seasonal adjustment which is satisfactory not only from the statistical point of view but also as an economic explanation of the movement of an economic time series. To attain this objective it will be necessary to replace purely mechanical explanations of the movement of an economic time series with explanations based on an explicit economic theory. This objective applies equally to trend and seasonal components of an economic time series. Both components must be explained on the basis of economic considerations. Such an explanation will provide the basis for a unification of statistical practice in economic statistics, viewed as the decomposition of an economic time series into seasonal, deterministic, and random components, and econometrics, viewed as the explanation of an economic time series through an explicit economic theory.

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SPECTRAL EVALUATION OF BLS AND CENSUS REVISED SEASONAL ADJUSTMENT PROCEDURES[1]

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This paper presents spectral criteria to evaluate seasonal adjustment procedures, and applies them to the estimated spectral properties of the monthly unemployment series, as adjusted by past and present methods of the Bureau of Labor Statistics (BLS) and of the Census Bureau. It also re-examines some conclusions reported in the Gordon Committee Report (1962) and in Nerlove (1964) on past methods of adjustment. The paper concludes that (1) the loss in spectral power between the unadjusted and adjusted series spectra over most of the frequency range could arise with a satisfactory adjustment, (2) the excessive loss of spectral power at seasonal and trend-cycle frequencies present in past methods of adjustment has been reduced in present methods, (3) the effect of deviations from desired spectral properties on the uses of seasonally adjusted data must be examined in the time domain; for certain applications they have been found to be unimportant.

1. INTRODUCTION AND SUMMARY

SPECTRAL techniques were used by the President's Committee to Appraise Employment and Unemployment Statistics in evaluating the BLS 1962 Seasonal Factor Method. Its findings were reported in the Gordon Committee Report (1962) and more fully by Nerlove (1964). A similar evaluation was made by Rosenblatt (1963) of Census Method II, Variant X-9. The latest revised methods are the BLS 1964 Seasonal Factor Method issued April 1964, and Census Method II, Variant X-11 issued September 1965. Since the Census and BLS methods are widely used to seasonally adjust economic time series, it seems appropriate to report on their latest versions in terms of spectral criteria for the benefit of those who have followed this type of analysis. The conclusion reached is that both these methods of seasonal adjustment show an overall improvement in terms of the spectral criteria used, although the improvement is not uniform. Further, a basis is given for a conjecture that an apparent excessive loss in spectral power at all frequencies between the unadjusted and seasonally adjusted series could be an expected result of a proper adjustment when significant moving seasonality occurs. This same explanation could also account for the differences in the coherence patterns, among the four unemployment series studied, between the unadjusted and seasonally adjusted series. The improvement in seasonal adjustment noted by the sensitive spectral measure is not obviously revealed in certain summary measures of the Census methods. Based on these summary measures the effect of deviations from spectral criteria in the old methods are considered to be unimportant.

The spectral criteria are developed from the expected properties of the true unobservable components of an economic time series. A goal for a good seasonal adjustment is, then, to produce estimates of the components which also satisfy

[1] Based on a paper presented at the meetings of the American Statistical Association, Los Angeles, California August 17, 1966.

these criteria. While this is a reasonable goal, it is not likely to be achieved completely by any method of seasonal adjustment. The question of importance is how do the deviations from spectral criteria affect the various uses of the seasonally adjusted and other component series. Except in very general terms, answers have not been advanced which show precisely how deviations from these goals measured in spectral terms can be applied to describe their effect on the use of the data in the time domain. Very little work has been done to show this effect. Rosenblatt (1963) shows different seasonal adjustments where one method more closely satisfies the criteria than the other, and Nettheim (1964) shows two seasonal adjustments where one is a modified version of the other after correcting for spectral criteria deviations. An eyeball test suggests little difference in these seasonal adjustments. However, a more critical examination needs to be made by economists by illustrating the various uses of seasonally adjusted data and indicating whether the different adjustments would lead to different conclusions.

2. SPECTRAL NOTIONS

The theory of spectral analysis and methods for estimating the spectrum will not be reviewed here since they have been amply described in the literature. See, for example, Blackman and Tukey (1958), Granger and Hatanaka (1964), Parzen (1963), and papers on seasonal adjustment by Nerlove (1964), (1965) and Rosenblatt (1963). A series of papers in *Technometrics*, Vol. 3, No. 2, May 1961, and in *Applied Statistics*, Vol. 14, No. 1, 1965 also discuss the subject.

The theory tells us that if a time series is a random sample from a discrete time, zero mean, covariance stationary stochastic process $\{X(t), t=1, 2, \cdots\}$ with spectral density function $f_x(\omega)$, $-\pi \leq \omega \leq \pi$, we may use the time series data X_t, $t=1, 2, \cdots, N$, to estimate the spectral density function of the stochastic process. Further, for two such time series which are also jointly covariance stationary, we may also estimate their coherence and phase from the co- and qu-spectral components[2] of their cross-spectral density function.

One can think of the power spectrum of a time series as distributing the total variance of the series over the frequency or period domain. For two series jointly, the co-spectrum distributes their covariance, the coherence essentially displays a correlation coefficient squared, and phase indicates their timing relationship, lead or lag, each as a function of frequency. This is analogous to the role played by the familiar color spectrum which displays the intensity of color producing wavelengths in a light source. Thus, the transformation to the spectrum, and inquiry into the frequency properties of a time series, provides an opportunity to search for additional information that otherwise might not be apparent.

In applying these notions to an economic time series, one is assuming that the series, or a simple transformation of it, is a sample from a stochastic process which satisfies the theory to the extent that it is useful for the purpose intended to look at the spectral estimates. Experimentation in the applications to seasonal adjustment cited above support this assumption.

The procedure used here to compute the spectral estimates is described in

[2] The co-spectrum and quadrature spectrum, respectively.

Rosenblatt (1963). The options selected, as defined in that paper, are deviations from the arithmetic mean, Parzen's autocovariance kernel, truncation point $M=36$, number of frequency bands $Q=36$, and no prewhitening.

3. TIME PROPERTIES OF COMPONENTS OF SEASONAL ADJUSTMENT

In the time domain the unadjusted economic time series $U(t)$, or its logarithm, is usually written as

$$U(t) = A(t) + S(t) \tag{1}$$

where

$$A(t) = C(t) + I(t) \tag{2}$$

is the seasonally adjusted series. The three basic components may be described as follows. $C(t)$, the trend-cycle component, is a relatively smooth function with rises and declines which are not particularly regular, but with duration about that of the business cycle or longer. $S(t)$, the seasonal component, is smooth, relatively regular with rises and declines repeating in a more or less similar manner each year, but possibly changing slowly over time. The irregular component, $I(t)$, is essentially random in character with occasional catastrophic events, such as strikes, superimposed. A necessary condition of the model is that there is no correlation between the seasonally adjusted series and the seasonal component; it is also expected that the trend-cycle and irregular components are uncorrelated. Hence, no correlation is expected between any two of the three components $C(t)$, $S(t)$ and $I(t)$.

An objective of a seasonal adjustment procedure is to decompose the unadjusted series into its three principal components $S(t)$, $C(t)$ and $I(t)$ without seriously changing the true composition of any of them. In effect, the operation is a two stage decomposition. One stage removes the seasonal component leaving the adjusted series $A(t)$. The other stage separates $A(t)$ into its two components $C(t)$ and $I(t)$. Usually, the operation is carried out in the order given, although it may be done in any other sequence, or even simultaneously. In any event the result of seasonal adjustment produces the estimates $\hat{A}(t)$, $\hat{S}(t)$, $\hat{C}(t)$ and $\hat{I}(t)$ of their respective true values, such that they are related in the same manner as the true values, that is

$$U(t) = A(t) + \hat{S}(t) \tag{3}$$

and

$$A(t) = \hat{C}(t) + \hat{I}(t). \tag{4}$$

For another discussion of the properties of seasonal adjustment see Lovell (1963).

4. SPECTRAL PROPERTIES OF COMPONENTS OF SEASONAL ADJUSTMENT

If we are to use spectral criteria to examine the adequacy of seasonal adjustment, we need to have an a priori judgment as to the general appearance of the spectra and cross-spectra measures to be expected from the unadjusted series and its true components. An evaluation of the adequacy of a seasonal adjustment then consists in comparing the observed spectral results obtained from the estimated components with that expected from their true values. As we shall see, it will not be difficult to make a useful and reasonable judgment as to the expected behavior of the spectra for the true time series values. First, let us indicate the relation between the spectra of the unadjusted series and its components. Corresponding to the additive models (1) and (2) we have, respectively,

$$f_U(\omega) = f_A(\omega) + f_S(\omega) + 2c_{AS}(\omega) \tag{5}$$

and

$$f_A(\omega) = f_C(\omega) + f_I(\omega) + 2c_{CI}(\omega) \tag{6}$$

where the co-spectrum between the adjusted series and the seasonal component is

$$c_{AS}(\omega) = c_{CS}(\omega) + c_{IS}(\omega), \tag{7}$$

and the spectra, $f(\omega)$, and co-spectra, $c(\omega)$, are for the series or components indicated by the subscripts. The general characteristics of the spectra and co-spectra can be deduced from the description previously given of the basic components in the time domain.

The co-spectrum for any pair of components may be negative, positive or zero depending on the behavior of the cross-correlation function. In particular, if the cross-correlation functions, $R_{AS}(\tau) = R_{CI}(\tau) = 0$ for all τ, as is expected, then the co-spectra $c_{AS}(\omega) = c_{CI}(\omega) = 0$, so that (5) and (6) reduce to

$$f_U(\omega) = f_A(\omega) + f_S(\omega) \tag{8}$$

and

$$f_A(\omega) = f_C(\omega) + f_I(\omega), \tag{9}$$

respectively, and (7) is eliminated.

If seasonality is present, the spectral density function of the seasonal component will have the following properties.

$$\begin{aligned} f_S(\omega) &> 0 \qquad \omega \in \Sigma \\ &= 0 \qquad \omega \in \Sigma^* = \Omega - \Sigma \end{aligned} \tag{10}$$

where, $\Omega = \{\omega: 0 \leq \omega \leq \pi\}$ is the set of all frequencies, Σ is the set of frequencies attributable to the seasonal component, and Σ^* is its complement. Where the seasonal component is stable, the set Σ will contain the frequencies $\omega_j = 2\pi j/12$, $j = 1, 2, \cdots, 6$ which are called the seasonal frequencies, and the spectrum will show a relative peak in the neighborhood of one or more of these frequencies, that is near one or more of the periods 12, 6, 4, 3, 2.4 and 2 months. If, in addition to the stable seasonal there is a moving seasonal component, the apparent frequencies resulting from moving seasonality will spread out from the seasonal frequencies and these will also be included in Σ. If the change is slow they will be close to the seasonal frequencies and also within the seasonal frequency bands, that is, within the frequency intervals $2\pi(j-\frac{1}{2})/12$ to $2\pi(j+\frac{1}{2})/12$. However, if the change in seasonal movement is rapid it is possible for these frequencies to be outside the seasonal frequency bands. This phenomena is illustrated succinctly by a model for moving seasonality suggested by D. K. Fairbarns of the Department of Labour, Ottawa, Canada in a private communication. Let $A_s \cos \omega_s t$ represent a stable seasonal harmonic. If moving seasonality is described by a changing seasonal amplitude represented by $A_a \cos \omega_a t$, then the moving seasonal contribution can be expressed by

$$A_a A_s \cos \omega_a t \cos \omega_s t = \tfrac{1}{2} A_a A_s \{\cos [(\omega_s + \omega_a)t] + \cos [(\omega_s - \omega_a)t]\} \quad (11)$$

so that the apparent frequencies $\omega_s+\omega_a$ and $\omega_s-\omega_a$ may or may not fall within the seasonal frequency band centered at ω_s depending on ω_a. The above model illustrates the point, although a general expression for the seasonal component including phase and all harmonics is more complicated.

The description of the trend-cycle suggests that its spectrum $f_C(\omega)$ principally consists of low frequency components much smaller than the seasonal frequencies. Over the high frequency range $f_\theta(\omega)$ may be zero or close to zero. Hence, we may deduce that the spectral density function of the trend-cycle component has the properties

$$\begin{aligned} f_C(\omega) &> 0 \qquad \omega \in \Gamma \\ &\doteq 0 \qquad \omega \in \Gamma^* = \Omega - \Gamma \end{aligned} \quad (12)$$

where $\Gamma = \{\omega: 0 \leq \omega \ll 2\pi/12\}$ is a set containing trend-cycle frequencies.

If the irregular component is essentially random in character its spectral density function $f_I(\omega)$ will be a constant for all ω, hence its graph will be flat. The occurrence of an occasional catastrophic event probably would have little effect on the appearance of the spectrum. If it has an effect it may emphasize the flatness of the spectrum, since the presence of a single large constant in a series tends to flatten the spectrum of the series. Hence, we may expect that

$$f_I(\omega) = R_I(0)/\pi \qquad \omega \in \Omega \quad (13)$$

where $R_I(0)$ is the variance of the irregular component.

Thus, we can expect that the spectrum of the seasonally adjusted series $A(t)$ will show a peak in the frequency range $0 \leq \omega < 2\pi/12$ and will be relatively flat over the range $\omega > 2\pi/12$. Over the low frequency range the peak will be due to the spectral power of the trend-cycle component, since all its power is in the low frequency range, but there will be added some amount of power, usually small, contributed by the irregular component. Over the high frequency range the entire spectral power of the seasonally adjusted series is due to the irregular component. If the unadjusted series $U(t)$ has a seasonal component, its spectrum will show a peak near one or more of the seasonal frequencies; this power and the power contributed by moving seasonality will be superimposed on the spectrum of the seasonally adjusted series. It should be noted that since the irregular component will add a constant level of spectral power over all frequencies it will necessarily contribute this same level of power at the seasonal frequencies as well.

5. SPECTRAL CRITERIA

The foregoing considerations lead us to list the following spectral criteria (SPC) for a satisfactory decomposition of an economic time series.

1. Spectral criteria for a good seasonal adjustment

   1.1 If there are seasonal peaks in the unadjusted series these peaks should be removed, and should not appear in the spectrum of the seasonally adjusted series. However, at seasonal periods, spectral power should not be reduced to zero but should be reduced to a uniform level consistent with the expectation for the irregular component. Hence, at least over the range of periods from 12 to 2 months, the spectrum of a good seasonally adjusted series should be relatively flat with no dominant peaks or troughs at seasonal periods $12/k$ months, $k = 1, 2, \cdots, 6$.

   1.2 The coherence between the seasonally adjusted and the unadjusted series should be very low at seasonal frequencies; at apparent non-seasonal frequencies, coherence should be very high, but may be reduced if significant moving seasonality occurs.

   1.3 Since the adjustment process should not alter the timing between the unadjusted and the seasonally adjusted series, phase should be zero. However, when the coherence is zero, phase is uniformly distributed, Goodman (1957); therefore, large deviations from zero phase are not unlikely to occur at seasonal frequencies where very low coherence is expected.

   1.4 The seasonally adjusted series and the seasonal component should have zero cross-correlation function, hence their co-spectrum should be zero at all periods.

2. Spectral criteria for separation of irregular component from the seasonally adjusted series

   2.1 The spectrum of the irregular component should be relatively flat over the entire period domain.

   2.2 Over the period range from 12 to 2 months the spectrum of the seasonally adjusted and the irregular series should be similar in appearance, that is, they should have the same spectral power, and both should be relatively flat.

   2.3 The coherence between the irregular and seasonally adjusted series over the period range 12 to 2 months should be very high, and phase should be zero; the coherence over the period range, say, greater than 12 months should be low, and phase arbitrary.

   2.4 The trend-cycle and irregular component should have zero cross-correlation function, hence their co-spectrum should be zero at all periods.

6. ANALYSIS

The figures which follow present correlation functions and spectra of various components of the unemployment series estimated by the Bureau of Labor Statistics and by the Census Bureau using their respective versions of the ratio-to-moving-average method. The specific model for the unadjusted series satisfied by their estimates is the multiplicative one

$$U(t) = A(t) \cdot \hat{S}(t) \tag{14}$$

where

$$A(t) = \hat{C}(t) \cdot \hat{I}(t) \tag{15}$$

is the estimated seasonally adjusted series. The trend-cycle component $\hat{C}(t)$, and the seasonally adjusted series $\hat{A}(t)$ carry the same units as the unadjusted series $U(t)$, but the seasonal component $\hat{S}(t)$ and the irregular component $\hat{I}(t)$ are unitless factors of order of magnitude 1.0. In order to write the unadjusted series as an additive model with all components measured in units of the original series so that spectral power for all components would also be comparable, the seasonal component is estimated as the difference

$$\hat{B}(t) = \hat{U}(t) - A(t) \tag{16}$$

and the irregular as the difference

$$\hat{D}(t) = A(t) - \hat{C}(t). \tag{17}$$

Thus we have the additive model

$$U(t) = A(t) + \hat{B}(t) \tag{18}$$

$$A(t) = \hat{C}(t) + \hat{D}(t) \tag{19}$$

in the form of (1) and (2). The additive model permits using the results for the relation among the estimated spectra and co-spectra

$$f_{U;N}(j) = f_{\hat{A};N}(j) + f_{\hat{B};N}(j) + 2c_{\hat{A}\hat{B};N}(j) \tag{20}$$

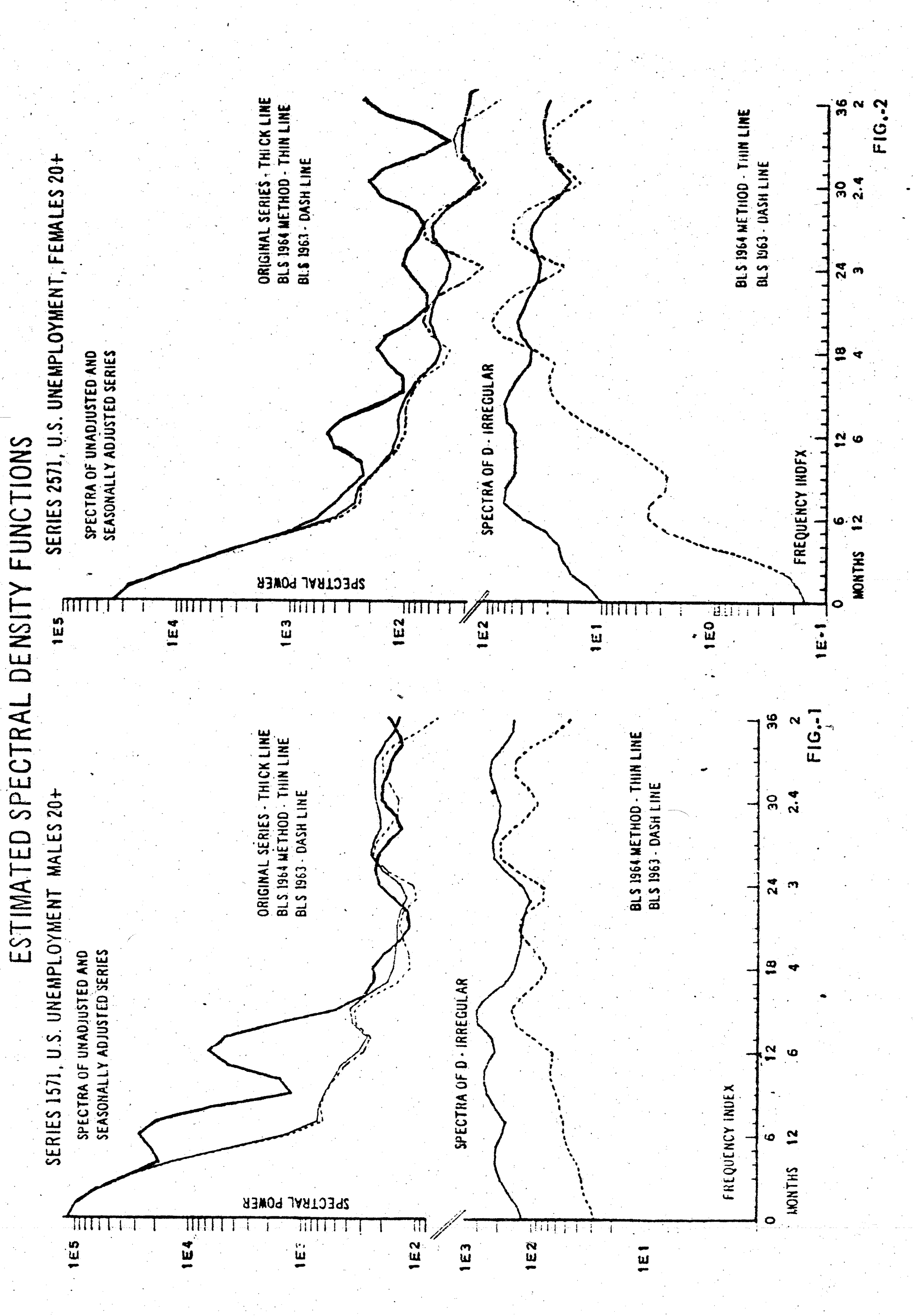
$$f_{\hat{A};N}(j) = f_{\hat{C};N}(j) + f_{\hat{D};N}(j) + 2c_{\hat{C}\hat{D};N}(j) \tag{21}$$

$$c_{\hat{A}\hat{B};N}(j) = c_{\hat{C}\hat{B};N}(j) + c_{\hat{D}\hat{B};N}(j) \tag{22}$$

as in (5), (6) and (7). The spectra and co-spectra, except for level, and the correlation properties of the multiplicative components I-irregular and S-seasonal, are very much the same as for the additive components D-irregular and B-seasonal.

All seasonal adjustments were based on unadjusted data covering the period April 1948 through December 1964. However, the BLS adjustment process does not provide estimates of the trend-cycle, seasonal, or irregular components for the first three and the last three months of the series. Since the beginning and ending dates had to be the same in computing co-spectra, all spectral runs were based on monthly data beginning in July 1948 and ending in September 1964. The spectral results are illustrated in a sequence of charts for the U. S. Unemployment Series for Males 20+, Males 14–19, Females 20+, and Females 14–19. We can now compare the spectral results for the estimated component time series with expectation according to the spectral criteria previously specified.

Figures 1–16, inclusive, show the spectral and correlation characteristics of the BLS method of seasonal adjustment. Basic spectra and co-spectra data for


ESTIMATED SPECTRAL DENSITY FUNCTIONS
SERIES 1571, U.S. UNEMPLOYMENT MALES 20+
SPECTRA OF UNADJUSTED AND SEASONALLY ADJUSTED SERIES
ORIGINAL SERIES - THICK LINE
BLS 1964 METHOD - THIN LINE
BLS 1963 - DASH LINE
SPECTRAL POWER
1E5
1E4
1E3
1E2
SPECTRA OF D - IRREGULAR
BLS 1964 METHOD - THIN LINE
BLS 1963 - DASH LINE
1E1
FREQUENCY INDEX
0
6
12
18
24
30
36
MONTHS
12
6
4
3
2.4
2
FIG.-1
SERIES 2571, U.S. UNEMPLOYMENT, FEMALES 20+
SPECTRA OF UNADJUSTED AND SEASONALLY ADJUSTED SERIES
ORIGINAL SERIES - THICK LINE
BLS 1964 METHOD - THIN LINE
BLS 1963 - DASH LINE
SPECTRAL POWER
SPECTRA OF D - IRREGULAR
BLS 1964 METHOD - THIN LINE
BLS 1963 - DASH LINE
1E0
1E-1
FREQUENCY INDEX
MONTHS
FIG.-2

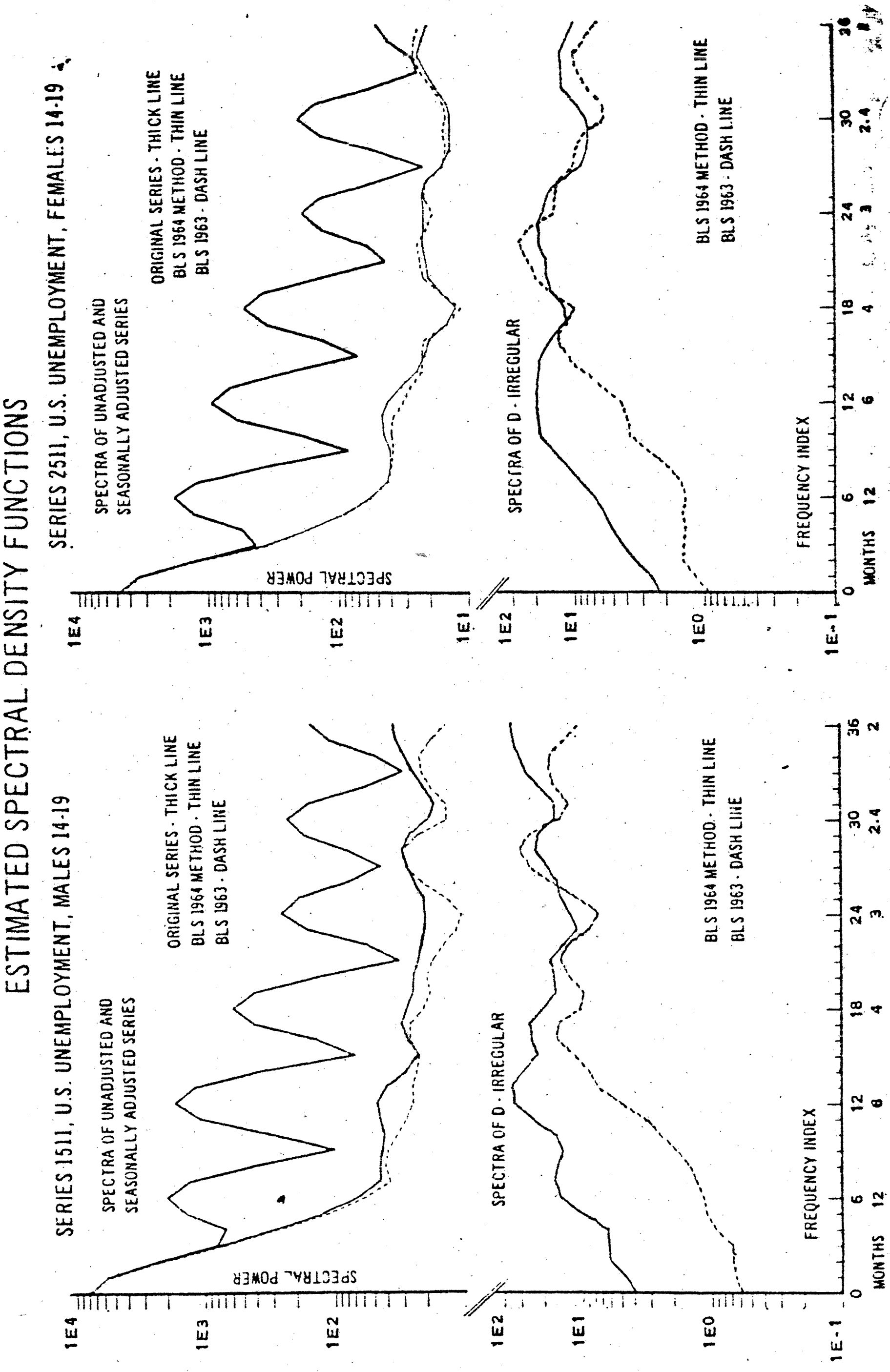

ESTIMATED SPECTRAL DENSITY FUNCTIONS
SERIES 1511, U.S. UNEMPLOYMENT, MALES 14-19
SERIES 2511, U.S. UNEMPLOYMENT, FEMALES 14-19
SPECTRA OF UNADJUSTED AND SEASONALLY ADJUSTED SERIES
ORIGINAL SERIES - THICK LINE
BLS 1964 METHOD - THIN LINE
BLS 1963 - DASH LINE
SPECTRAL POWER
SPECTRA OF D - IRREGULAR
BLS 1964 METHOD - THIN LINE
BLS 1963 - DASH LINE
FREQUENCY INDEX
MONTHS
1E4
1E3
1E2
1E1
1E0
1E-1
0
6
12
18
24
30
36
12
6
4
3
2.4
2

FIG.-3

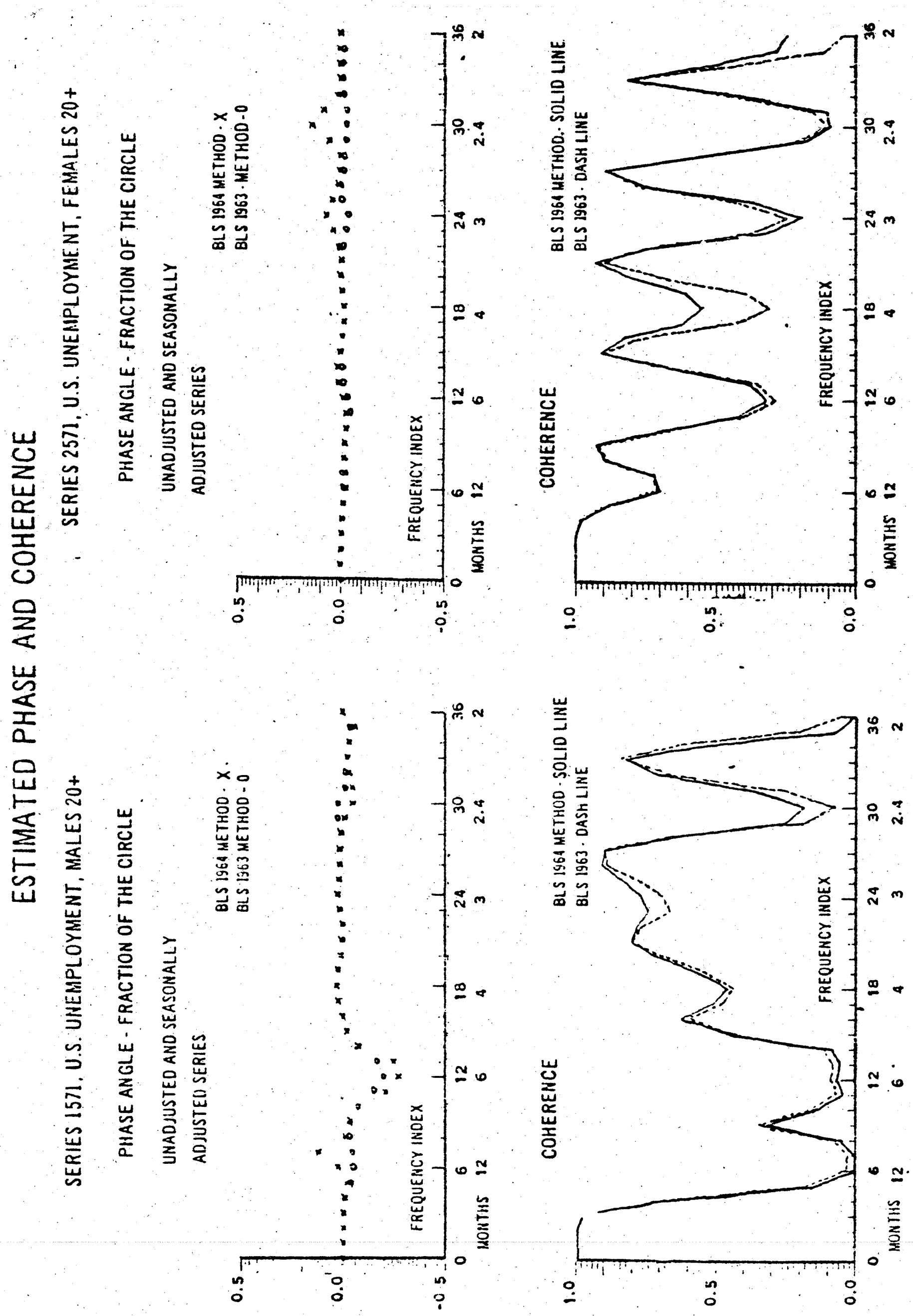

ESTIMATED PHASE AND COHERENCE
SERIES 1571, U.S. UNEMPLOYMENT, MALES 20+
SERIES 2571, U.S. UNEMPLOYMENT, FEMALES 20+
PHASE ANGLE - FRACTION OF THE CIRCLE
UNADJUSTED AND SEASONALLY
ADJUSTED SERIES
BLS 1964 METHOD - X
BLS 1963 METHOD - 0
0.5
0.0
-0.5
FREQUENCY INDEX
0 6 12 18 24 30 36
MONTHS 12 6 4 3 2.4 2
COHERENCE
BLS 1964 METHOD - SOLID LINE
BLS 1963 - DASH LINE
1.0
0.5
0.0

FIG.-5

FIG.-6

ESTIMATED PHASE AND COHERENCE

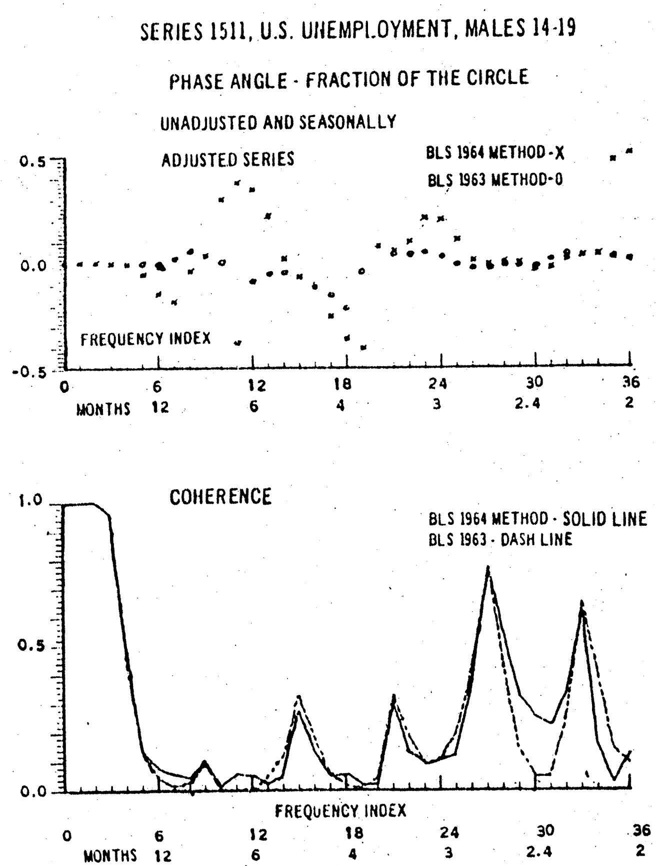


FIG.-7

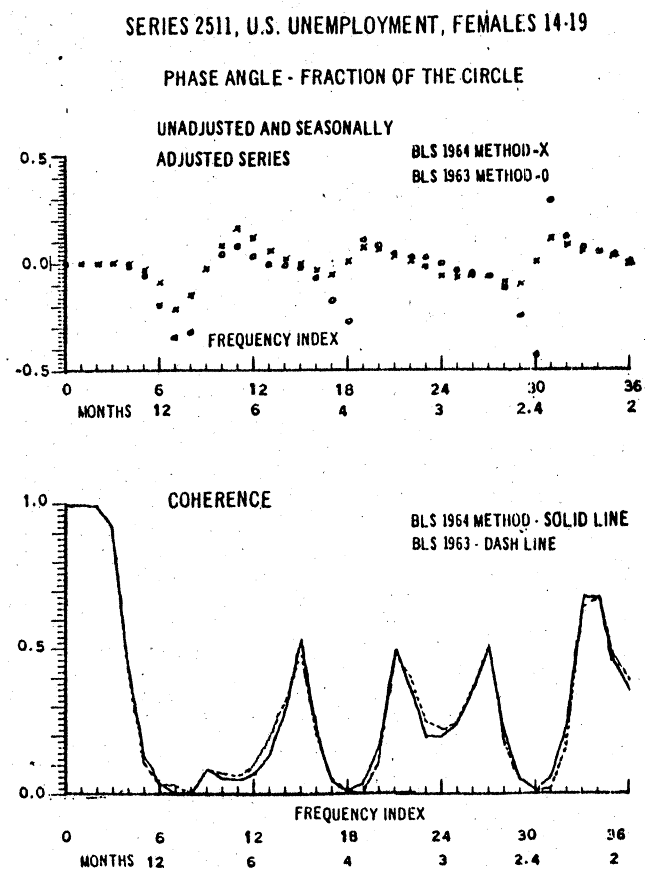


FIG.-8

ESTIMATED PHASE AND COHERENCE

SERIES 1571, U.S. UNEMPLOYMENT, MALES 20+

PHASE ANGLE - FRACTION OF THE CIRCLE

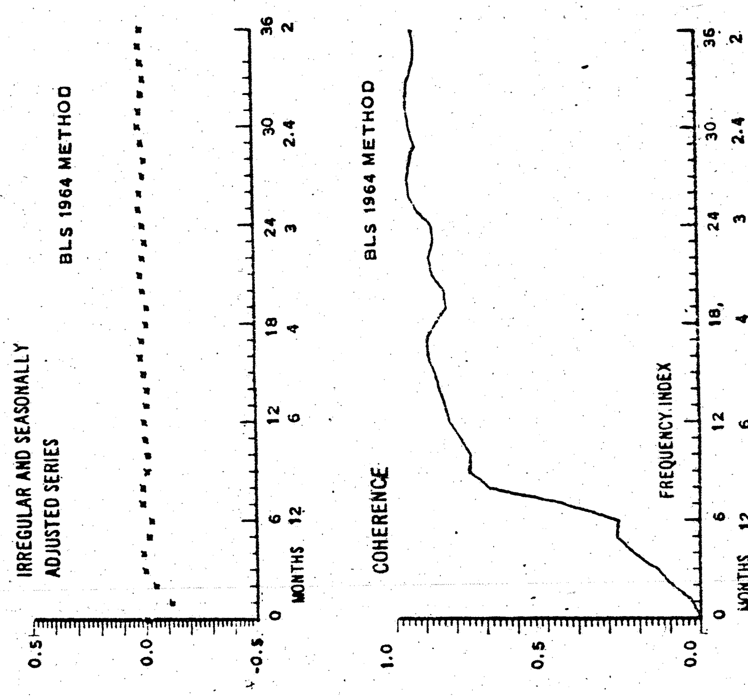


FIG.-9

SERIES 1511, U.S. UNEMPLOYMENT, MALES 14-19

PHASE ANGLE - FRACTION OF THE CIRCLE

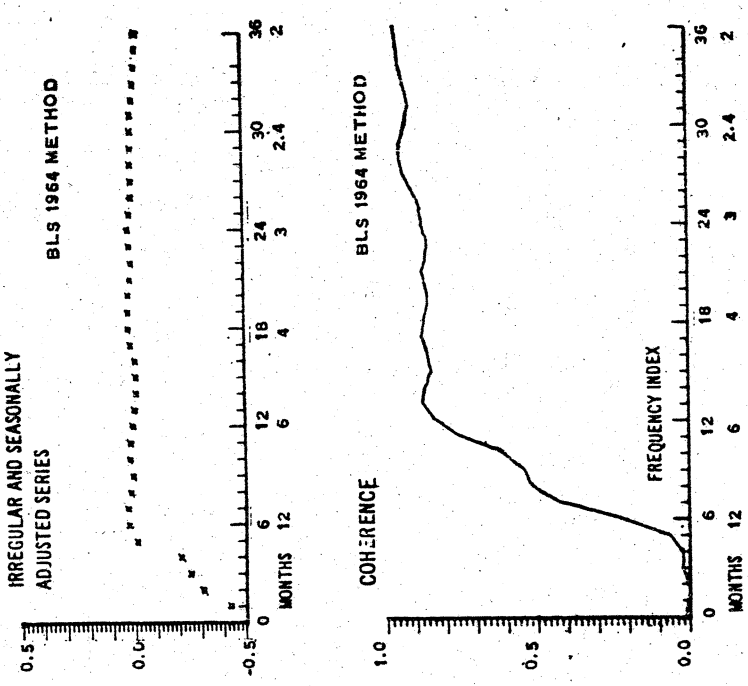
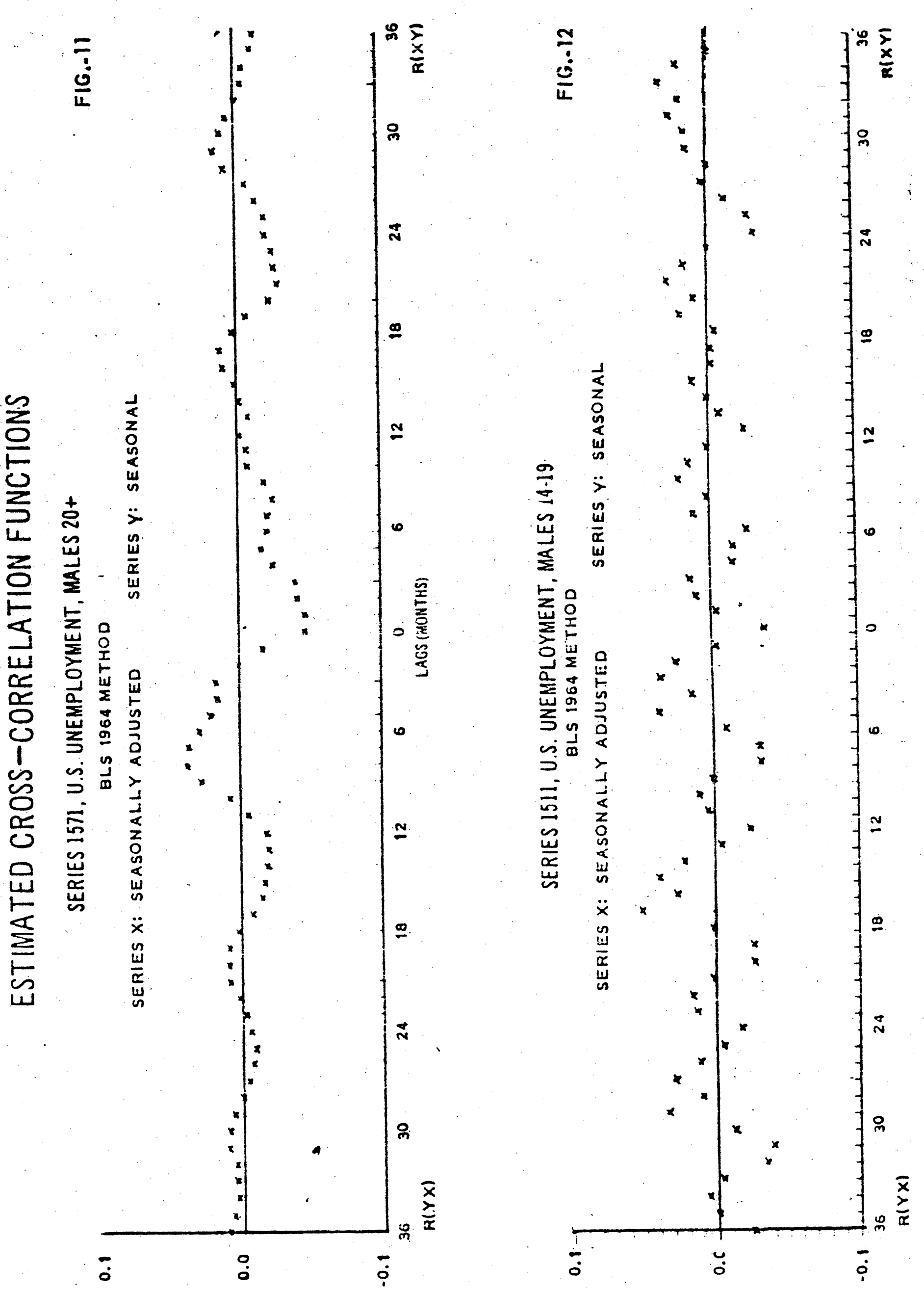


FIG.-10


ESTIMATED CROSS-CORRELATION FUNCTIONS
SERIES 1571, U.S. UNEMPLOYMENT, MALES 20+
BLS 1964 METHOD
SERIES X: SEASONALLY ADJUSTED
SERIES Y: SEASONAL
FIG.-11
0.1
0.0
-0.1
36
30
24
18
12
6
0
6
12
18
24
30
36
R(YX)
R(XY)
LAGS (MONTHS)
SERIES 1511, U.S. UNEMPLOYMENT, MALES 14-19
BLS 1964 METHOD
SERIES X: SEASONALLY ADJUSTED
SERIES Y: SEASONAL
FIG.-12
0.1
0.0
-0.1
R(YX)
R(XY)

ESTIMATED CROSS—CORRELATION FUNCTIONS

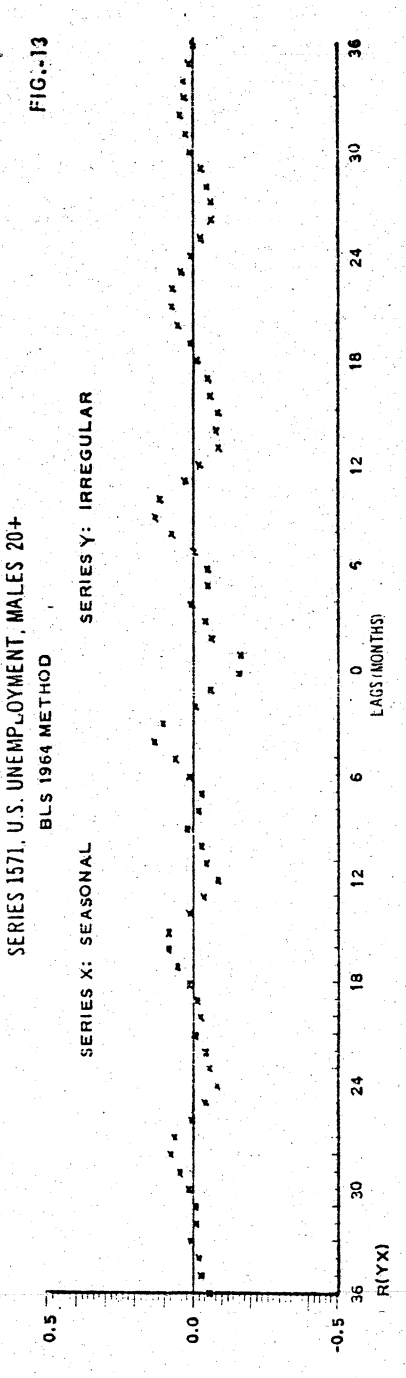


FIG.-13

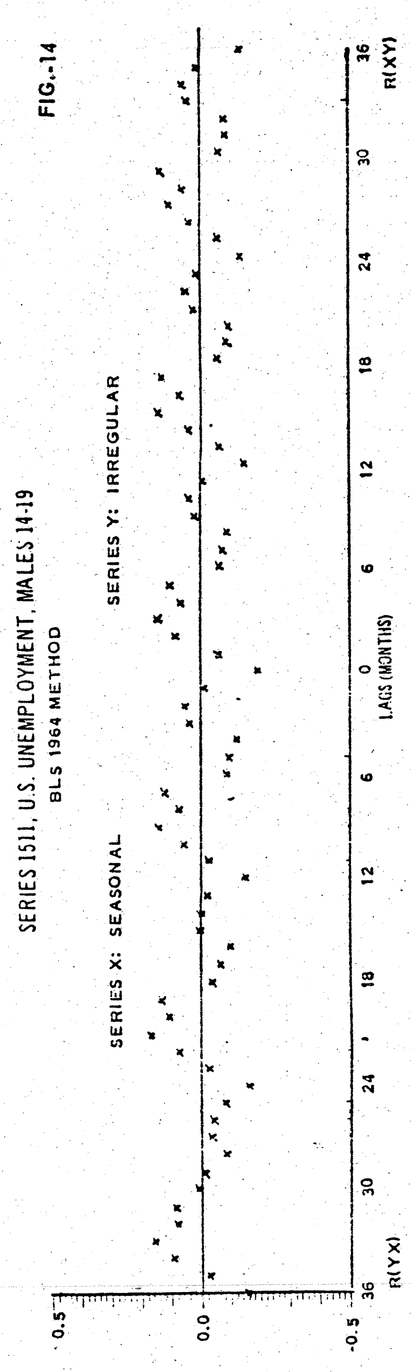
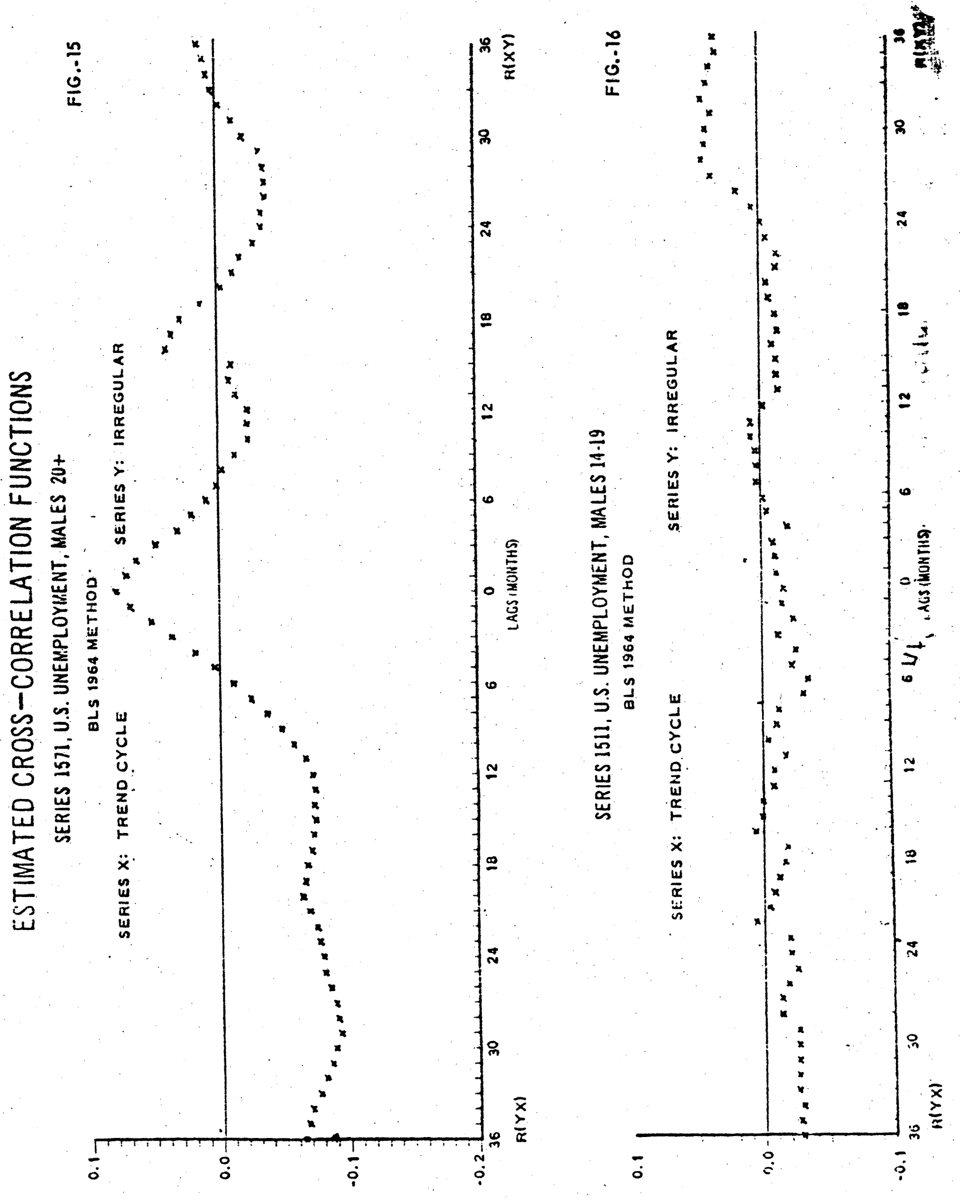


FIG.-14


ESTIMATED CROSS—CORRELATION FUNCTIONS
SERIES 1571, U.S. UNEMPLOYMENT, MALES 20+
BLS 1964 METHOD
SERIES X: TREND CYCLE
SERIES Y: IRREGULAR
FIG.-15
0.1
0.0
-0.1
-0.2
36
30
24
18
12
6
0
LAGS (MONTHS)
R(YX)
R(XY)
SERIES 1511, U.S. UNEMPLOYMENT, MALES 14-19
BLS 1964 METHOD
SERIES X: TREND CYCLE
SERIES Y: IRREGULAR
FIG.-16
0.1
0.0
-0.1
R(YX)
R(XY)

the BLS 1964 method are illustrated in Tables 1 and 2, respectively, for Males 20+ and Males 14–19.

It is clear from Figures 1–4 that the BLS 1964 method shows an overall improvement over the BLS 1963 method with respect to the spectral criteria, SPC(1.1) for removal of the seasonal peaks, and SPC(2.1) concerning the flatness of the irregular spectrum. This follows from the observation that, at seasonal and trend-cycle frequencies, an improvement over the older method is observed with respect to the phenomena sometimes referred to as "dips" (i.e., excessive removal of spectral power) at these frequencies. An examination of Figures 5–8, suggests that there is little choice between the two methods in so far as coherence and phase between the unadjusted and seasonally adjusted series is concerned, (SPC(1.2) and SPC(1.3)). The fact that coherence is not very high at apparent nonseasonal periods, especially for Males and Females 14–19, may be due to significant moving seasonality. Some evidence for the occurrence of moving seasonality is presented subsequently in discussing another spectral property.

Spectral criteria SPC(1.4) suggests that the seasonally adjusted series and the seasonal component have zero cross-correlation, hence zero co-spectrum at all periods. The cross-correlation between the estimates of these components are indeed very small, within ±.05, however, they exhibit an approximate 12 month periodic pattern. To show this pattern the correlation scale has been magnified five fold; the resulting plot is given in Figure 11 for Males 20+ and in Figure 12 for Males 14–19. Of the two components which comprise the seasonally adjusted series, we find that the seasonal component is more closely correlated with the irregular than with the trend-cycle.

The cross-correlation function between the seasonal and trend-cycle components has a negligible roughly 12 month oscillation within ±.01 for Males 20+ and within ±.03 for Males 14–19. The correlations are not plotted, but the co-spectra are shown in Tables 1 and 2. The correlations between the seasonal and the irregular components are larger, within ±.20, and the periodic pattern is clearly visible over a standard correlation scale. This is seen in Figures 13 and 14 for Males 20+ and Males 14–19, respectively. Comparable cross-correlation plots for females are similar to those given for males and are not shown. A. Rothman, formerly of the Bureau of Labor Statistics, who is partly responsible for the BLS methods of adjustment, suggests that the 12 month periodic pattern is to be expected because the estimation of the seasonal component by a moving average of seasonal-irregular values over years for each month results in seasonal values which include a portion of the irregular values from adjacent years.

Although the cross-correlations between the estimated seasonal component and seasonally adjusted series are very small, their co-spectrum, see (5) and (7), is not close to zero, nor is it always small relative to the power in the unadjusted series spectrum; in fact, it is negative for many frequency bands. Since we expected zero co-spectrum for the true components, the fact that the co-spectrum for the estimated components is not zero alters the simple additive relation, (8), between these two spectra to that of equation (5). Thus, at some frequencies the sum of these component power spectra may be less than, or

TABLE 1. COMPONENTS OF SPECTRUM OF UNADJUSTED SERIES FOR U.S. UNEMPLOYMENT, MALES 20+

BLS 1964 Method of Adjustment

		Spectra				$f_U - f_{\hat{A}} =$	Spectra		Co-spectra		
J	Period	Unadjusted $f_{U;N}(J)$	Seas. Adj. $f_{\hat{A};N}(J)$	Seasonal $f_{\hat{B};N}(J)$	Co-spectrum $2c_{\hat{A}\hat{B};N}(J)$	$f_{\hat{B}} + 2c_{\hat{A}\hat{B}}$	Trend-Cycle $f_{\hat{C};N}(J)$	Irregular $f_{\hat{D};N}(J)$	$2c_{\hat{C}\hat{D};N}(J)$	$2c_{\hat{C}\hat{B};N}(J)$	$2c_{\hat{D}\hat{B};N}(J)$
0		117114	118584	161	-1631	-1470	118512	120	-48	-1563	-68
1		100511	101798	120	-1407	-1287	100961	133	704	-1343	-64
2		65356	66454	197	-1296	-1099	64713	163	1578	-1203	-87
3		33631	34648	797	-1814	-1017	32888	193	1567	-1621	-193
4		17541	14450	5524	-2433	3091	13035	208	1206	-1967	-466
5		21461	4930	18899	-2368	16531	4091	203	636	-1484	-884
6	12	28630	1569	28415	-1355	27060	1209	178	182	-373	-982
7		20237	725	19702	-190	19512	430	171	124	344	-535
8		6833	679	6001	153	6154	266	209	204	279	-126
9		1628	611	1040	-23	1017	186	246	179	86	-108
10		1798	468	1648	-318	1330	126	262	80	39	-357
11		4330	365	4804	-839	3965	82	247	36	-105	-734
12	6	6138	312	6936	-1110	5826	60	206	47	-227	-883
13		4299	316	4742	-759	3983	54	226	36	-116	-643
14		1573	348	1508	-282	1226	53	298	-3	49	-332
15		534	328	277	-70	207	45	304	-21	81	-151
16		375	280	107	-12	95	34	233	12	46	-58
17		365	242	112	11	123	29	168	45	22	-11
18	4	332	194	129	9	138	28	139	27	6	3
19		214	140	82	-8	74	25	128	-13	-4	-4
20		143	124	38	-19	19	22	122	-20	-3	-16
21		145	147	22	-24	-2	20	120	7	-2	-22
22		156	155	22	-21	1	19	110	25	-6	-16
23		166	123	36	2	38	17	102	9	-7	9
24	3	201	123	52	27	79	16	119	-13	-1	28
25		242	186	42	14	56	17	164	5	6	8
26		262	273	24	-34	-10	17	210	45	4	-39
27		222	282	25	-85	-60	16	218	48	3	-87
28		157	218	70	-132	-62	14	196	9	13	-145
29		161	175	176	-190	-14	13	183	-20	19	-209
30	2.4	215	197	249	-232	17	13	188	-3	0+	-232
31		230	254	190	-213	-23	13	209	31	-21	-192
32		217	276	86	-145	-59	13	233	31	-15	-130
33		182	226	49	-93	-44	12	216	-2	5	-98
34		135	164	98	-128	-30	12	169	-17	8	-135
35		134	159	233	-259	-26	12	142	5	-20	-239
36	2	152	173	321	-342	-21	12	138	23	-40	-302
Weighted sum*		377,387	291,322	102,692	-16,629		277,823	6,745	6,754	-8,315	-8,315

*The weighted sum of the spectra and of twice the co-spectra correspond to the variance and twice the covariance, respectively, of the subscripted time series. Weights are ½ for j=0,36 and 1 for j=1,2,...,35.

TABLE 2. COMPONENTS OF SPECTRUM OF UNADJUSTED SERIES FOR U.S. UNEMPLOYMENT, MALES 14-19

BLS 1964 Method of Adjustment

		Spectra				$f_U - f_{\hat{A}}$	Spectra		Co-spectra		
		Unadjusted	Seas. Adj.	Seasonal	Co-spectrum		Trend-Cycle	Irregular			
J	Period	$f_{U;N}(J)$	$f_{\hat{A};N}(J)$	$f_{\hat{B};N}(J)$	$2c_{\hat{A}\hat{B};N}(J)$	$f_{\hat{B}}+2c_{\hat{A}\hat{B}}$	$f_{\hat{C};N}(J)$	$f_{\hat{D};N}(J)$	$2c_{\hat{C}\hat{D};N}(J)$	$2c_{\hat{C}\hat{B};N}(J)$	$2c_{\hat{D}\hat{B};N}(J)$
0		8307	8021	21	265	286	8064	4	-47	264	2
1		5979	5789	16	174	190	5820	5	-36	175	0-
2		2376	2314	16	46	62	2327	6	-19	48	-2
3		835	761	50	24	74	768	6	-13	23	0+
4		754	311	373	70	443	314	6	-9	76	-7
5		1533	146	1260	128	1388	136	10	-1	164	-36
6	12	2082	95	1847	139	1986	75	15	5	184	-44
7		1381	61	1243	77	1320	37	17	8	92	-15
8		418	45	363	9	372	22	16	7	8	1
9		99	33	74	-8	66	15	14	3	-2	-6
10		322	27	322	-26	296	10	17	0+	15	-41
11		1083	26	1162	-106	1056	7	24	-5	21	-127
12	6	1639	32	1783	-177	1606	5	33	-7	-3	-175
13		1140	39	1251	-150	1101	5	36	-2	-31	-119
14		342	35	390	-83	307	5	28	2	-30	-52
15		52	27	76	-51	25	5	22	1	-18	-34
16		115	27	143	-55	88	4	25	-1	-13	-42
17		384	28	447	-91	356	3	26	-1	-19	-72
18	4	571	23	661	-113	548	3	20	0+	-24	-89
19		393	19	451	-77	374	3	16	0+	-18	-59
20		125	18	137	-30	107	2	17	-2	-9	-20
21		30	18	28	-15	13	2	18	-2	-6	-9
22		51	16	54	-19	35	2	14	0-	-5	-14
23		151	14	165	-29	136	2	11	1	-1	-28
24	3	231	16	245	-29	216	2	13	1	6	-34
25		171	18	169	-16	153	2	15	1	6	-22
26		68	21	54	-7	47	2	16	3	1	-8
27		38	26	14	-1	13	2	19	5	0+	-1
28		72	29	35	9	44	2	23	4	3	6
29		158	26	110	22	132	1	22	3	6	16
30	2.4	208	20	165	23	188	2	17	2	3	20
31		146	18	119	9	128	2	16	0-	-5	14
32		61	20	44	-3	41	2	22	-3	-7	5
33		34	23	17	-6	11	2	27	-6	-6	0+
34		51	26	51	-26	25	2	32	-7	-6	-20
35		106	32	161	-87	74	1	35	-4	-9	-78
36	2	143	36	235	-128	107	1	36	-2	-12	-116
Weighted sum*		27,424	14,207	13,623	-407		13,626	679	-96	745	-1,149

*The weighted sum of the spectra and of twice the co-spectra correspond to the variance and twice the covariance, respectively, of the subscripted time series. Weights are ½ for J=0,36 and 1 for J=1,2,...,35.

more than, the spectral power for the unadjusted series, according as the co-spectrum is positive, or negative; the power for either component may even be greater than the power for the unadjusted series; or, the power for the adjusted series may be less than that for the unadjusted series at nonseasonal frequencies, and less than expected at seasonal frequencies after removal of seasonal power.

The finding that spectra for the seasonally adjusted series was almost always less at all frequencies than the corresponding spectra for the unadjusted series for unemployment was considered serious by the Gordon Committee Report (1962, p. 177), and the most striking finding by Nerlove (1964, p. 267) in their study of the BLS 1962 method. To show this apparent loss in power their studies presented graphs similar to Figures 1–4. Except for Males 20+, where the adjusted series spectrum actually exceeds the unadjusted at the high frequency range, these charts reveal that the spectrum of the adjusted series lies below the spectrum of the unadjusted series over the entire frequency range, especially for the younger age groups.

However, whether there really is an excessive loss in power, and if so, whether the effect is serious has never been adequately demonstrated. The question of its seriousness, it would seem, should be carefully examined in the time domain in the spirit suggested in the second paragraph of Section 1. Whether the loss is real actually depends on the model appropriate for the seasonal component. For example, if the seasonal component changes rapidly and the model (11) is reasonable, a proper seasonal adjustment would remove power over much of the low frequency range and not only at the stable seasonal frequencies, so that the spectra as seen in Figures 1–4 could be appropriate. To check the extent of moving seasonality, we examined a measure provided by Census Method II, X-11 since these data were readily available. There should be little difference for the comparable BLS data. Table F2 SUMMARY MEASURES of the X-11 computer print-out shows the average year to year percent change without regard to sign of the estimated seasonal component. For each series, this measure and its magnitude relative to that for Males 20+ is shown in Table 3. We note that between the two age groups these measures increase in

TABLE 3. MOVING SEASONALITY MEASURES, U. S. UNEMPLOYMENT SERIES

	Males 20+	Females 20+	Males 14–19	Females 14–19
Measure	.61	.68	.96	1.05
Relative Magnitude	1.00	1.11	1.57	1.72

magnitude in the direction of the extent of apparent loss in spectral power as seen in Figures 1–4, so that they are not inconsistent with an hypothesis which cites moving seasonality as a possible explanation.

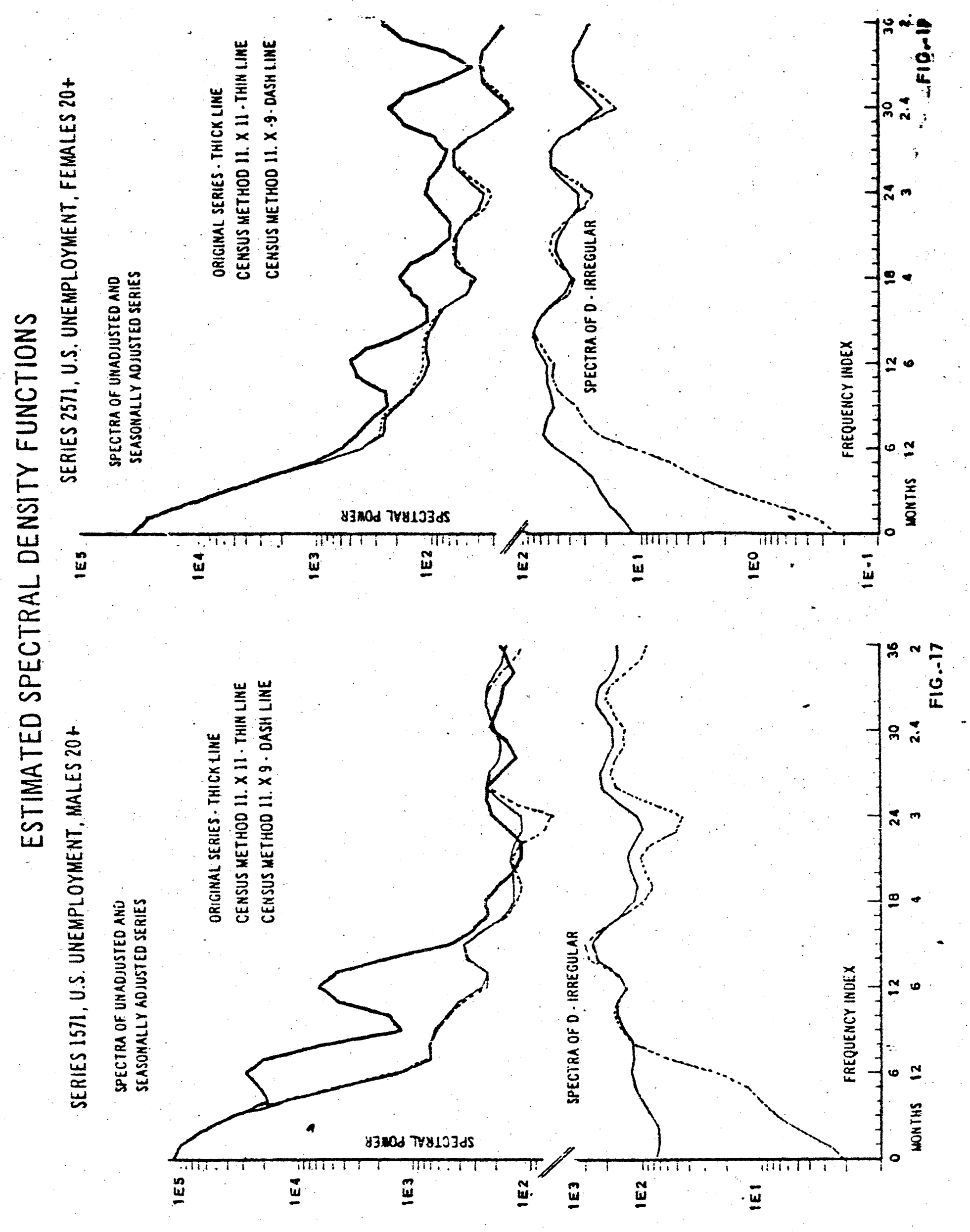
The similarity between the spectra of the irregular component and the seasonally adjusted series seen in Figures 1–4 over the period range 12 to 2 months is clearly evident for either BLS method, in accordance with SPC(2.2). This is unlike the loss in spectral power noted between the unadjusted and seasonally adjusted series. In addition, the coherence and phase between the irregular and adjusted series, shown in Figures 9 and 10 for the BLS 1964 method, have the expected properties of SPC(2.3). The cross-correlation function between the trend-cycle and irregular component is very small, between ±.10 for Males 20+ and between ±.05 for Males 14–19. These are shown in Figures 15 and 16, respectively, on a correlation scale expanded five fold. The 12 month periodic pattern, observed for the small correlations between the seasonally adjusted series and seasonal component, is not noted here. The fact that, over the high frequency range, the co-spectrum between these components is very small, and the trend-cycle spectral power is also very small, explains why the spectra of the irregular and seasonally adjusted series are similar over this range. This suggests that SPC(2.4) is reasonably well satisfied.

A set of graphs for Census Method II seasonal adjustment procedure corresponding to those for the BLS method is shown in Figures 17–24 comparing the X-11 and X-9 versions and in Figures 25–32 for X-11 alone. Comments on the spectral properties of the Census Method II seasonal adjustment procedure would be similar to those made for the BLS method. One notes an overall improvement in the X-11 spectral properties as compared with X-9.

This paper would not be complete without attempting some comment on the effect of the greater deviations from spectral criteria for the earlier methods since they have been in wide use. For example, using the X-11 variant with its improved spectral properties as a standard, a judgment may be made by comparing in the time domain the older X-9 variant to the X-11 based on summary statistics computed from the adjustments. The Census Method is discussed since summary statistics pertaining to its seasonal adjustment are readily available. An example of some data for comparing the two methods is given in Table 4. Other data may occur to the reader. Since the BLS and Census methods generally give similar results, comparable data for the BLS methods, and the conclusion for either approach, should not be different. Except for the data on percent differences between the Census X-9 and X-11 variants, all entries in Table 4 have been extracted from the SUMMARY MEASURES tables of the computer print-out of the Census seasonal adjustments. The SUMMARY MEASURES data are in general use by those who follow the Census methods to extract information from the seasonal adjustments.

For example, the MCD, or months for cyclical dominance, is the first span in months for which the absolute average percent change in the trend-cycle curve exceeds the absolute average percent change in the irregular movement and remains so. The MCD curve is the result of applying a moving average of MCD length to a seasonally adjusted series. Because the average trend-cycle changes exceed the average irregular changes, differences of MCD span in the seasonally adjusted series, likewise differences in adjacent values on the MCD curve, are considered to be of economic importance to users of this concept.

One measure of the difference between the X-9 and X-11 variants is the difference between their month to month percent change averages for each component. It is seen from Table 4 that these differences are small relative to the magnitude of the values. Another measure is the average percent difference between the X-9 and X-11 seasonally adjusted series as compared to changes


ESTIMATED SPECTRAL DENSITY FUNCTIONS
SERIES 1571, U.S. UNEMPLOYMENT, MALES 20+
SPECTRA OF UNADJUSTED AND SEASONALLY ADJUSTED SERIES
ORIGINAL SERIES - THICK LINE
CENSUS METHOD II. X 11 - THIN LINE
CENSUS METHOD II. X 9 - DASH LINE
SPECTRAL POWER
SPECTRA OF D - IRREGULAR
FREQUENCY INDEX
MONTHS
FIG.-17
SERIES 2571, U.S. UNEMPLOYMENT, FEMALES 20+
SPECTRA OF UNADJUSTED AND SEASONALLY ADJUSTED SERIES
ORIGINAL SERIES - THICK LINE
CENSUS METHOD II. X 11 - THIN LINE
CENSUS METHOD II. X 9 - DASH LINE
SPECTRAL POWER
SPECTRA OF D - IRREGULAR
FREQUENCY INDEX
MONTHS
FIG.-18

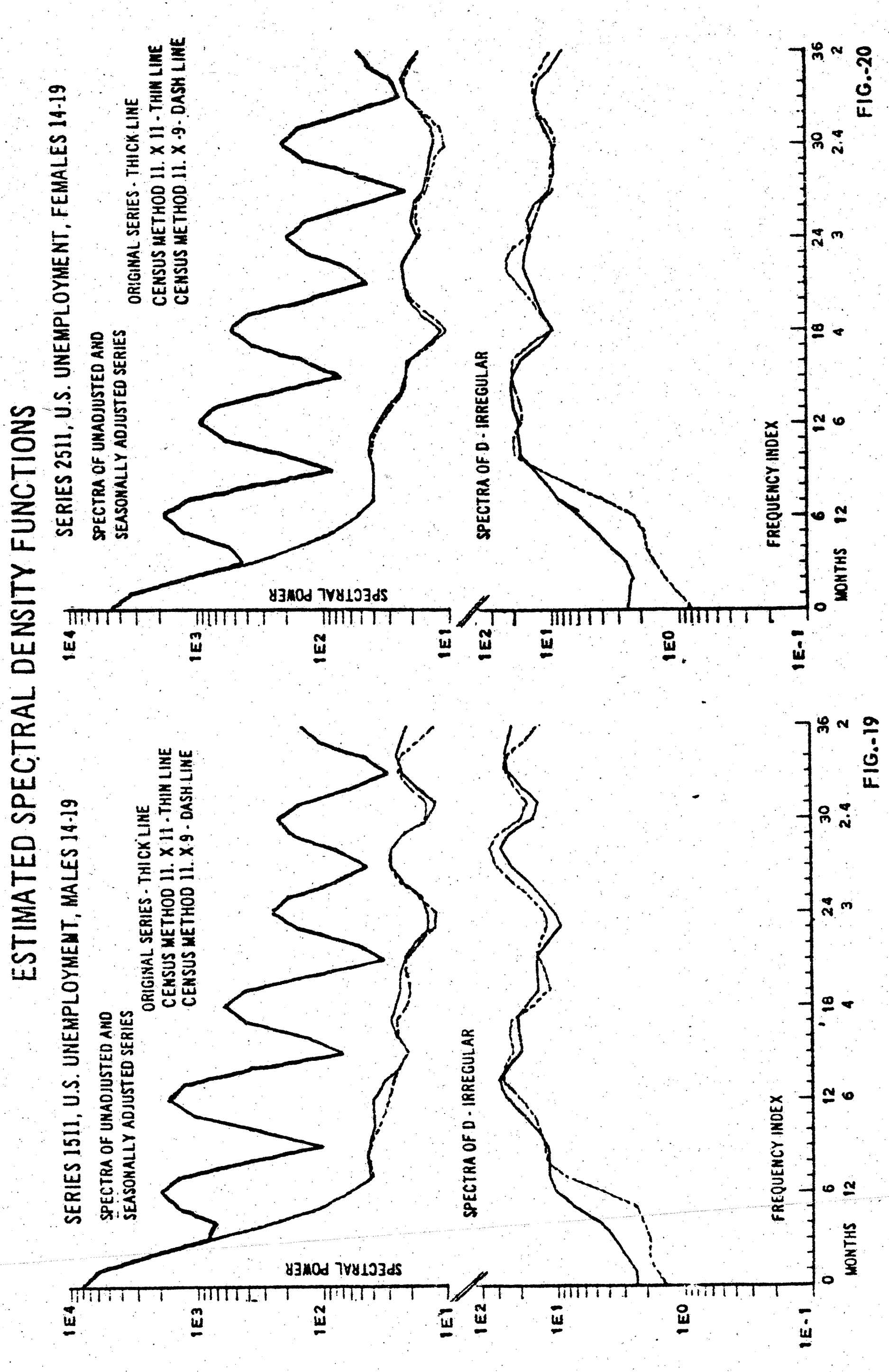
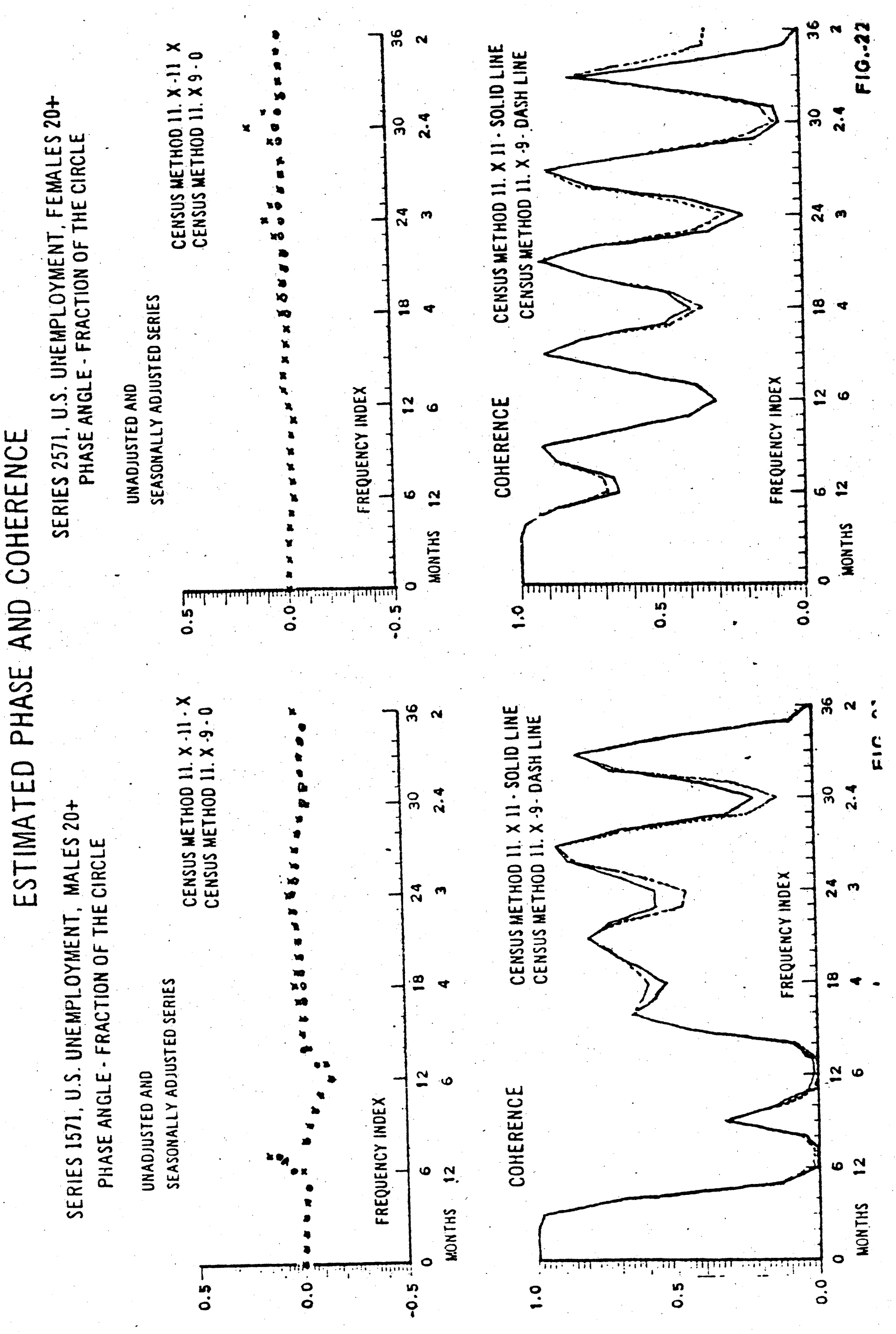

ESTIMATED SPECTRAL DENSITY FUNCTIONS
SERIES 1511, U.S. UNEMPLOYMENT, MALES 14-19
SPECTRA OF UNADJUSTED AND SEASONALLY ADJUSTED SERIES
ORIGINAL SERIES - THICK LINE
CENSUS METHOD 11. X 11 - THIN LINE
CENSUS METHOD 11. X 9 - DASH LINE
SPECTRA OF D - IRREGULAR
SPECTRAL POWER
1E4
1E3
1E2
1E1
1E2
1E1
1E0
1E-1
FREQUENCY INDEX
0
6
12
18
24
30
36
MONTHS
12
6
4
3
2.4
2
SERIES 2511, U.S. UNEMPLOYMENT, FEMALES 14-19
SPECTRA OF UNADJUSTED AND SEASONALLY ADJUSTED SERIES
ORIGINAL SERIES - THICK LINE
CENSUS METHOD 11. X 11 - THIN LINE
CENSUS METHOD 11. X 9 - DASH LINE
SPECTRA OF D - IRREGULAR
SPECTRAL POWER
FREQUENCY INDEX
MONTHS

FIG.-19

FIG.-20


ESTIMATED PHASE AND COHERENCE
SERIES 1571, U.S. UNEMPLOYMENT, MALES 20+
PHASE ANGLE - FRACTION OF THE CIRCLE
UNADJUSTED AND
SEASONALLY ADJUSTED SERIES
CENSUS METHOD 11. X-11-X
CENSUS METHOD 11. X-9-0
FREQUENCY INDEX
MONTHS
COHERENCE
CENSUS METHOD 11. X 11 - SOLID LINE
CENSUS METHOD 11. X-9- DASH LINE
SERIES 2571, U.S. UNEMPLOYMENT, FEMALES 20+
PHASE ANGLE - FRACTION OF THE CIRCLE
UNADJUSTED AND
SEASONALLY ADJUSTED SERIES
CENSUS METHOD 11. X-11 X
CENSUS METHOD 11. X 9-0
FREQUENCY INDEX
MONTHS
COHERENCE
CENSUS METHOD 11. X 11 - SOLID LINE
CENSUS METHOD 11. X-9- DASH LINE
FIG.-22

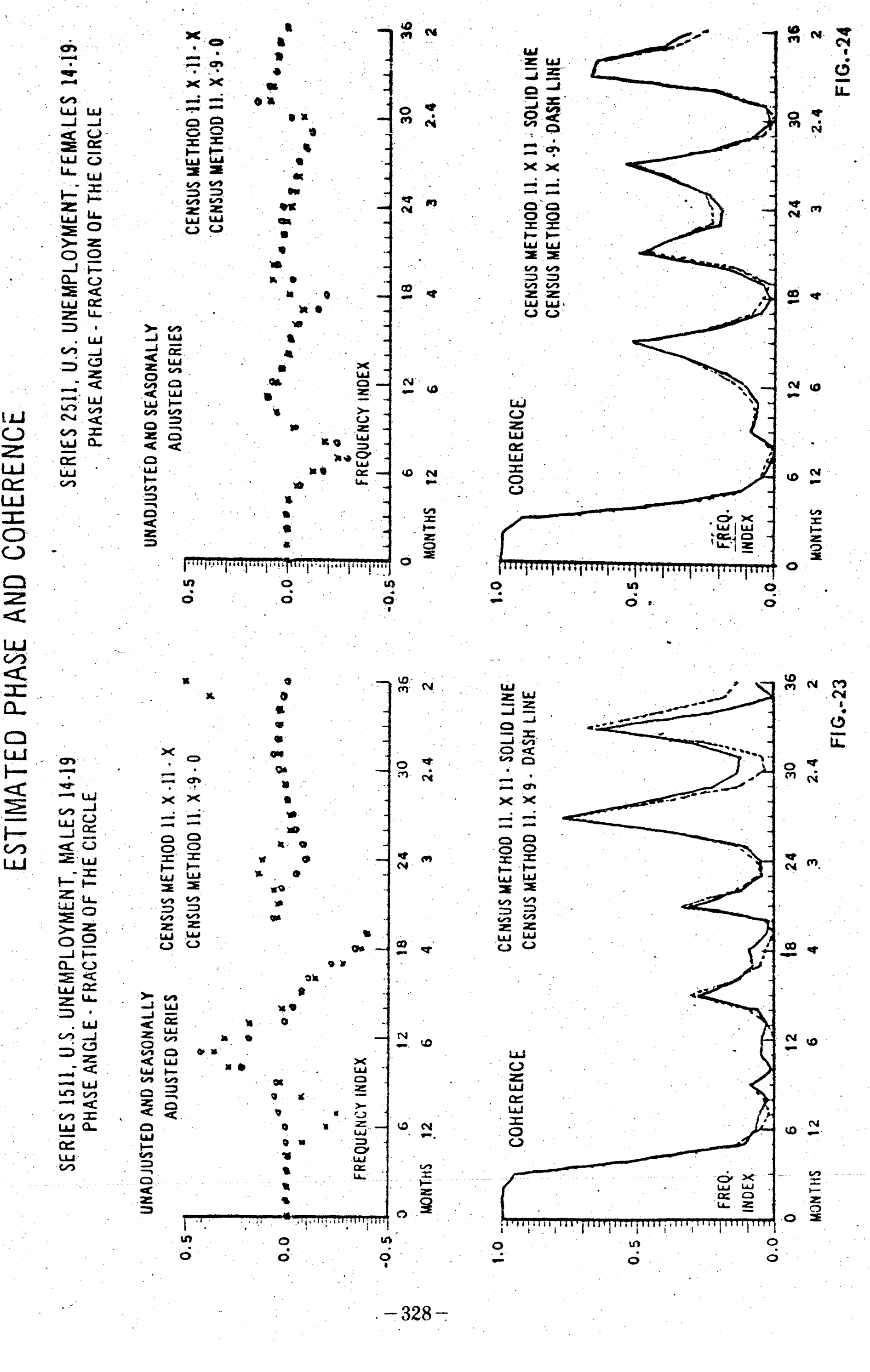
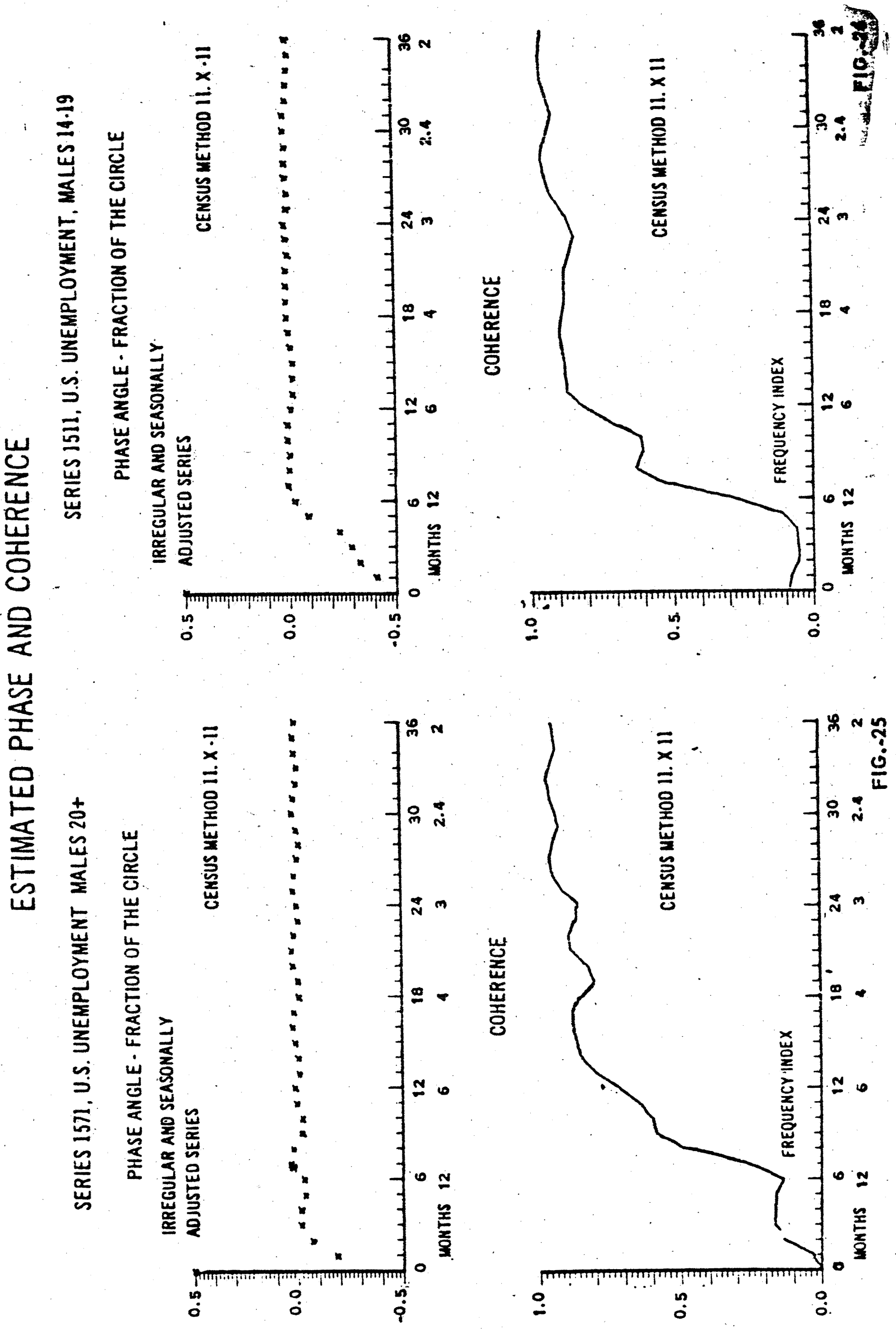
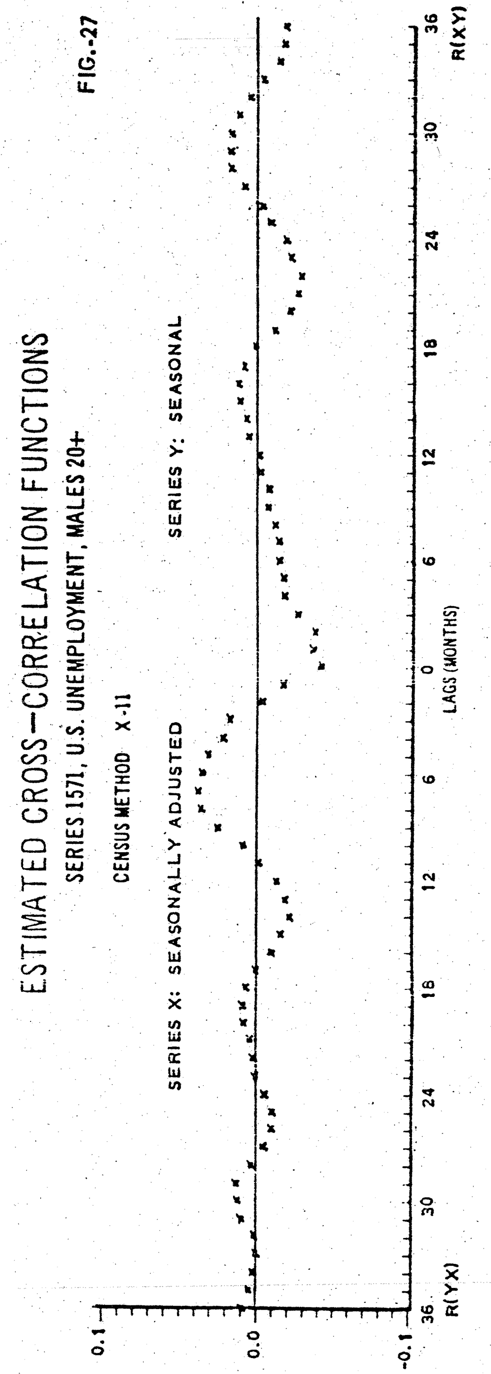
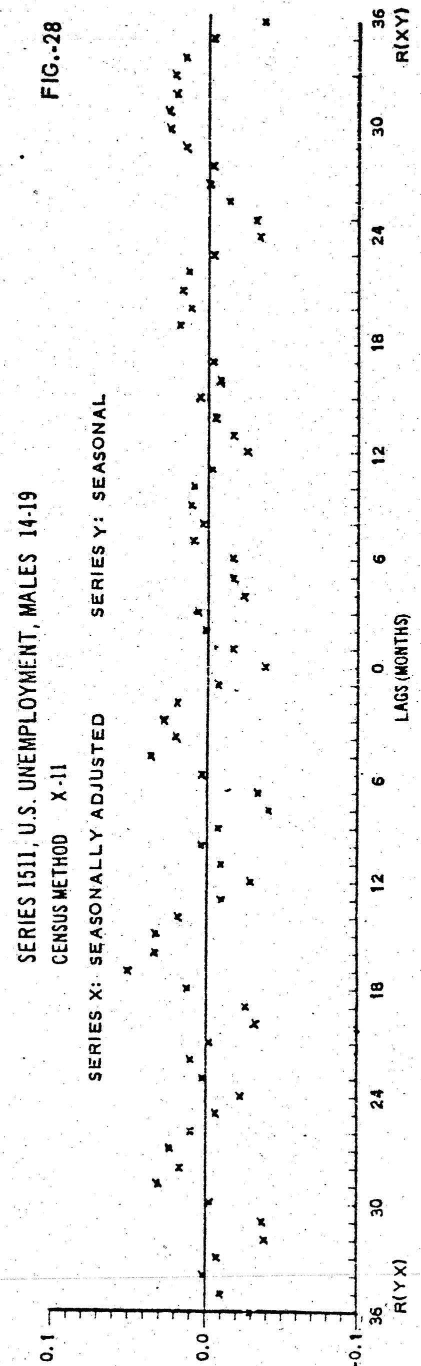

ESTIMATED PHASE AND COHERENCE
SERIES 1511, U.S. UNEMPLOYMENT, MALES 14-19
PHASE ANGLE - FRACTION OF THE CIRCLE
UNADJUSTED AND SEASONALLY ADJUSTED SERIES
CENSUS METHOD II. X-11 - X
CENSUS METHOD II. X-9 - 0
FREQUENCY INDEX
MONTHS
COHERENCE
CENSUS METHOD II. X 11 - SOLID LINE
CENSUS METHOD II. X 9 - DASH LINE
FREQ. INDEX
SERIES 2511, U.S. UNEMPLOYMENT, FEMALES 14-19
PHASE ANGLE - FRACTION OF THE CIRCLE
UNADJUSTED AND SEASONALLY ADJUSTED SERIES
CENSUS METHOD II. X-11 - X
CENSUS METHOD II. X-9 - 0
FREQUENCY INDEX
MONTHS
COHERENCE
CENSUS METHOD II. X 11 - SOLID LINE
CENSUS METHOD II. X-9 - DASH LINE
FREQ. INDEX

FIG.-23

FIG.-24


ESTIMATED PHASE AND COHERENCE
SERIES 1571, U.S. UNEMPLOYMENT MALES 20+
PHASE ANGLE - FRACTION OF THE CIRCLE
IRREGULAR AND SEASONALLY ADJUSTED SERIES
CENSUS METHOD II. X-11
COHERENCE
CENSUS METHOD II. X 11
FREQUENCY INDEX
MONTHS
FIG.-25
SERIES 1511, U.S. UNEMPLOYMENT, MALES 14-19
PHASE ANGLE - FRACTION OF THE CIRCLE
IRREGULAR AND SEASONALLY ADJUSTED SERIES
CENSUS METHOD II. X-11
COHERENCE
CENSUS METHOD II. X 11
FREQUENCY INDEX
MONTHS


ESTIMATED CROSS-CORRELATION FUNCTIONS
SERIES 1571, U.S. UNEMPLOYMENT, MALES 20+
CENSUS METHOD X-11
SERIES X: SEASONALLY ADJUSTED
SERIES Y: SEASONAL
FIG.-27
LAGS (MONTHS)
R(YX)
R(XY)
0.1
0.0
-0.1
36
30
24
18
12
6
0
6
12
18
24
30
36


SERIES 1511, U.S. UNEMPLOYMENT, MALES 14-19
CENSUS METHOD X-11
SERIES X: SEASONALLY ADJUSTED
SERIES Y: SEASONAL
FIG.-28
LAGS (MONTHS)
R(YX)
R(XY)
0.1
0.0
-0.1
36
30
24
18
12
6
0
6
12
18
24
30
36

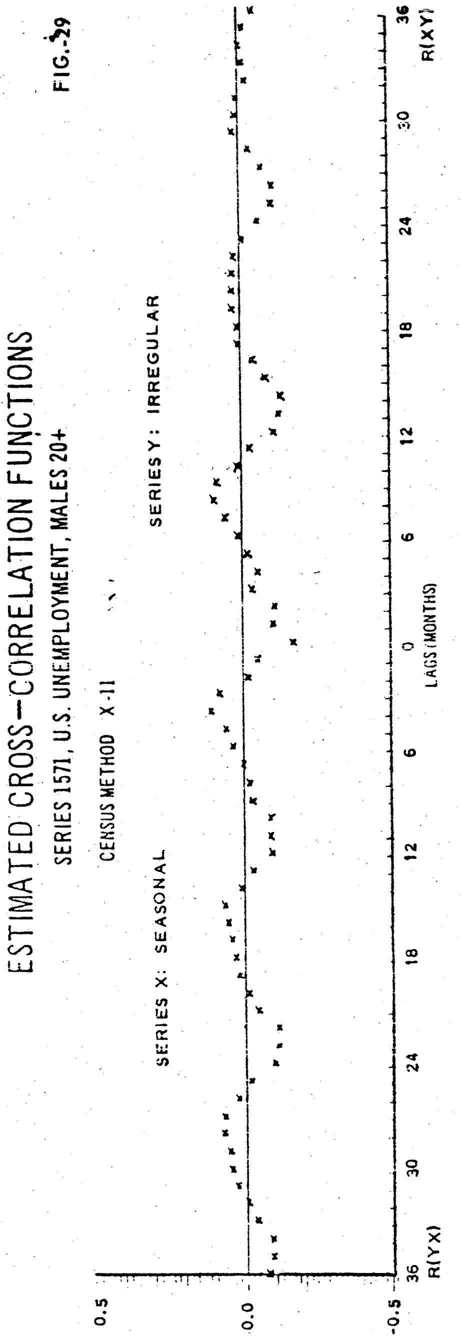


FIG. 29

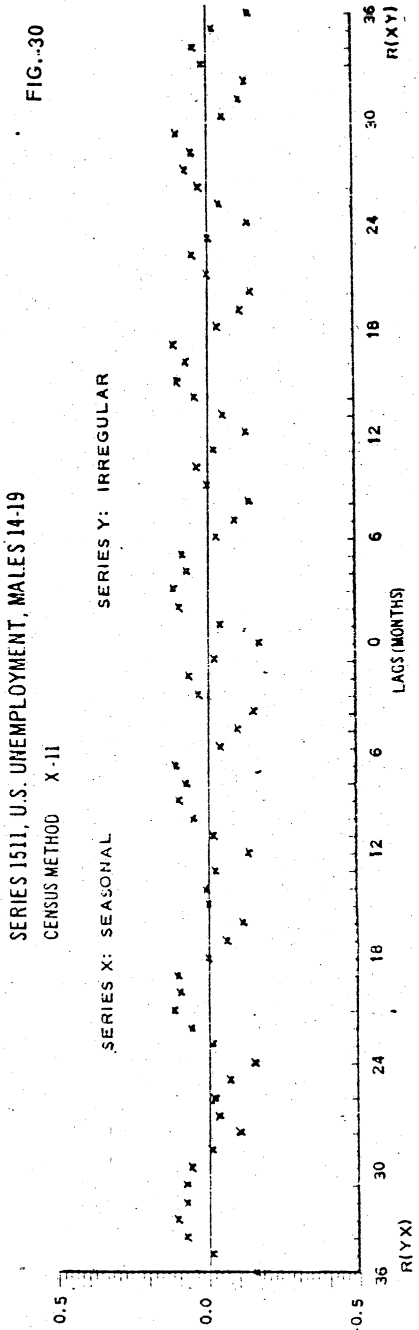


FIG. 30

ESTIMATED CROSS—CORRELATION FUNCTIONS

FIG. 31

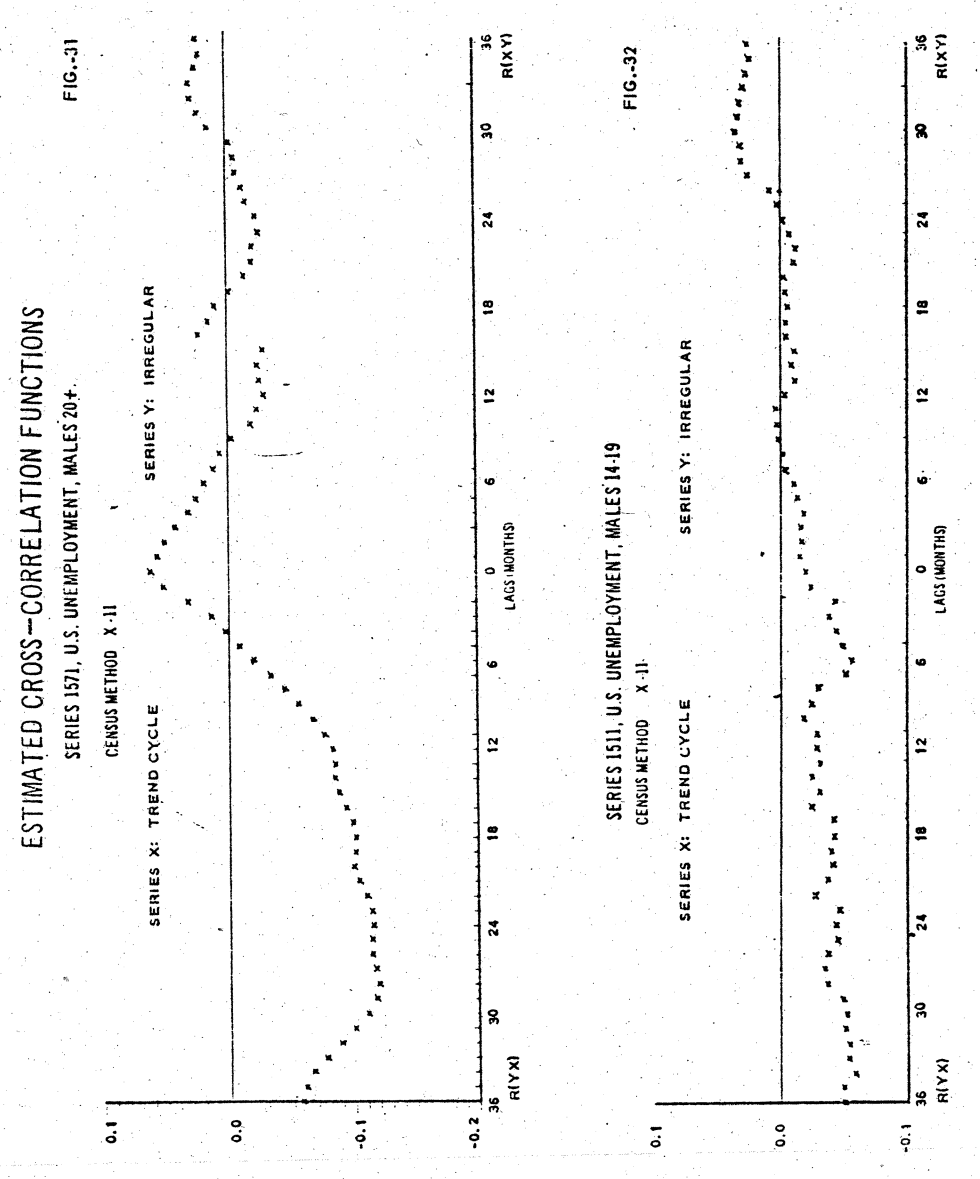

ESTIMATED CROSS—CORRELATION FUNCTIONS
SERIES 1571, U.S. UNEMPLOYMENT, MALES 20+
CENSUS METHOD X-11
SERIES X: TREND CYCLE
SERIES Y: IRREGULAR
FIG.-31
0.1
0.0
-0.1
-0.2
36
30
24
18
12
6
0
6
12
18
24
30
36
R(YX)
R(XY)
LAGS (MONTHS)
SERIES 1511, U.S. UNEMPLOYMENT, MALES 14-19
CENSUS METHOD X-11
SERIES X: TREND CYCLE
SERIES Y: IRREGULAR
FIG.-32
0.1
0.0
-0.1
R(YX)
R(XY)
LAGS (MONTHS)

TABLE 4. DATA FOR COMPARING CENSUS METHOD II *X*-9 AND *X*-11 SEASONAL ADJUSTMENTS OF U. S. UNEMPLOYMENT SERIES

Data Identification	Males 20+	Females 20+	Males 14–19	Females 14–19
	Absolute Average* Month to Month Percent Change			
Seasonally Adjusted Series, *CI*				
X-9	5.41	5.55	8.11	10.66
X-11	5.52	5.63	8.26	10.79
Seasonal Component, *S*				
X-9	9.43	5.82	19.43	21.90
X-11	9.40	6.28	19.77	21.47
Trend-Cycle Component, *C*				
X-9	3.09	2.45	2.26	2.69
X-11	3.17	2.33	2.29	2.65
Irregular Component, *I*				
X-9	4.07	4.91	7.65	10.14
X-11	3.85	5.03	7.68	10.21
MCD Curve				
X-9	4.27	3.07	2.92	3.61
X-11	4.26	3.03	2.99	3.65
	Other Data			
MCD: Span in Months	2	2	4	4
X-9 and *X*-11				
Seasonally Adjusted Series, *CI*				
X-11: Percent Change MCD Span				
Absolute Average	8.53	9.36	12.29	15.13
Arithmetic Mean	.74	1.89	2.64	3.68
Standard Deviation	11.10	13.72	16.02	20.18
X-9 to *X*-11: Percent Difference				
Absolute Average	.75	.76	1.40	1.41
Arithmetic Mean	−.04	−.02	−.18	.14
Standard Deviation	.97	1.06	2.20	1.79

* Average without regard to sign.

of economic importance in the *X*-11 adjusted series, namely changes of MCD span. It is seen from Table 4 that these differences are also relatively small. One may conclude that where the use to which the time series components are put is reflected by these measures, the differences pointed out by the spectra are not serious. In particular, for these applications, the greater deviations from spectral criteria of the *X*-9 variant, namely the "dips" at seasonal and trend-cycle frequencies, cannot be important. Although the spectral data are more sensitive and provide information about differences not revealed by these measures, the spectral data alone are not sufficient to deduce the importance of the differences in the practical use of the estimates.

Nevertheless, spectral analysis is an additional source of information for the economist which has considerable potential. Although much can now be learned from an examination of the spectrum itself, e.g., in identifying a seasonal component, in judging the relative adequacy of an adjustment, in comparing adjustments, in studying the properties of changes made in an adjustment procedure, and in studying lead-lag relationships, its value would be increased if an adequate calibration was available between the spectral properties of an economic time series and its meaning for a specific application of the time series data. No panacea should be expected, but hopefully, progress can be made with an assist from the economist.

7. ACKNOWLEDGMENTS

Acknowledgment is made to A. Rothman, formerly of the Bureau of Labor Statistics and to Julius Shiskin of the Census Bureau for providing the data from their seasonal adjustments and for many valuable discussions. Thanks are also due to Morris Hansen, William N. Hurwitz, and Marc Nerlove for reading an advanced draft of the paper and for contributing to its improvement. The computer programming for spectral analysis is due to Edward L. Melnick who also provided many helpful comments. The preparation of graphs is due to Harry T. Sturgis. Lorraine B. Hughes assisted in data preparation for the computer, and Beatrice L. Warren typed the final manuscript. I am deeply grateful for this assistance.

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A COMPARISON OF A MODIFIED "HANNAN" AND THE BLS SEASONAL ADJUSTMENT FILTERS*

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In a previous paper [12], an attempt was made to show how spectral techniques could be used to compare the effects of two seasonal adjustment procedures on the series to which they were applied. The two procedures compared were: (a) the technique currently used by the Bureau of Labor Statistics for seasonally adjusting employment, unemployment, and labor force monthly statistics, and (b) the so-called "residual" method, proposed by Brittain [2], Samuelson [16], and others. Spectra of the original and the seasonally adjusted series and the cross spectrum of the two were used to aid in the assessment of whether either procedure removed more than could be considered seasonal, introduced spurious regularities, and/or distorted temporal relationships. It was concluded that both techniques removed more than seasonal effects from, and produced some temporal distortion in, the series to which they were applied. Neither method appeared to be superior to the other.

It is the purpose of this paper to carry the previous analysis one step further and to compare the BLS procedure with a modified version of the regression method of seasonal adjustment suggested by Cowden [3] and Mendershausen [11], and recently revived by Hannan [9, 10] in an exceptionally sophisticated form. In addition to Hannan's work along the lines suggested, Nettheim [13] and Rosenblatt [15] have made studies. Rosenblatt [15] has carried out analyses similar to those reported here.

1. INTRODUCTION

SEASONALLY adjusted figures are usually published in two separate forms: (1) Seasonal factors of some sort are computed on the basis of a stretch of past data, e.g., 1947–63, and these factors are then used to adjust data for a year *not* included in the period used for the computation of the seasonal factors, e.g., 1964. This is the sort of information one sees published in the newspaper and in current reports of economic statistics. For this reason, I call the method producing them the *current form* of the seasonal adjustment procedure. (2) Seasonal factors may be computed on the basis of a stretch of past data and used to adjust that same data. This is the form one generally finds published in statistical compendia. The method producing such seasonally adjusted series may be termed the *historical form*.

In their historical form, I believe that most currently used methods of seasonal adjustment which operate only upon a single series at a time produce less useful and important results than they do operating in their current form. In

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historical form, such methods take little account of possible interrelations among seasonal patterns in several time series or of most underlying causes of seasonal variation and changes in seasonal patterns. (Trading day adjustments and the like, however, are important steps in accounting for basic causes of variations.) On the other hand, these methods in their current form are designed to present the information contained in an observation on an important economic variable in a useful way for multiple purposes of policy formulation and forecasting, and such techniques appear useful even if based solely upon the analysis of past observations on a single series.[1] The present paper is designed to compare a current form of the BLS procedure[2] with a suitably modified form of the Hannan procedure. The modification introduced is to derive seasonal factors for adjusting the data in a given year from a Hannan-type regression for the preceding five years. In other words, the procedure compared with a current form of the BLS procedure is not the two-sided, mostly asymmetrical regression filter originally suggested by Hannan, but a one-sided moving regression filter that could be employed for the same purpose as the current form of the BLS filter. It is clear that such modification of Hannan's method is necessary since, as is well-known, one-sided filters always cause some form of temporal distortion, whereas it is possible to design a two-sided filter free of this defect. In any case, a two-sided filter could certainly produce no worse effects than a one-sided one since there is always the option of discarding one of the two sides.

In Section 2 of this paper, the main spectral analytic results employed in the analysis are briefly summarized. Section 3 contains descriptions of the two seasonal adjustment procedures. In Section 4, these two procedures are compared according to the same criteria used in reference [12]. The Appendix contains derivations of certain of the more mathematical results used in the computations.

2. SUMMARY OF THE MAIN SPECTRAL ANALYTIC RESULTS USED IN THIS PAPER[3]

The fundamental result underlying spectral techniques is that any covariance stationary process (i.e., one whose covariance function depends only on the lag) may be written in the form

$$x(t) = \int_0^\infty \cos \lambda t dU(\lambda) + \int_0^\infty \sin \lambda t dV(\lambda), \qquad (2.1)$$

if it is supposed that the series is continuous in time. If $x(t)$ is, *in principle*, measured only at discrete points in time, the upper limit should be π. In more convenient complex form, (2.1) is

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[1] From an information theoretic point of view, one might argue that the only *new* information present in an observation on an economic time series is that part which could not have been predicted on the basis of knowledge of the past values of the series in question, and related series. Since unemployment, for example, "usually" rises in January and February, the fact that it does is not news. Seasonal adjustment in a current context can thus be viewed as the construction of a kind of prediction which, when netted out of the series, removes a certain type of "non-information." Naturally, the question of seasonal adjustment in this context is relative to the class of techniques which operate only on past values of the series in question.

[2] Data supplied by the staff of the President's Committee to Appraise Employment and Unemployment Statistics in early 1962.

[3] More detailed treatment may be found in Granger and Hatanaka [7], Hannan [8], and Nerlove [12].

Consider two stationary time series, $x(t)$ and $y(t)$; according to (2.2), they may be written

$$\left.\begin{aligned} x(t) &= \int_{-\infty}^{\infty} e^{i\lambda t} dZ_x(\lambda) \\ y(t) &= \int_{-\infty}^{\infty} e^{i\lambda t} dZ_y(\lambda) \end{aligned}\right\}, \tag{2.2'}$$

where $dZ_x(\lambda)$ and $dZ_y(\lambda)$ individually have the property (2.3); let their cumulative spectra be $F_{xx}(\lambda)$ and $F_{yy}(\lambda)$. But what of the relationship *between* the two random increments $dZ_x(\lambda)$ and $dZ_y(\lambda)$ occurring in the spectral representations of the two series? There is in general no simple relationship unless the two series, in addition to each being covariance stationary, are *jointly* covariance stationary; that is, the covariances between the two series, e.g., $Ex(t)y(t+\tau)$, depend only on the lag. In this case, and only in this case,

$$EdZ_y(\lambda)\overline{dZ_x(\lambda')} = \overline{E\overline{dZ_y(\lambda)}dZ_x(\lambda')} = \begin{cases} 0, & \lambda \neq \lambda' \\ dF_{xy}(\lambda)/2, & \lambda = \lambda', \end{cases} \tag{2.6}$$

where $F_{xy}(\lambda)$, a generally complex function of frequency, is the cumulative cross-spectrum. As before, we neglect all but the absolutely continuous and differentiable part of the function, and write

$$dF_{xy}(\lambda) = dC_{xy}(\lambda) - idQ_{xy}(\lambda) = [c_{xy}(\lambda) - iq_{xy}(\lambda)]d\lambda. \tag{2.7}$$

The function $c_{xy}(\lambda)$ is called the *co-spectrum* of x and y, and $q_{xy}(\lambda)$ the *quadrature spectrum*.[6] Again making use of the properties (2.6) and the spectral representations of the two jointly stationary time series, we find, after some manipulation, that the cross lag-covariance of the two series may be represented as

$$\begin{aligned} \gamma_{xy}(\tau) &= Ey(t)x(t-\tau) \\ &= \int_{-\infty}^{\infty} e^{i\lambda\tau} \frac{dF_{xy}(\lambda)}{2} \\ &= \frac{1}{2}\int_{-\infty}^{\infty} e^{i\lambda\tau} f_{xy}(\lambda)d\lambda. \end{aligned} \tag{2.8}$$

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[6] The reason for this terminology can be seen most clearly by expressing the expected representations of $x(t)$ and $y(t)$ in real form:

(i)
$$\begin{aligned} x(t) &= \int_0^{\infty} \cos \lambda t dU_x(\lambda) + \int_0^{\infty} \sin \lambda t dV_x(\lambda), \\ y(t) &= \int_0^{\infty} \cos \lambda t dU_y(\lambda) + \int_0^{\infty} \sin \lambda t dV_y(\lambda). \end{aligned}$$

Making use of the fact that, in both cases,

(ii) $$dZ(\lambda) = \tfrac{1}{2}[dU(\lambda) - idV(\lambda)],$$

we find

(iii) $$2EdZ_y(\lambda)\overline{dZ_x(\lambda)} = \tfrac{1}{2}[EdU_x(\lambda)dU_y(\lambda) + EdV_x(\lambda)dV_y(\lambda)] - i\tfrac{1}{2}[EdU_x(\lambda)dV_y(\lambda) - EdU_y(\lambda)dV_x(\lambda)].$$

Thus, the real part of $dF_{xy}(\lambda)$ is essentially the covariance of the in-phase components of the two series as represented in the frequency domain, and the complex part is essentially the covariance of the components 90° out of phase or in *quadrature*.

Setting $\tau=0$ shows that $f_{xy}(\lambda)$ is the frequency decomposition of the contemporaneous covariance between $x(t)$ and $y(t)$.

The interpretation of the cross-spectrum is simpler, however, in the special case considered in this paper. In this case we may divide by the spectrum of one of the two series, and regard the result essentially as a complex regression coefficient exhibiting a linear conditional expectation, or approximation thereto, in the frequency domain. To examine this notion in more detail, it is useful to introduce the concept of a time-invariant, linear filter with kernel $K(\tau)$. If we regard $x(t)$ as the input of such a filter, and $y(t)$ as the output, then we may write

$$y(t) = Lx(t) = \int_{-\infty}^{\infty} K(\tau)x(t-\tau)d\tau. \tag{2.9}$$

Provided $x(t)$ is covariance stationary, $y(t)$ will be too. Making use of the spectral representations of both series, we find:

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} e^{i\lambda t}dZ_y(\lambda) \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K(\tau)[e^{i\lambda(t-\tau)}dZ_x(\lambda)]d\tau \\ &= \int_{-\infty}^{\infty} e^{i\lambda t}l(\lambda)dZ_x(\lambda), \end{aligned} \tag{2.10}$$

where

$$l(\lambda) = \int_{-\infty}^{\infty} K(\tau)e^{-i\lambda\tau}d\tau, \tag{2.11}$$

the complex Fourier transform of the filter kernel, is called the *frequency response function* of the filter. Thus, equating coefficients of common values of $e^{i\lambda t}$, the relation

$$dZ_y(\lambda) = l(\lambda)dZ_x(\lambda) \tag{2.12}$$

is obtained. Equation (2.12) shows, incidentally, that $x(t)$ and $y(t)$ are such that (2.6) holds for $l(\lambda)\neq 0$, for, multiplying both sides by $\overline{dZ_x(\lambda')}$, we find

$$EdZ_y(\lambda)\overline{dZ_x(\lambda')} = l(\lambda)EdZ_x(\lambda)\overline{dZ_x(\lambda')}\begin{cases}\neq 0, & \lambda=\lambda' \\ =0, & \lambda\neq\lambda'\end{cases} \tag{2.13}$$

by (2.3). The interpretation of $l(\lambda)$ as a regression coefficient follows from (2.12) and (2.13). According to (2.12), the frequency response function of a filter is the *slope*, at each frequency, of the linear function connecting the complex stochastic increments of the spectral representation of the input with that of the output at the corresponding frequency. For a common frequency $\lambda=\lambda'$, equation (2.13) shows that

$$x(t) = \int_{-\infty}^{\infty} e^{i\lambda t} dZ(\lambda), \tag{2.2}$$

where the complex stochastic increment $dZ(\lambda)$ possesses the following properties:

$$EdZ(\lambda)\overline{dZ(\lambda')} = \begin{cases} 0, & \lambda \neq \lambda' \\ \dfrac{dF(\lambda)}{2}, & \lambda = \lambda'. \end{cases} \tag{2.3}$$

The notation "$\lambda \neq \lambda'$" should be read: "λ and λ'" lie in non-overlapping intervals." The importance of the representation (2.2) is that it reduces the problem of studying a time series, which typically exhibits a complex pattern of dependence in the *time domain*, to the study of the stochastic increments $dZ(\lambda)$, which are orthogonal in the frequency domain.[4]

Twice the variance of $dZ(\lambda)$ is the increment $dF(\lambda)$. The function $F(\lambda)$ is the cumulative power spectrum. It is a monotonically increasing function which can be written in the form

$$F(\lambda) = F_1(\lambda) + F_2(\lambda) + F_3(\lambda), \tag{2.4}$$

where F_1, F_2, and F_3 are non-decreasing functions of frequency. $F_1(\lambda)$ is absolutely continuous with a derivative $f(\lambda)$; $F_2(\lambda)$ is a step function, and $F_3(\lambda)$ is the so-called singular component, a function which is constant except on a set of measure zero. $F_3(\lambda)$ does not appear to have any economic significance and I shall henceforth disregard it. $F_2(\lambda)$, the step function, would occur if $x(t)$ contained components such as

$$\alpha \cos \omega t,$$

where α was a random variable with variance σ^2. $F(\lambda)$ would then have a jump of σ^2 at $\lambda = \omega$. In economic time series, it might be thought that the commonest cause of $F_2(\lambda)$, *not* identically zero, would be the phenomenon of seasonality. However, I have argued elsewhere [12] that it is highly unlikely that seasonality will produce jumps in the cumulative spectrum, but rather that it will produce peaks of greater or lesser breadth in the power spectrum $f(\lambda)$. For this reason I shall also disregard the component $F_2(\lambda)$ and write $dF(\lambda) = f(\lambda)d\lambda$.[5]

Under the above assumptions, manipulation of (2.2), using the properties (2.3), yields the autocovariance of the series

$$\gamma(\tau) = Ex(t)x(t + \tau) = \int_{-\infty}^{\infty} e^{i\lambda\tau} \frac{f(\lambda)}{2} d\lambda. \tag{2.5}$$

Setting $\tau = 0$, we see at once that the spectrum is nothing more than a decomposition of the total variance of the series into components attributable to different frequencies.

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[4] Hannan [8, p. 19], Gnedenko [6, p. 377], and other standard works.

[5] Spectra for which neither $F_1(\lambda)$ nor $F_2(\lambda)$ are identically zero are called mixed spectra, and present difficult problems in estimation. Mr. George Hext of Stanford University is currently developing appropriate techniques in his doctoral dissertation. While I do not believe mixed spectra are actually of much relevance in economics, Hext's work should shed much light on the estimation of continuous spectra containing relatively narrow peaks.

$$l(\lambda) = \frac{EdZ_y(\lambda)\overline{dZ_x(\lambda)}}{EdZ_x(\lambda)\overline{dZ_x(\lambda)}} = \frac{dF_{xy}(\lambda)}{dF_{xx}(\lambda)} = \frac{c_{xy}(\lambda)}{f_{xx}(\lambda)} - i\frac{q_{xy}(\lambda)}{f_{xx}(\lambda)} \tag{2.14}$$

where $dF_{xx}(\lambda)/d\lambda = f_{xx}(\lambda)$ is the spectrum of the input series. Hence, estimates of the cross-spectra of the output and the input of a linear time-invariant filter and the spectrum of the latter permit estimation of the frequency response function of the filter.

It is worth noting at this point that, just as the cross-spectrum between two series can be interpreted as the numerators of regression coefficients in linear relations between the stochastic elements in the spectral representations of two jointly covariance stationary series, a concept precisely analogous to the correlation between two random variables also exists. Making appropriate substitutions in the usual formula for the squared correlation between two variables with zero means, and always using products of factors with complex conjugates rather than ordinary squares or cross products, we obtain the spectral analogue of the correlation coefficient:

$$\rho^2(\lambda) = \frac{[EdZ_y(\lambda)\overline{dZ_x(\lambda)}]\overline{[EdZ_y(\lambda)\overline{dZ_x(\lambda)}]}}{EdZ_x(\lambda)\overline{dZ_x(\lambda)}][EdZ_y(\lambda)dZ_y(\lambda)]} = \frac{c^2_{xy}(\lambda) + q^2_{xy}(\lambda)}{f_{xx}(\lambda)\cdot f_{yy}(\lambda)}, \tag{2.15}$$

or the *coherence*, as it is known in spectral circles.

Why one should wish to estimate a frequency response function for a filter such as the BLS seasonal adjustment filter, which is not time-invariant and is non-linear, may be better understood in the context of the following brief remarks concerning this function. From its very definition in equation (2.11), it may be seen that the real part of $l(\lambda)$ describes the output of the filter when a cosine wave of unit amplitude, zero phase, and frequency λ is used as the input to the filter. The complex part describes the output when a similar sine wave is the input. Since, by the spectral representation theorem, any covariance stationary time series can be regarded as a superposition of sine and cosine waves of random amplitude and zero phase angle, the frequency response function completely describes the effects of time-invariant linear filters. If $l(\lambda)$ is written in polar form, we have

$$\begin{aligned} l(\lambda) &= u(\lambda) + iv(\lambda) \\ &= G(\lambda)[\cos\phi(\lambda) + i\sin\phi(\lambda)] \\ &= G(\lambda)e^{i\phi(\lambda)} \end{aligned} \tag{2.16}$$

where

$$G(\lambda) = \sqrt{u^2(\lambda) + v^2(\lambda)}$$

$$\phi(\lambda) = \text{arc tan} \frac{v(\lambda)}{u(\lambda)} \cdot$$

$G(\lambda)$ is called the *gain* of the filter and is the amplitude of the output for a sinusoidal input of unit amplitude. $\phi(\lambda)$ is called the *phase angle* of the filter and shows the shift in phase in the output as compared with the input. The gain and phase angle of a filter *measure* its effects at different frequencies. If a filter is not linear, we may still wish to form the estimate on the right-hand side of (2.14), i.e., the ratio of the estimated cross-spectrum of two series to the estimated spectrum of the one regarded as the input of the filter. Appropriate functions of these estimates measure the gain and phase angle, at each frequency, of a filter which is in some sense a linear, time-invariant approximation to the actual filter as it operated on the body of data used to generate the estimates.[7]

Coherence is a measure which is free, to some extent, of the restrictions of this interpretation. It may be viewed simply as a measure of the degree of association at each frequency between the stochastic increments in the spectral representations of the two time series.

Inasmuch as the practical problems of spectral estimation have been widely discussed elsewhere,[8] such material is not included in this paper. However, certain technicalities associated with the estimates presented in Part 4 of this paper are treated in the Appendix.

3. DESCRIPTIONS OF THE SEASONAL ADJUSTMENT PROCEDURES COMPARED

The technique used by the Bureau of Labor Statistics to adjust the monthly series of employment, unemployment, and labor force issued by that agency has been described in detail elsewhere; e.g., [14, Appendix G]. Consequently, only a bare and incomplete outline is given here. The model underlying this method is the familiar one: Each value of the series is the product (or sum) of three factors: (1) trend-cycle, (2) seasonal, and (3) irregular. The problem is to disentangle these three components, then eliminate the seasonal by division (or subtraction). In spectral terminology, and as argued above and in [12], trend-cycle may be regarded as consisting largely (but not solely) of low frequency components of the time series, seasonal of those components which produce peaks at the six seasonal frequencies, and irregular of all the rest.[9]

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[7] This remark shows incidentally why it may be a bit dangerous to draw firm conclusions from results based on artificially generated series. The effects of non-linear and/or non-time-invariant filters depend crucially on the frequency characteristics of the input series. Unless these are known with some accuracy, erroneous conclusions may be drawn about the operation of the filter by studying its effects on an artificially generated series with markedly different characteristics. One way to obtain a series with the same characteristics as the one to which the filter is commonly applied is to use past values of the very same series. This is precisely the approach of [12] and the present paper. An alternative approach is to approximate the characteristics of a non-linear filter in the frequency domain by a linear filter. This approach has been taken in G. Hext, "Transfer Functions for Two Seasonal Adjustment Filters," Technical Report No. 3 (NSF GS-142), Institute for Mathematical Studies in the Social Sciences, Serra House, Stanford University (September 5, 1964).

[8] For example, Blackman and Tukey [1] and Nerlove [12] discuss certain problems of estimation of special relevance in the study of economic time series.

[9] This interpretation is closely connected with the question of non-stationarity. The type of non-stationarity which is likely to be prevalent in economic time series cannot be distinguished from the presence of high power at low frequencies in the spectrum of a stationary series, and may be treated *as if* it were simply that.

The seasonal frequencies referred to are the frequencies $\pi/12$, $2\pi/12$, $3\pi/12$, $4\pi/12$, $5\pi/12$, and $6\pi/12$, when

The first step in the procedure is an attempt to isolate the components at seasonal frequencies. This is done by computing a 12-month moving average of the series and finding the ratio of the series for each month to the average centered on the month. The forming of ratios to moving average is a higher non-linear filter designed to remove the low frequency components, leaving only the higher frequency seasonal and irregular components.

The next step in the procedure attempts to separate seasonal components from the other high frequency components with which they are mixed. This is done in the BLS method by computing a five-term moving average of the ratios to moving average for each month. The reason a five-term moving average is used rather than a longer one (in the extreme all the ratios available) is to allow for the possibility of slowly varying seasonals.[10] The length of the average reflects a judgment on the speed of this variation: the shorter the average, the faster the seasonal factor may vary and the broader the band of frequencies about each seasonal frequency is supposed to be.

At this point in the procedure there are available a series of estimates of the seasonal component of the time series. If these are now divided into the original series, the result will be an estimate of the trend-cycle and irregular components, i.e., a preliminary seasonally adjusted series. However, the procedure does not stop with this preliminary adjustment because it is felt that the original attempt to remove the low frequency components, without distorting the higher frequency ones by the formation of ratios to moving average, may not have been completely successful.[11] Consequently, the resulting estimate of the trend-cycle and irregular components is again averaged, this time with a weighted seven-term formula, in order to arrive at a new estimate of the trend-cycle component. If the ratios of the original series to this new estimate of the trend-cycle component are formed, the result will be a new estimate of the seasonal and irregular components. The second step in the procedure can then be repeated to arrive at a new estimate of the seasonal components, and so on indefinitely. The BLS procedure, however, is to repeat this operation only twice after the initial estimate of the trend-cycle components by means of a twelve-month moving average.

The most recent complete set of seasonal factors obtained by the method is used to filter the series for the current year in the current form of the procedure.

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monthly data are considered. (Different frequencies are seasonal if a different unit of observation is considered.) No particular importance should be attached to these frequencies individually; they are all necessary to make up an arbitrary 12-month pattern.

One could argue that the three components should be orthogonal to one another, although this is perhaps an empirical question. In any case, the irregular component, practically by definition, should, it seems to me, be a sequence of purely random numbers, i.e., independent at different points in time, or what in spectral terminology is called *white noise*. (Although, as H. Rosenblatt has pointed out, the problem of how to take account of catastrophic events such as strikes creates certain difficulties with this position.) White noise has a perfectly flat spectrum. Unfortunately, the BLS procedure does not always produce an irregular component with this spectrum; often a spectrum with markedly low power near the origin is obtained, thus suggesting the presence of negative serial correlation. This sort of thing would occur if too much of the power at low frequencies were attributed to trend-cycle.

[10] It is this slow variation which causes seasonality to show up as peaks in the spectrum, rather than as lines (jumps in the cumulative spectrum). See [12, Section 2.1].

[11] Indeed, as Hannan [10, p. 35] shows, there will generally be a bias in the estimates of the seasonal factors. This bias can be reduced by iteration as the BLS attempts to do, but it is more efficient to design a more appropriate filter for trend-cycle to begin with.

The above description does not attempt to do justice to a number of details of the procedure, which often involve considerable sophistication in the treatment of outliers, end-point corrections, and the like. One further point, however, deserves special mention. The employment, unemployment, and labor force series, as estimated from a sample survey, actually consist of a large number of components. Distinctions are made on the basis of age and sex, major industry attachment (agricultural versus non-agricultural), and, in recent years, color. In the unemployment statistics there is also a classification on the basis of the length of unemployment. Thus, in seasonally adjusting these series, it is possible either to adjust them all individually, including such major aggregates as total unemployment, employment, and labor force, or to adjust certain components, deriving seasonally adjusted values for the aggregates by implication. In the published series for the U. S., all aggregates and components are adjusted separately except total unemployment. This series is seasonally adjusted by adjusting four major age-sex categories of unemployed (male and female, ages 14–19 and 20 and over) and summing the resulting four seasonally adjusted components to arrive at the adjusted total.[12]

In broad outline the procedure suggested by Hannan ([9] and [10]) is not basically different from the BLS procedure, although there is considerable difference in detail, and superficially the two procedures hardly resemble one another at all. Hannan begins with the following model for the time series $x(t)$:

$$x(t) = p(t) + s(t) + u(t). \tag{3.1}$$

$p(t)$ represents the trend-cycle, low frequency components, or even a deterministic function of time; $s(t)$ is a periodic function of time, which may or may not be slowly evolving, and which we seek to estimate and remove; $u(t)$ is assumed to be a covariance stationary time series, the so-called irregular component. If one takes the position that $p(t)$ is stochastic, not deterministic, then one can argue that the irregular component should be *white noise*, i.e., a time series whose values are serially independent. But this is not essential to Hannan's development, and he does not make the assumption. If $x(t)$ represents the logarithm of an actual series, (3.1) clearly represents the same sort of multiplicative model which underlies the BLS procedures.

The differences between Hannan's suggested procedure and the BLS procedure lie, first, in the details of the treatment of the problem of removing $p(t)$ before estimating $s(t)$; and second, in the *form* in which $s(t)$ is expressed. This last leads to a parametric treatment of slowly changing seasonals in an historical context, but this treatment is not especially useful in a current context.[13]

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[12] Examination of the unemployment series themselves and the power spectra of the components reveals striking differences in both the pattern and the relative importance of the seasonal components. The contribution of the four categories to the total volume of unemployment has been changing over time. Separate adjustment is a way of narrowing the frequency bands within which seasonal adjustment is supposed to reduce power, and so increasing the hoped-for accuracy of the adjustment.

[13] This special treatment for time-trending seasonals is suggested in Hannan's first paper on the subject [9]; in his second paper, however [10, pp. 39–40], he rejects his earlier suggestion on the grounds that "These procedures are computationally arduous and their usefulness also seems dubious to the writer because they are founded upon a very special model." Instead he proposes a procedure very similar to that used by Nettheim [13, pp. 77–104]. This last, however, has considerable disadvantages in terms of testing the significance of the supposed evolution of the seasonal, and in an historical context it may be worth paying the price of a less flexible model (although it is not all that inflexible) for the possibility of restraining the imagination by a few formal statistical tests. In a current con-

Let us take up these differences in reverse order, assuming initially that the seasonal pattern is stable. We also assume monthly data throughout.

In this case, $s(t)$ may be represented as a series of twelve seasonal factors which repeat themselves each year. If deviations from the mean are taken, and if the mean itself is absorbed in $p(t)$, the trend-cycle component, the seasonal component may be taken as a series of twelve constants, one for each month, which sum to zero. Let these constants be a_j; then we have

$$s(t) = \begin{cases} a_j, & \text{for } t = j \text{ or } t - j \text{ divisible by 12} \\ 0, & \text{otherwise} \end{cases}$$
$$\sum_1^{12} a_j = 0. \tag{3.2}$$

The key to Hannan's approach is to recognize that $s(t)$, as defined in (3.2), can be written in apparently different, but precisely equivalent, form:

$$s(t) = \sum_{k=1}^{6} [\alpha_k \cos \lambda_k t + \beta_k \sin \lambda_k t], \tag{3.3}$$

where

$$\lambda_k = \frac{2\pi k}{12}$$
$$\beta_6 \equiv 0.$$

The coefficients $\alpha_1, \cdots, \alpha_6$ and $\beta_1, \cdots, \beta_5$ uniquely determine, and are uniquely determined by, the 12 seasonal shift factors $a_1, \cdots, a_{12}$, which are constrained to sum to zero.[14]

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text, however, such refinements are likely to be unnecessary, and I would argue that most of the evolution of seasonals can be adequately handled by the moving regression modification to the Hannan procedure proposed below.

[14] To see this, let $t = j = 1, \cdots, 12$, and write down (3.3) in extensive form:

$$s(1) = a_1 = \sum_{k=1}^{6} \alpha_k \cos \lambda_k \cdot 1 + \sum_{k=1}^{5} \beta_k \sin \lambda_k \cdot 1$$
$$s(2) = a_2 = \sum_{k=1}^{6} \alpha_k \cos \lambda_k \cdot 2 + \sum_{k=1}^{5} \beta_k \sin \lambda_k \cdot 2$$
$$\vdots$$
$$s(12) = a_{12} = \sum_{k=1}^{6} \alpha_k \cos \lambda_k \cdot 12 + \sum_{k=1}^{5} \beta_k \sin \lambda_k \cdot 12.$$

We have thus 12 equations in the 11 unknowns $\alpha_1, \cdots, \alpha_6$, and $\beta_1, \cdots, \beta_5$, but the matrix of coefficients,

$$\begin{bmatrix} \cos\left(\frac{2\pi}{12}\cdot 1\cdot 1\right) & \cdots & \sin\left(\frac{2\pi}{12}\cdot 5\cdot 1\right) \\ \vdots & & \\ \cos\left(\frac{2\pi}{12}\cdot 1\cdot 12\right) & \cdots & \sin\left(\frac{2\pi}{12}\cdot 5\cdot 12\right) \end{bmatrix}$$

has rank 11. If we eliminate the extra equation, using the restriction $\Sigma_1^{12} a_j = 0$, we obtain the 11 equations

Once the equivalence of (3.3) and (3.2) is observed, the whole of Hannan's procedure follows. First note that there is no reason to assume $\alpha_1, \cdots, \alpha_6$ and $\beta_1, \cdots, \beta_5$ to be constants; they can just as well be assumed to be known functions of time; then Hannan's first suggestion for handling evolving seasonals follows ([9, p 13–5]). Alternatively, one can assume that these coefficients change sufficiently slowly that values can be reasonably well estimated by a moving regression. This is the basis for the modification of the Hannan procedure adopted below for comparison with the BLS procedure.

Second, the representation of the seasonal factors in trigonometric form leads, via the result (2.12) of the previous section, to the conclusion that if trend is eliminated by any *linear* filter, the seasonal factors appropriate to the trend-contaminated series can nonetheless be recovered. This then forms the basis for Hannan's contention that iterations such as those employed in the BLS procedure are entirely unnecessary.

As we saw in the previous section, the frequency response function specifies the precise effects of a linear-time invariant filter upon sinusoidal fluctuations. Since (3.3) expresses any stable seasonal pattern in terms of sines and cosines, it follows that the precise effects on the seasonal component of applying any *linear* time-invariant filter to the series can be specified exactly.[15] Given this specification, it is possible to reverse the procedure; i.e., to recover seasonal factors appropriate to the trend-contaminated series from estimates of factors based on a series filtered so as to be relatively trend-free. Such reversal is not, in general, possible in the case of a non-linear filter of, for example, the type used by the BLS. If only for this reason, non-linear filters should never be used for trend removal if a linear alternative exists.

To see how this recovery process works in general, consider the typical term of the summation on the right-hand side of (3.3). Making use of a well-known trigonometrical identity, this term may be written

$$\begin{aligned} \alpha_k \cos \lambda_k t + \beta_k \sin \lambda_k t &= r_k[\cos \theta_k \cos \lambda_k t + \sin \theta_k \sin \lambda_k t] \\ &= r_k \cos [\lambda_k t - \theta_k], \end{aligned} \tag{3.4}$$

where

$$\lambda_k = \frac{2\pi k}{12}, \qquad k = 1, \cdots, 6,$$

$$s(1) - s(12) = a_1 - a_{12} = \sum_{k=1}^{6} \alpha_k(\cos[\lambda_k \cdot 1] - 1) + \sum_{k=1}^{5} \beta_k \sin[\lambda_k \cdot 1]$$

$$\vdots \qquad \vdots$$

$$s(11) - s(12) = a_{11} - a_{12} = \sum_{k=1}^{6} \alpha_k(\cos[\lambda_k \cdot 11] - 1) + \sum_{k=1}^{5} \beta_k \sin[\lambda_k \cdot 11],$$

since $\cos 2\pi \cdot k = 1$ and $\sin 2\pi k = 0$ for all integral k. The matrix of coefficients

$$\| [\cos \lambda_k \cdot j] - I : [\sin \lambda_k j] \|$$

has rank 11; its inverse is

$$\left\| \begin{matrix} [\tfrac{1}{6}(1 - \delta_{6,k} \cdot \tfrac{1}{2}) \cos \lambda_k \cdot j] + I \\ [\tfrac{1}{6} \sin \lambda_{k'} \cdot j] \end{matrix} \right\|$$

where $k = 1, \cdots, 6$, $k' = 1, \cdots, 5$, $j = 1, \cdots, 11$, and $\delta_{6,k}$ is zero for $k \neq 6$, and one for $k = 6$. Thus, the a_j's are uniquely determined by, and uniquely determine, $\alpha_1, \cdots, \alpha_6$ and $\beta_1, \cdots, \beta_5$.

[15] As shown in the Appendix, it is also possible to do this when the coefficients of the sinusoidal terms are trending linearly with time. However, it is not generally possible to specify exactly the effects of non-linear filters, or filters which are not time-invariant in so simple a fashion.

$$\beta_6 \equiv 0,$$

$$r_k = \sqrt{\alpha_k^2 + \beta_k^2},$$

$$\theta_k = \arctan \beta_k/\alpha_k.$$

Thus, the effect of the linear filter L on the typical seasonal term in the trigonometric representation (3.3) of $s(t)$ is its effect on a cosine wave of amplitude r_k and phase angle $-\theta_k$. As we know from the previous section, the effects of any linear, time-invariant filter are completely described by its frequency response function: in particular, the amplitude is multiplied at each frequency by the *gain* of the filter and the phase of any sinusoidal wave is shifted by the *phase angle* of the filter at each frequency. The gain and phase angle of the filter are, of course, just the modulus and argument (or phase) of the complex function $l(\lambda)$ written in polar form. It follows, then, that the effect on the typical term is

$$\begin{aligned} Lr_k \cos[\lambda_k t - \theta_k] &= r_k G(\lambda_k) \cos[\lambda_k t - \theta_k + \phi(\lambda_k)] \\ &= r_k G(\lambda_k) \cos[\theta_k - \phi(\lambda_k)] \cos \lambda_k t \\ &\quad + r_k G(\lambda_k) \sin[\theta_k - \phi(\lambda_k)] \sin \lambda_k t, \qquad k = 1, \cdots, 6. \end{aligned} \tag{3.5}$$

Let α_k' and β_k' be the coefficients of $\cos \lambda_k t$ and $\sin \lambda_k t$, respectively, in the seasonal component appropriate to the transformed and presumably trend-free series. It follows from (3.5) that

$$\left.\begin{aligned} \alpha_k' &= r_k G(\lambda_k) \cos[\theta_k - \phi(\lambda_k)] \\ &= [G(\lambda_k) \cos \phi(\lambda_k)]\alpha_k + [G(\lambda_k) \sin \phi(\lambda_k)]\beta_k \\ \beta_k' &= r_k G(\lambda_k) \sin[\theta_k - \phi(\lambda_k)] \\ &= -[G(\lambda_k) \sin \phi(\lambda_k)]\alpha_k + [G(\lambda_k) \cos \phi(\lambda_k)]\beta_k, \qquad k = 1, \cdots, 6 \end{aligned}\right\} \tag{3.6}$$

Solving (3.6) for α_k and β_k completes the proof that recovery of seasonal factors for the trend-contaminated series is always possible, provided trend is eliminated in the first instance by a linear, time-invariant filter:

$$\left.\begin{aligned} \alpha_k &= \frac{\alpha_k' \cos \phi(\lambda_k) - \beta_k' \sin \phi(\lambda_k)}{G(\lambda_k)} \\ \beta_k &= \frac{\alpha_k' \sin \phi(\lambda_k) + \beta_k' \cos \phi(\lambda_k)}{G(\lambda_k)}, \qquad k = 1, \cdots, 6.^{16} \end{aligned}\right\}. \tag{3.7}$$

More complicated relations are obtained when the coefficients α_k and β_k are assumed to trend with time. These are given for a special case in the Appendix to this paper. However, the moving regression technique adopted below as the

[16] In the special case of a symmetric linear filter, this result reduces to that given by Hannan [10, p. 36]. Durbin [5, p. 13, eq. (25)] gives the corresponding general result in terms not of the coefficients in a trigonometric representation of the seasonal but in terms of the shift-variable representation. The above discussion in terms of the frequency response function is thought to be both more understandable and more elegant.

appropriate modification of Hannan's procedure seems to remove the need for introducing trending coefficients into the analysis of seasonality.

The whole question of trend removal is the subject of an extensive literature. Most notable in the present context are the two recent papers of Durbin ([4] and [5]), which discuss the effects of moving-average and variate-difference filters on the estimates of periodogram ordinates and seasonal components from the filtered series. Hannan [10] and Nettheim [13] consider several types of centered and non-centered moving averages;[17] Durbin [4] gives several reasons for preferring variate-difference filters; and Parzen, in the slightly different context of spectral estimation, suggests fitting an autoregressive scheme and using the result to filter. Unfortunately, it has not yet been possible to explore more than a very few of the possible alternatives. Indeed, this particular question is the subject of much current investigation. Suffice it to say that, for the purpose of trend removal prior to estimation of seasonal components in the research presented below, the scheme of *iterated quasi-differences* proposed in [12] for another purpose has been employed. This is as follows: The γ quasi-difference of order p is defined as the quasi-difference

$$\Delta_\gamma x(t) = x(t) - \gamma x(t-1) \tag{3.8}$$

iterated p times, where γ is a constant which will usually be between zero and one.[18] Both p and γ are chosen in such a way that *the estimated spectrum of the series to be adjusted is relatively flat except for seasonal peaks*. Thus, the scheme provides a flexible method for trend-cycle removal which can be applied to a wide variety of different series. The point at which trend-cycle is in fact removed is judged empirically.[19]

The second modification of Hannan's procedure which has been made in the results reported below is designed to permit the procedure to be used as a one-sided filter. As argued in the previous section, such modification is necessary if the comparison is to be against the current form of the BLS procedure. Thus a regression is taken over a five-year period and the seasonal factors so obtained are used to adjust the year *following* the period; e.g., data for 1955–59 are used to adjust 1960, data for 1956–60 to adjust 1961, and so on.

The following summarizes the procedure which is compared with the BLS procedure: Let $x(t)$ be the original series to be adjusted:

(a) First, form the series $y(t)$ from $x(t)$ by the use of the iterated quasi-difference filter in such a way that the estimated spectrum of $y(t)$ is flat except for seasonal peaks.

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[17] Various orders of *differences* can all be represented as one-sided moving averages.

[18] In all of the analyses reported below, $\gamma = 0.75$.

[19] The method of iterated quasi-differences is very likely an approximation to the fitting autoregressive scheme suggested by Parzen for removal of high power, especially at low frequencies, preparatory to spectral estimation. In the present work, only iterated quasi-differences are used in the seasonal estimation, but the Parzen technique has been employed in the estimation of spectra and cross-spectra. See Panel A of the charts presented in the next section for the appearance of the prewhitened series using an autoregressive filter.

In an unpublished technical report ["A Note on Prewhitening and Recolouring," Technical Report No. 5 (NSF GS-142), Institute for Mathematical Studies in the Social Sciences, Serra House, Stanford University (September 24, 1964)], George Hext argues that the scheme of iterated quasi-differences is satisfactory for the purpose of spectral estimation only under rather special circumstances. Fortunately, these circumstances are approximately encountered in connection with the unemployment data.

(b) Next, for each five-year period (60 observations), compute estimates of $\alpha_1', \cdots, \alpha_6'$, and $\beta_1', \cdots, \beta_5'$ by means of the following formulae:

$$\left.\begin{aligned}
\hat{\alpha}_k' &= \frac{1}{6}\sum_{j=1}^{12}\cos\left[2\pi\left(\frac{k}{12}\right)j\right]\bar{y}(j), \qquad k = 1, \cdots, 5,\\
\hat{\alpha}_6' &= \frac{1}{12}\sum_{j=1}^{12}\cos\left[2\pi\left(\frac{6}{12}\right)j\right]\bar{y}(j),\\
\hat{\beta}_k' &= \frac{1}{6}\sum_{j=1}^{12}\sin\left[2\pi\left(\frac{k}{12}\right)j\right]\bar{y}(j), \qquad k = 1, \cdots, 5,\\
\hat{\beta}_6' &\equiv 0,
\end{aligned}\right\} \tag{3.9}$$

where $\bar{y}(j)$ is the mean of $y(t)$ for the jth month in the five-year period. We assume the overall mean for the period has been removed initially from the series $y(t)$.[20]

(c) Recover estimates of the coefficients $\alpha_1, \cdots, \alpha_6$, and $\beta_1, \cdots, \beta_5$ appropriate to the trend-contaminated series $x(t)$ from $\alpha_1', \cdots, \beta_5'$ by using the formulae corresponding to (3.7) in the special case of a pth order quasi-difference with parameter γ:

$$\left.\begin{aligned}
\hat{\alpha}_k &= \frac{\hat{\alpha}_k' A_k + \hat{\beta}_k' B_k}{\left[1 - 2\gamma\cos\frac{2\pi k}{12} + \gamma^2\right]^p}\\
\hat{\beta}_k &= \frac{-\hat{\alpha}_k' B_k + \hat{\beta}_k A_k}{\left[1 - 2\gamma\cos\frac{2\pi k}{12} + \gamma^2\right]^p}, \qquad k = 1, \cdots, 6,
\end{aligned}\right\} \tag{3.10}$$

where

$$\left.\begin{aligned}
A_k &= \sum_{r=0}^{p}\binom{p}{r}(-\gamma)^{p-r}\cos\left[\frac{2\pi k}{12}(p-r)\right]\\
B_k &= \sum_{r=0}^{p}\binom{p}{r}(-\gamma)^{p-r}\sin\left[\frac{2\pi k}{12}(p-r)\right], \qquad k = 1, \cdots, 6.
\end{aligned}\right\} \tag{3.11}$$

(This result is derived in the Appendix.)

(d) Compute seasonal factors

$$s(j) = \sum_{k=1}^{6}[\hat{\alpha}_k\cos\lambda_k j + \hat{\beta}_k\sin\lambda_k], \qquad j = 1, \cdots, 12, \tag{3.12}$$

and subtract these from the values for corresponding months in the year immediately following the period used for estimation.

[20] That the estimates given by (3.9) are actually the regression coefficients in

$$y(t) = \sum_{k=1}^{6}[\alpha_k'\cos\lambda_k t + \beta_k'\sin\lambda_k t] + v(t)$$

when the number of observations is an integral multiple of 12 (e.g., 60) can readily be demonstrated by a series of elementary trigonometric manipulations. The proof is, however, too standard to be repeated here.

If logarithms were initially taken to justify the additive model, it is, of course, necessary at this point to take antilogs to arrive at the seasonally adjusted series.

It is interesting to note that while this method of seasonal adjustment is linear, it is not time-invariant. For a filter to be time-invariant it must be true that a shift in all the inputs by τ periods shifts the output by exactly τ periods. While the Hannan filter gives an output that is a weighted sum of current and past values of the series to be adjusted, the weights are trigonometric with frequencies integral multiples of $2\pi/12$; hence, unless τ is an integer mod 12, shifting the inputs by τ will not produce the output appropriate to the period $t+\tau$. This means, of course, that analytical methods cannot be fully utilized to explore the properties of the modified Hannan filter, and that some such discussion as is contained in the next section is needed.

In the next section, five unemployment series seasonally adjusted in the way described above are compared with the corresponding results of the BLS procedure.

4. SPECTRAL COMPARISONS OF THE MODIFIED HANNAN AND BLS SEASONAL ADJUSTMENT PROCEDURES

Spectral comparisons of the modified Hannan and BLS procedures discussed above are presented in Figures 1.1–5.2. Initial figure numbers refer to the category of unemployment considered:

1. Total U. S. unemployment
2. Unemployment, Male, 14–19
4. Unemployment, Male, 20+
4. Unemployment, Female, 14–19
5. Unemployment, Female, 20+

Secondary figure numbers refer to the type of procedure used for the adjustment: 1 for the BLS procedure, and 2 for the modified Hannan.

Panel A of each figure shows the estimated power spectra of the original and the appropriate seasonally adjusted series. In order to obtain these estimates, as well as estimates of the cross-spectra between the original and seasonally adjusted series upon which panels C and D are based, the series were prewhitened, using the Parzen autoregressive filter. The resulting estimates for the prewhitened series were then recolored according to the formulae derived in the Appendix.

The Parzen autoregressive prewhitening routine amounts to the following: We compute an autoregression of the series $x(t)$ to be prewhitened by least squares:

$$x(t) = a_1x(t-1) + a_2x(t-2) + \cdots + a_Mx(t-M). \qquad (4.1)$$

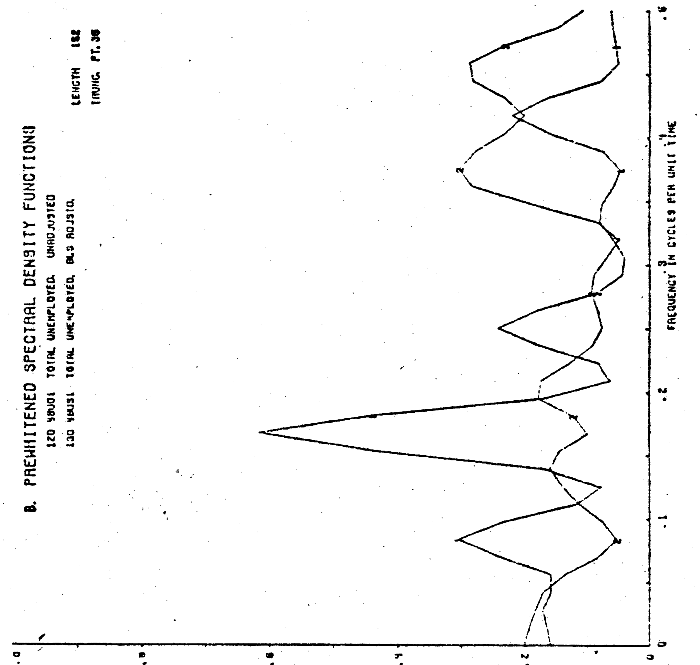
Note that we assume the mean has been removed and that all the coefficients to some maximal lag are computed. However, it is inadvisable to use the very large number of coefficients so obtained to prewhiten the series; hence, a smaller number are selected from those appearing in (4.1) by requiring that those used are significant at some level using the ordinary least squares estimates of their standard errors. Since these standard errors will normally be severely biased towards zero if there is any serial correlation present in the residuals of (4.1), it is advisable to use quite a high significance level. We found that 99% worked markedly better than 95%, and the former is the level used for prewhitening in all the analyses reported here. The autoregressive coefficients which resulted from this rule, and which were used in the analyses, are presented in Table 1 for all fifteen series. It can be seen that most of the schemes are very nearly first- or second-order quasi-differences with some seasonal effects in the original series for the two younger age groups. For comparison, it should be mentioned that the quasi-differences, used in eliminating trend before computing the Hannan adjusted series, were all second-order except for unemployment of females, 14–19, which was first-order. In every case, $\gamma=0.75$, which implies that the first two autoregressive coefficients are -1.500 and 0.562 for second-order differences, and $-.7500$ for first-order differences.

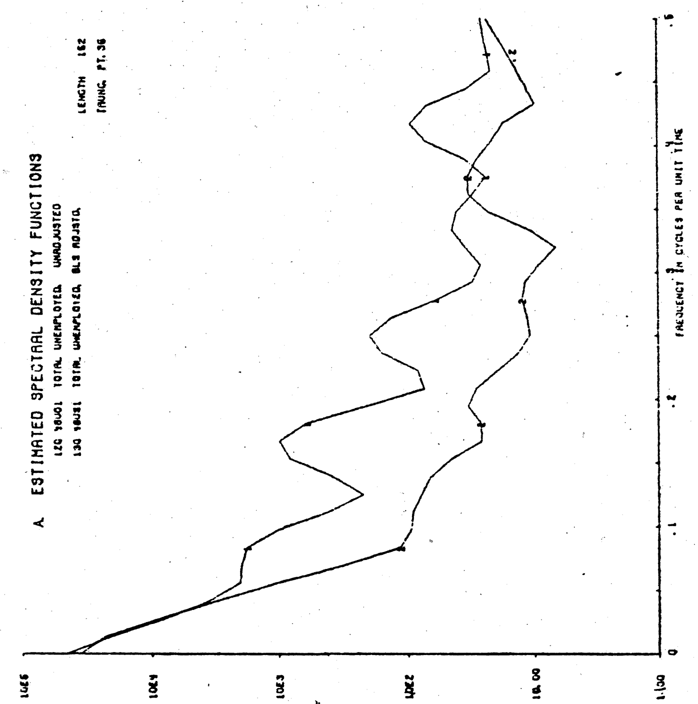
Panel B of each figure shows the spectra of the two prewhitened series, original and seasonally adjusted. It should be noted that despite the inclusion of autoregression coefficients for lags of 12 and 13 months for the two younger age groups, quite marked seasonal peaks remain in the spectra of the prewhitened, but unadjusted, series. Apart from this, however, it is clear that the autoregressive filter does the job for which it was intended—namely, to reduce the extraordinarily high power present at low frequencies relative to that present at high frequencies in economic time series, including those under consideration here. The estimates suggest that in the original series the power may be as much as 10,000 or 100,000 times greater at low frequencies than at high; prewhitening reduces this difference to a factor of 10 or less.

Panel C of each figure shows the estimated phase angle (as a fraction of $\pi=180°$) and the estimated coherence for the filter concerned. The scale for phase is to the right, that for coherence is to the left; both are arithmetic.

Panel D of each figure shows the gain of each series over the other. The lower curve is the one appropriate in the present case; it shows the gain (almost always less than one) of the adjusted series over the original series. The reason for the second curve is that our computing and plotting routine is designed to be multipurpose, and in many situations one series cannot so easily be identified as the input and the other as the output.

Comparison of the relation between the spectra of original and seasonally adjusted series for BLS and Hannan methods of adjustment, as presented in Panel A of the figures, is subject to severe limitations but nonetheless is useful. The upper curve in each panel is the estimated spectrum of the row series, the lower that of the adjusted series. The upper curve contains the typical peaks occurring at one or more of the six frequencies, 0.083, 0.167, 0.250, 0.333, 0.417, and 0.500, which can be associated with the phenomenon of seasonality. These are particularly marked for the two younger age groups. Those with sharp eyes will note that the upper curves in corresponding figures for the BLS and Hannan adjustments are slightly different; these differences result from differences in the scale on the left. This scale is logarithmic but the number of cycles varies in such a way that no space is wasted; hence, the graphs for a BLS comparison typically contain more cycles than for the Hannan procedure, since the BLS


B. PREWHITENED SPECTRAL DENSITY FUNCTIONS
TOTAL UNEMPLOYED, UNADJUSTED
TOTAL UNEMPLOYED, BLS ADJSTD.
LENGTH 162
TRUNC. PT. 36
FREQUENCY IN CYCLES PER UNIT TIME


A. ESTIMATED SPECTRAL DENSITY FUNCTIONS
TOTAL UNEMPLOYED, UNADJUSTED
TOTAL UNEMPLOYED, BLS ADJSTD.
LENGTH 162
TRUNC. PT. 36
FREQUENCY IN CYCLES PER UNIT TIME

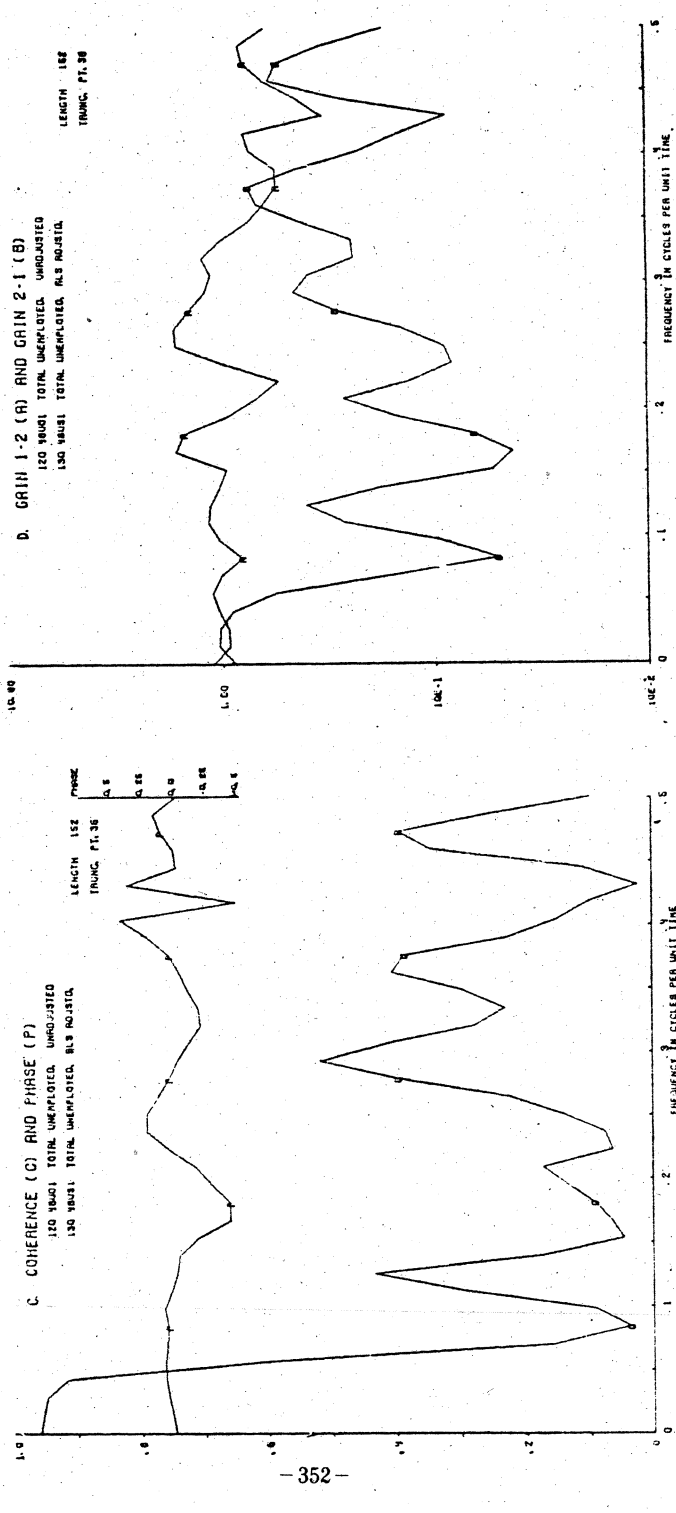
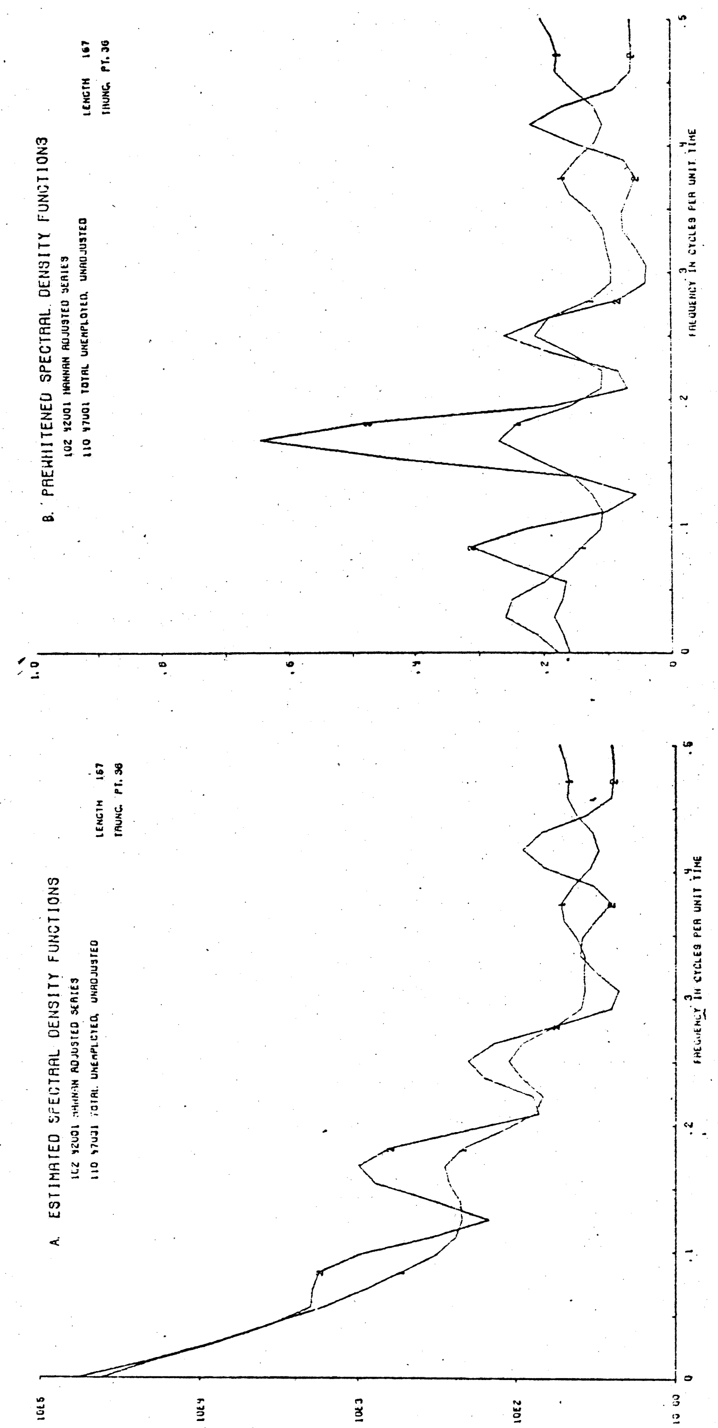


FIG. 1.1. Power spectra, coherence, phase and gains, total U. S. unemployment, original and BLS, seasonally adjusted.


A. ESTIMATED SPECTRAL DENSITY FUNCTIONS
LENGTH 167
TRUNC. PT. 36
FREQUENCY IN CYCLES PER UNIT TIME
B. PREWHITENED SPECTRAL DENSITY FUNCTIONS
LENGTH 167
TRUNC. PT. 36
FREQUENCY IN CYCLES PER UNIT TIME

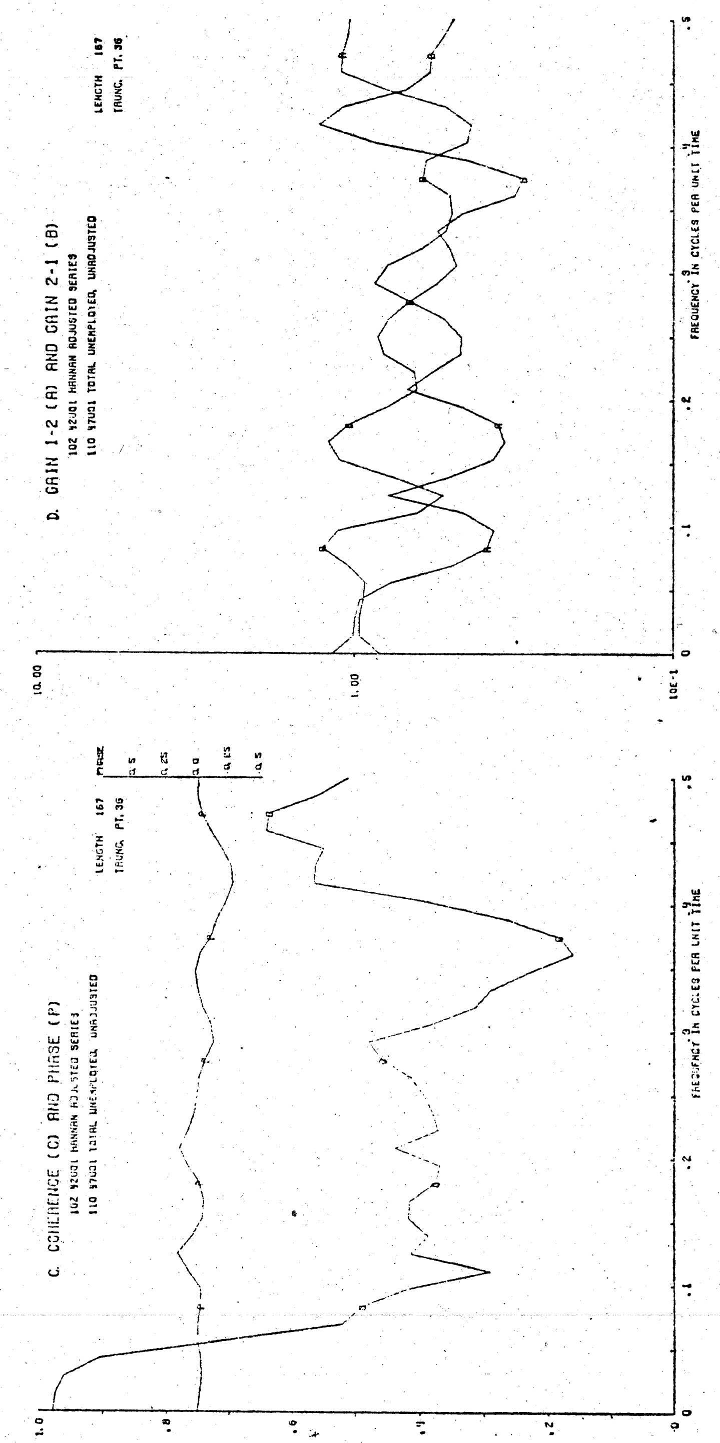
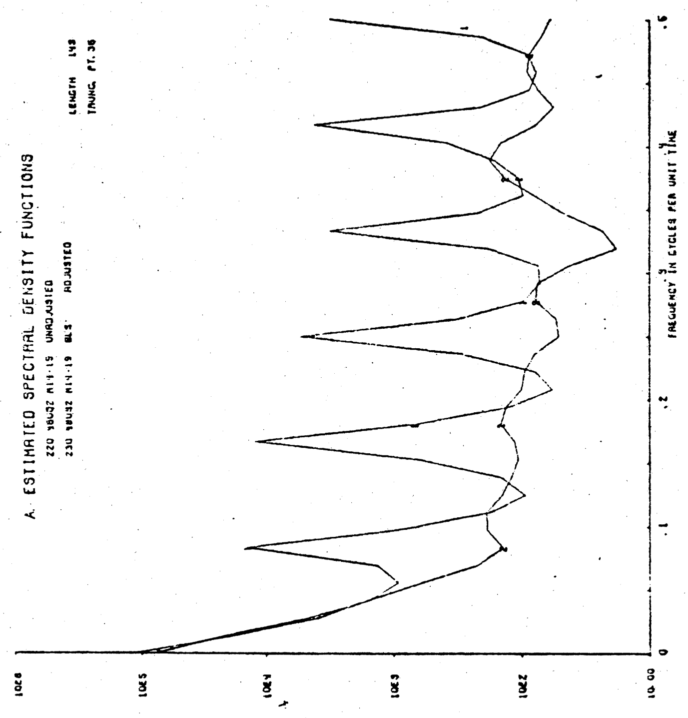
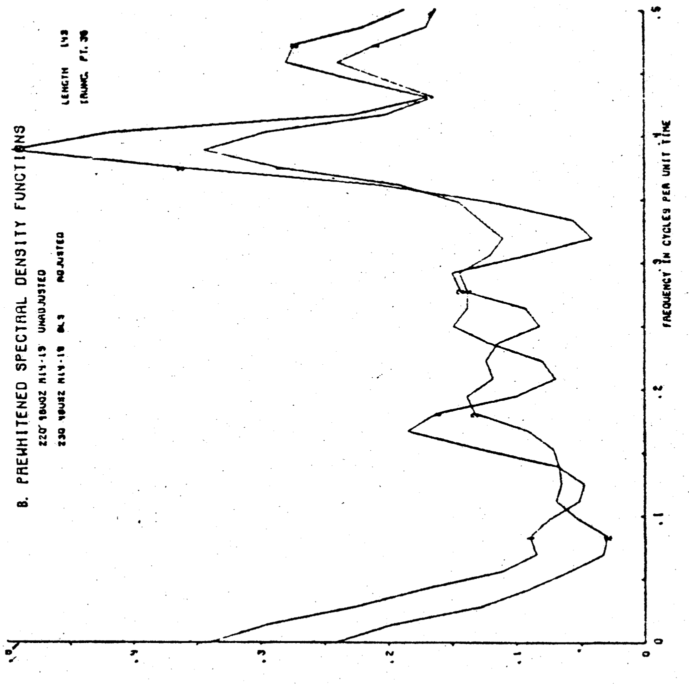


FIG. 1.2. Power spectra, coherence, phase and gains, total U. S. unemployment, original and "Hannan" adjusted.


A. ESTIMATED SPECTRAL DENSITY FUNCTIONS
UNADJUSTED
ADJUSTED
FREQUENCY IN CYCLES PER UNIT TIME


B. PREWHITENED SPECTRAL DENSITY FUNCTIONS
UNADJUSTED
ADJUSTED
FREQUENCY IN CYCLES PER UNIT TIME

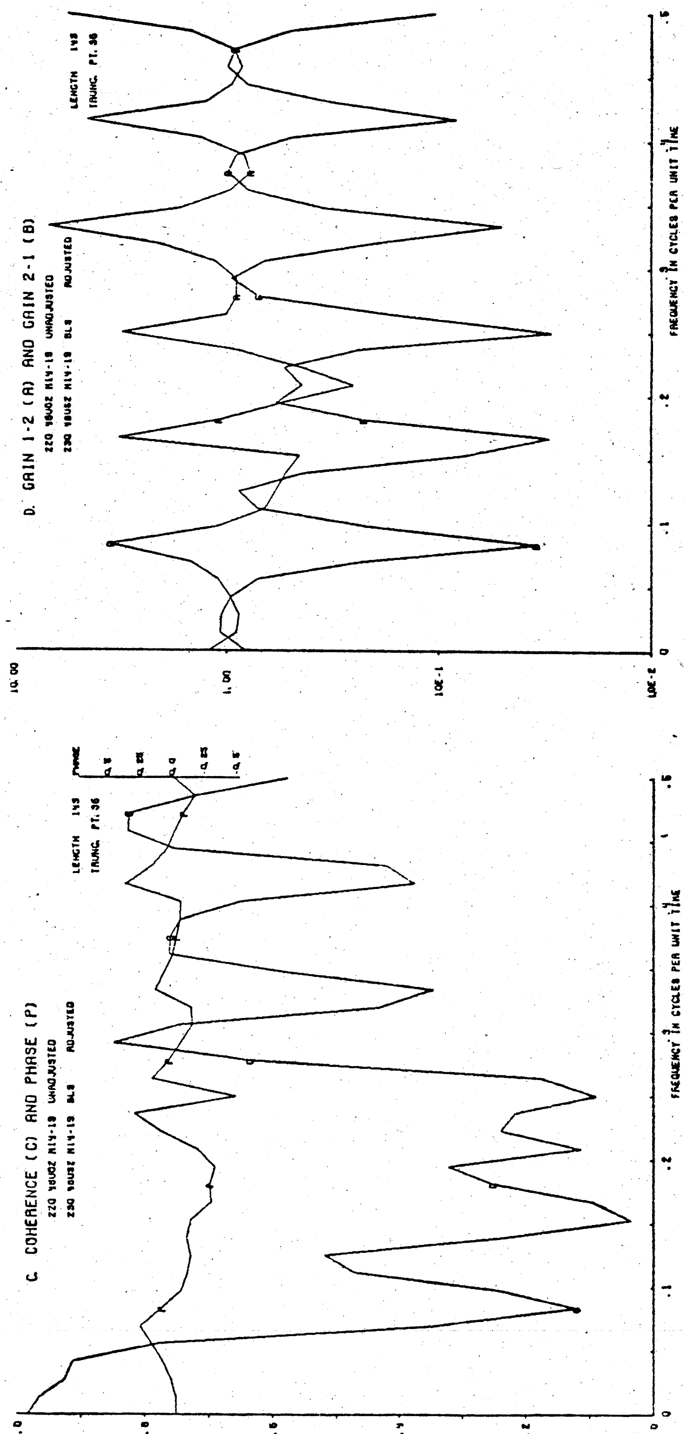
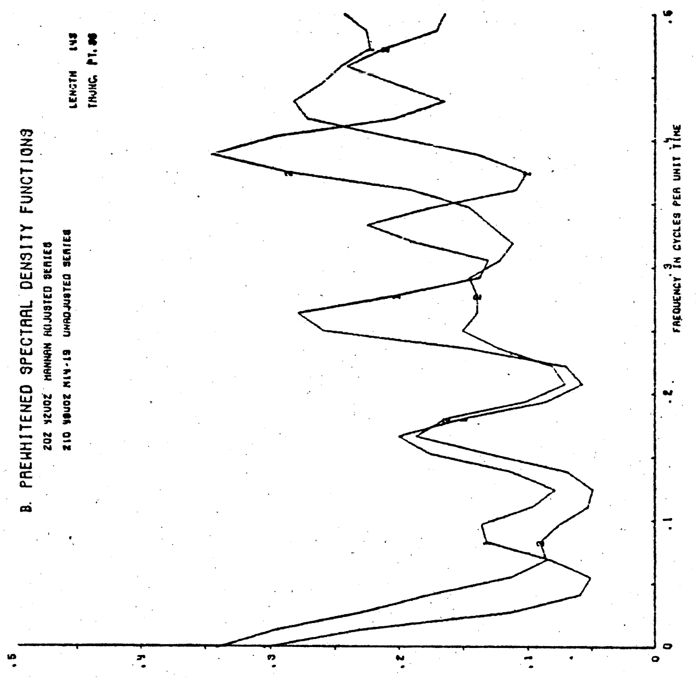
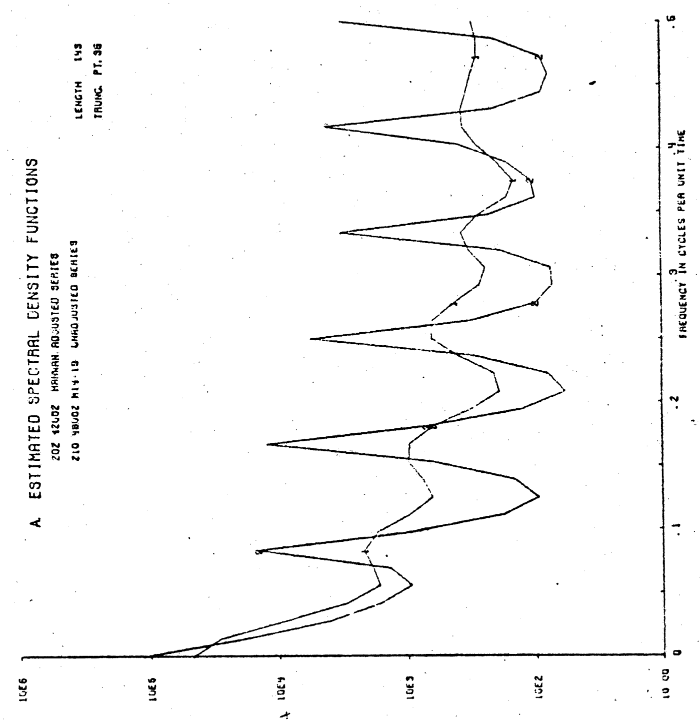


FIG. 2.1. Power spectra, coherence, phase and gains, U. S. unemployment, male, 14–19, original and BLS, seasonally adjusted.


B. PREWHITENED SPECTRAL DENSITY FUNCTIONS
LENGTH 143
TRUNC. PT. 36
FREQUENCY IN CYCLES PER UNIT TIME


A. ESTIMATED SPECTRAL DENSITY FUNCTIONS
LENGTH 143
TRUNC. PT. 36
FREQUENCY IN CYCLES PER UNIT TIME

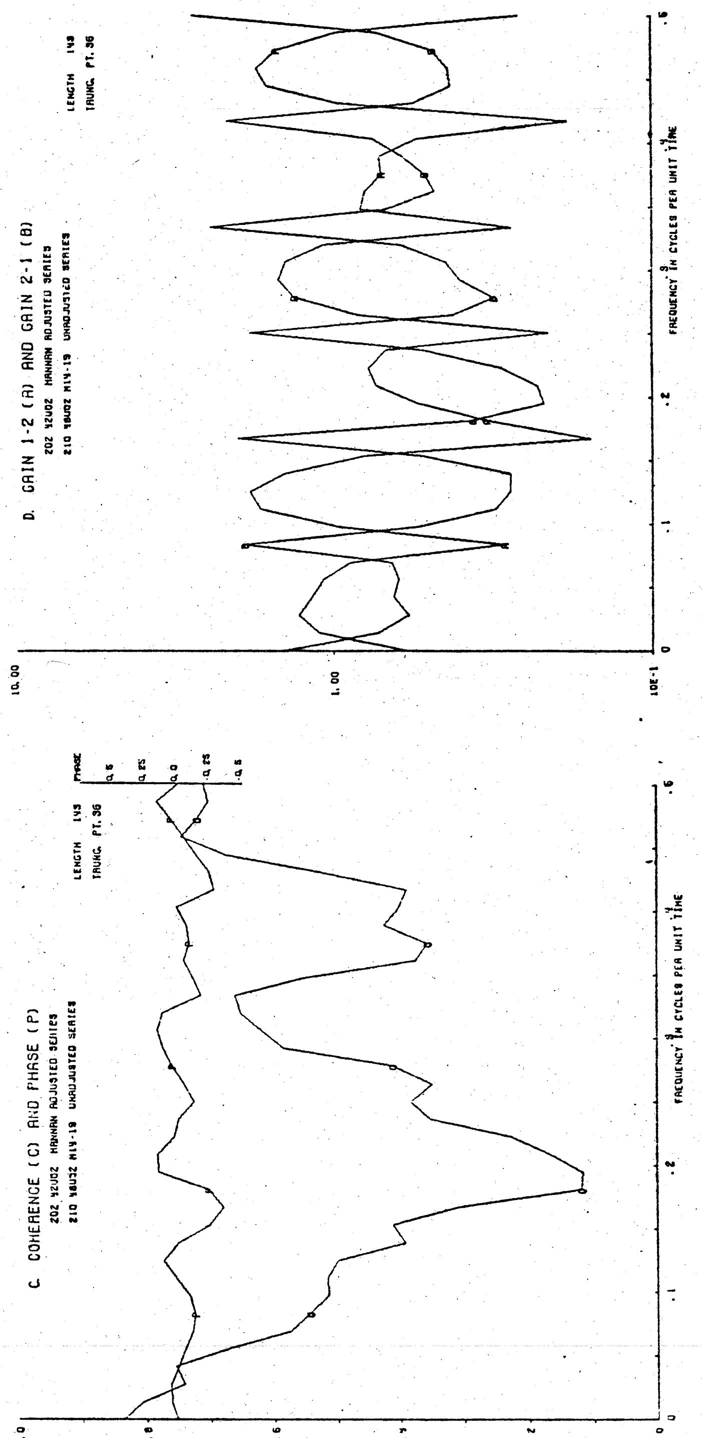
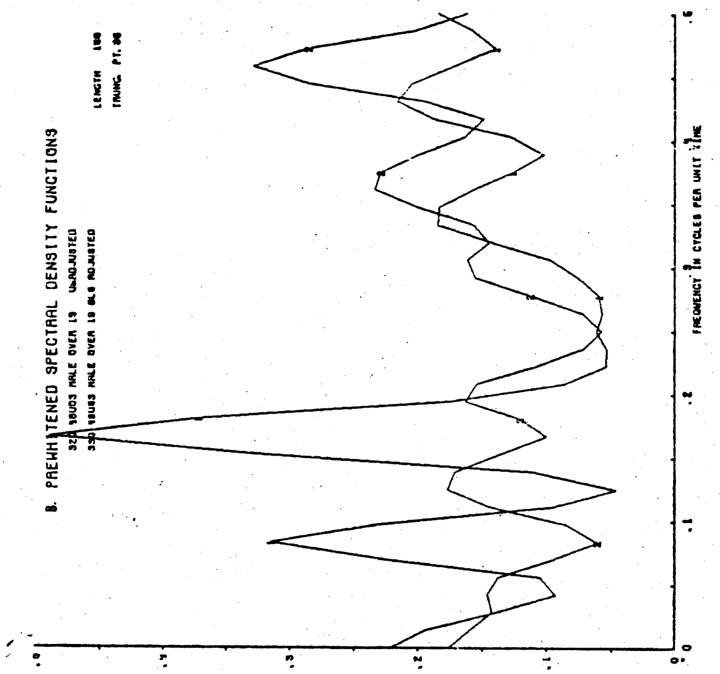
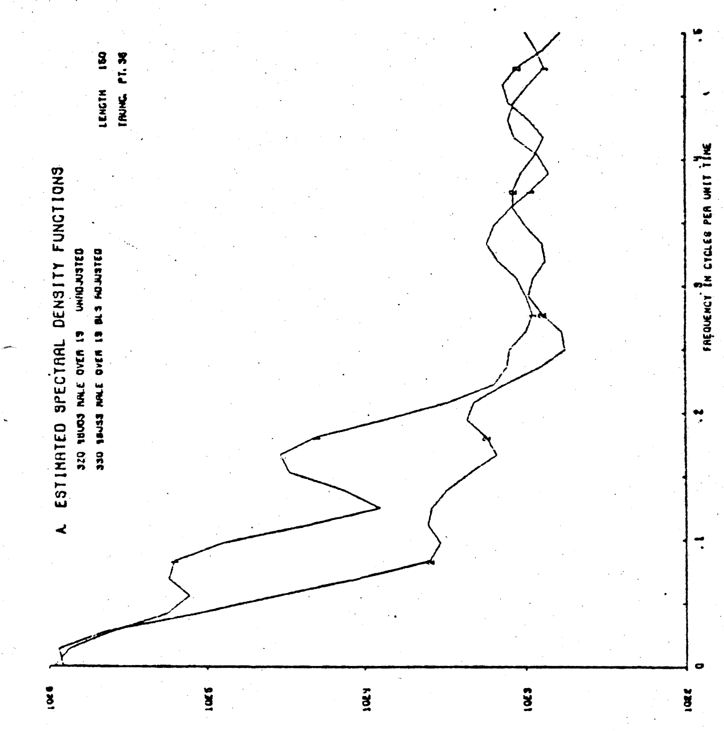


Fig. 2.2. Power spectra, coherence, phase and gains, U. S. unemployment, male, 14–19, original and "Hannan" adjusted.


B. PREWHITENED SPECTRAL DENSITY FUNCTIONS
FREQUENCY IN CYCLES PER UNIT TIME


A. ESTIMATED SPECTRAL DENSITY FUNCTIONS
LENGTH 150
FREQUENCY IN CYCLES PER UNIT TIME

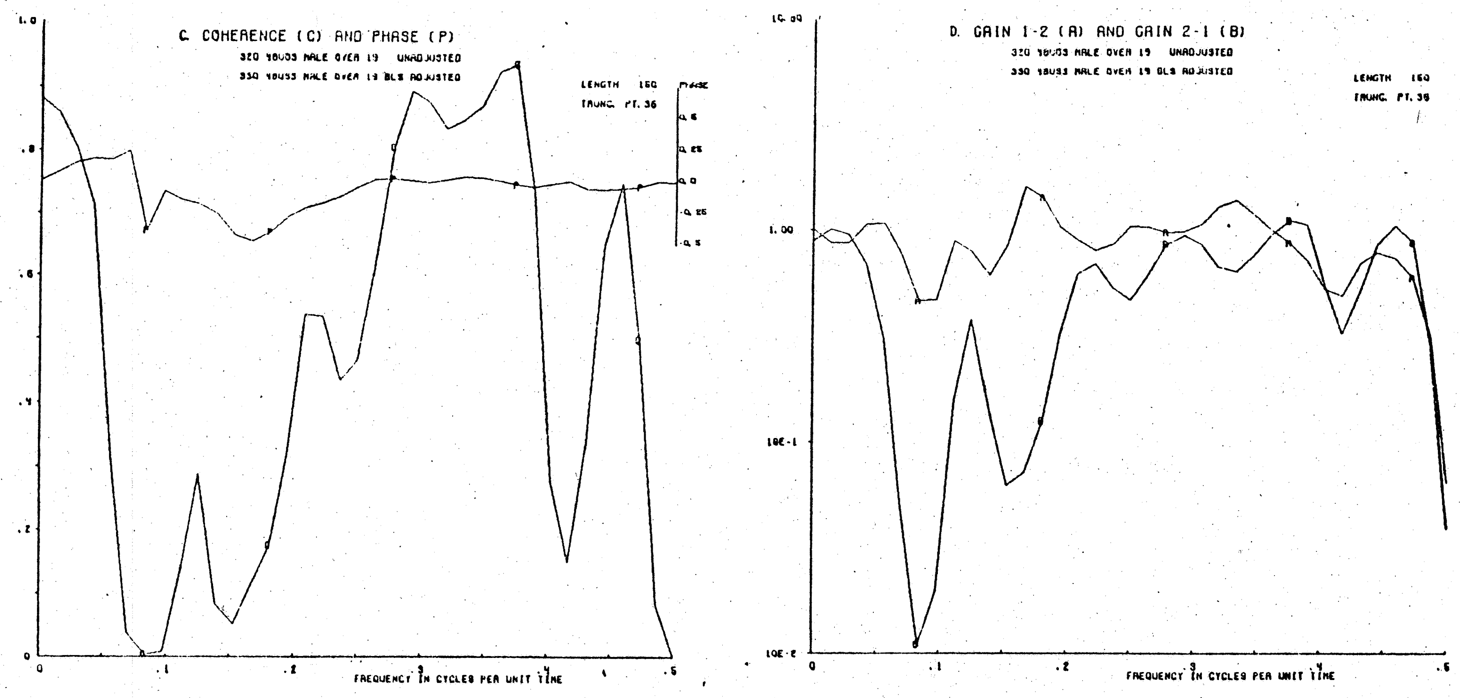
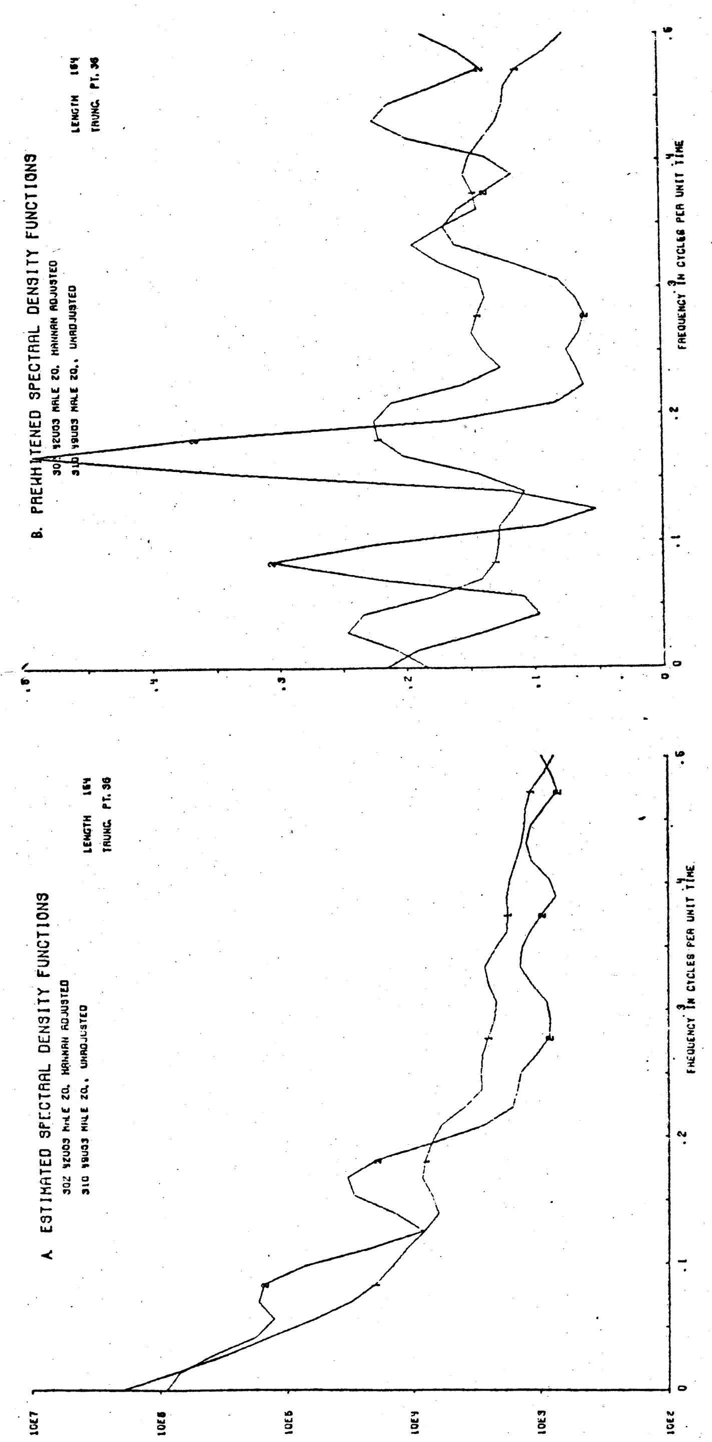


FIG. 3.1. Power spectra, coherence, phase and gains, U. S. unemployment, male, 20+, original and BLS, seasonally adjusted.


A. ESTIMATED SPECTRAL DENSITY FUNCTIONS
B. PREWHITENED SPECTRAL DENSITY FUNCTIONS
HANNAH ADJUSTED
UNADJUSTED
LENGTH 164
TRUNC. PT. 36
FREQUENCY IN CYCLES PER UNIT TIME

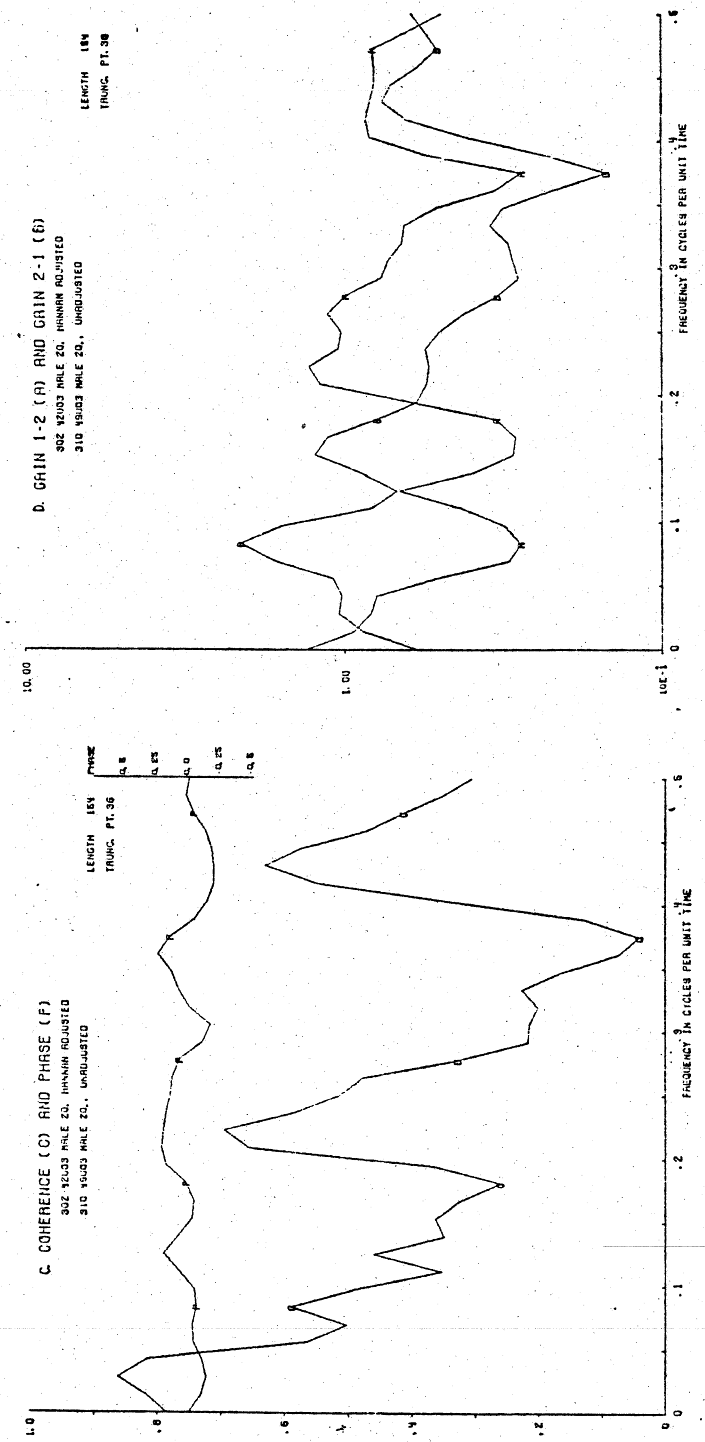
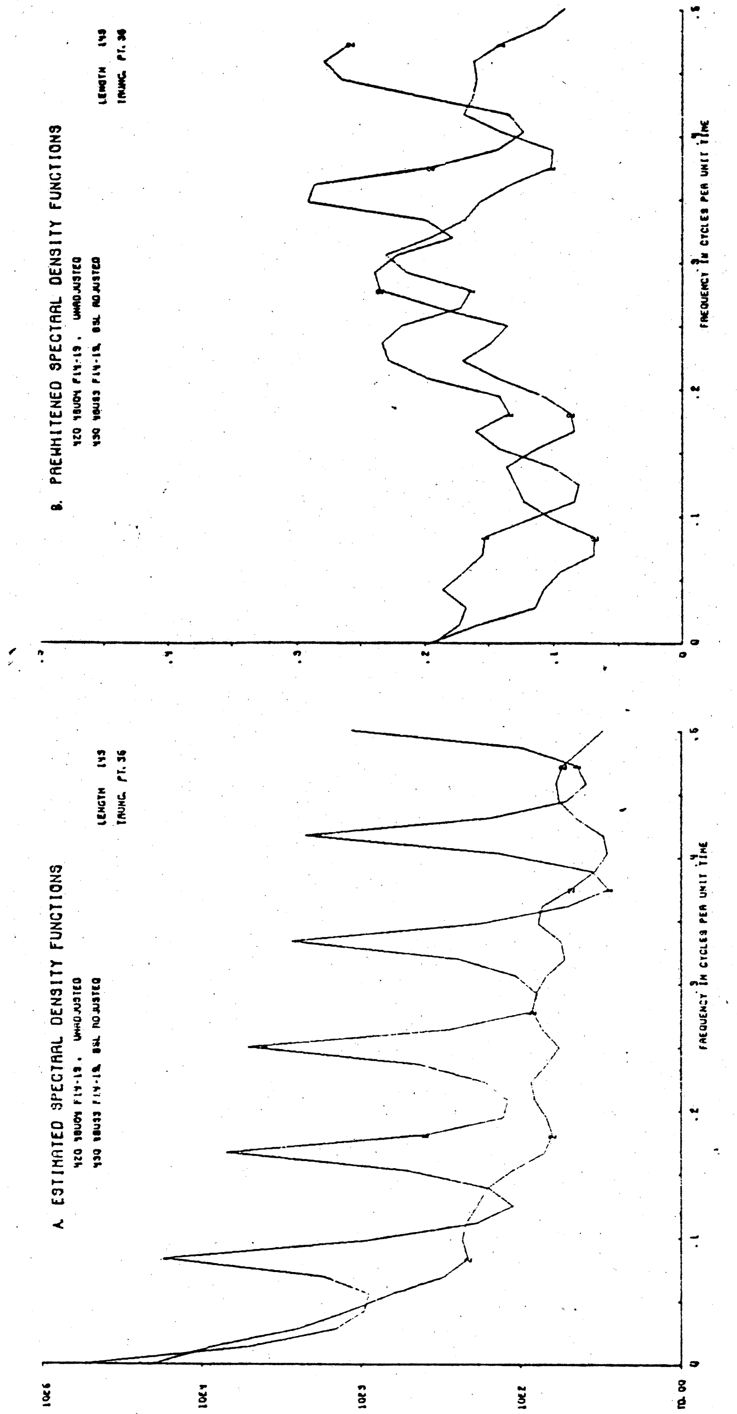


FIG. 3.2. Power spectra, coherence, phase and gains, U. S. unemployment, male, 20+, original and "Hannan" adjusted.


A. ESTIMATED SPECTRAL DENSITY FUNCTIONS
LENGTH 143
TRUNC. PT. 36
FREQUENCY IN CYCLES PER UNIT TIME
B. PREWHITENED SPECTRAL DENSITY FUNCTIONS
LENGTH 143
TRUNC. PT. 36
FREQUENCY IN CYCLES PER UNIT TIME

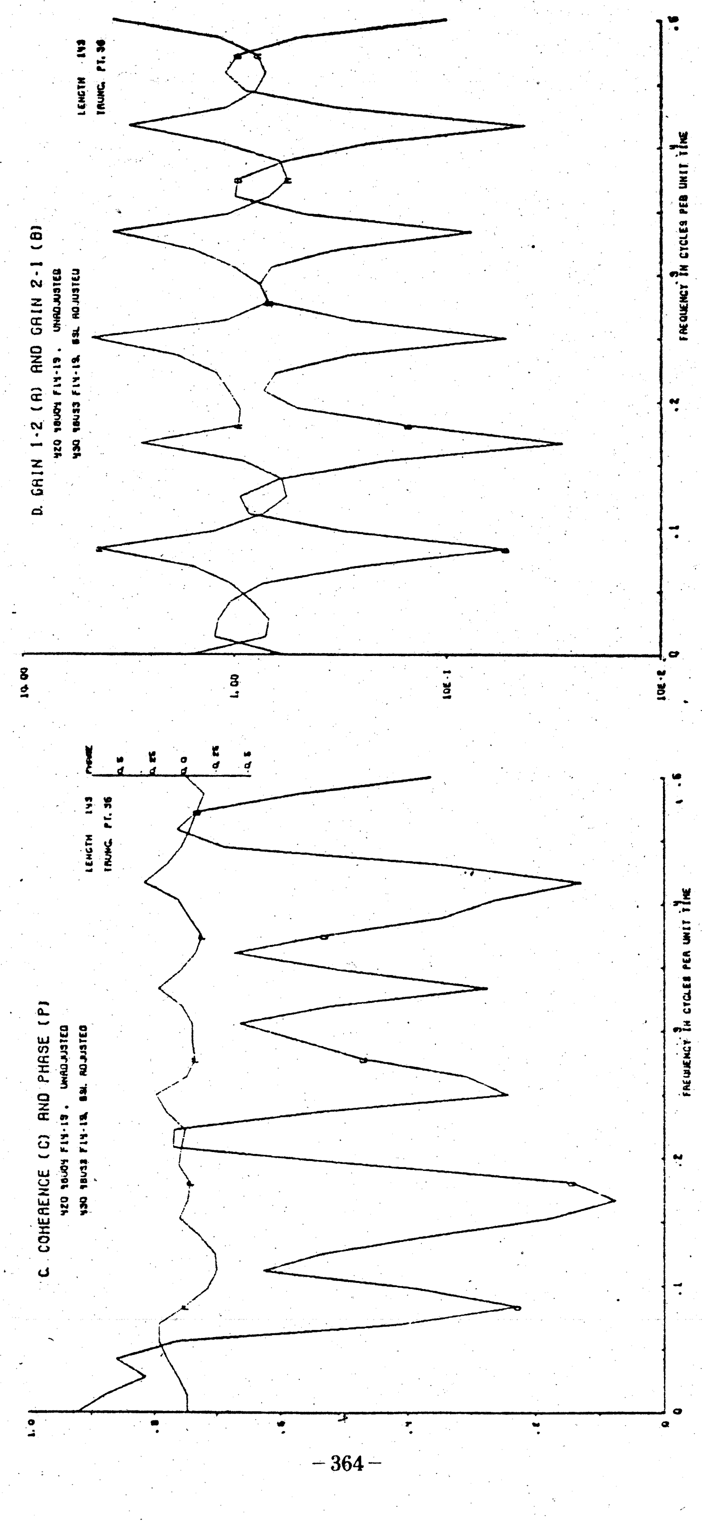
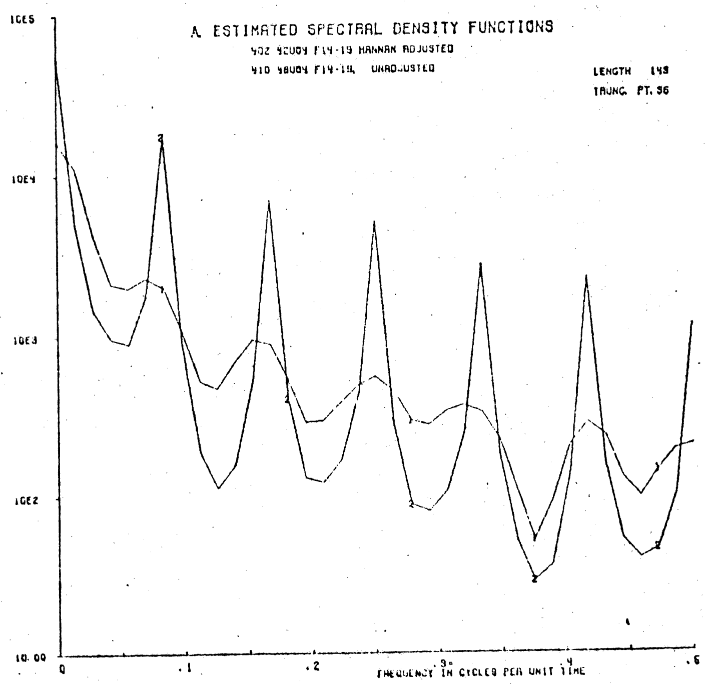
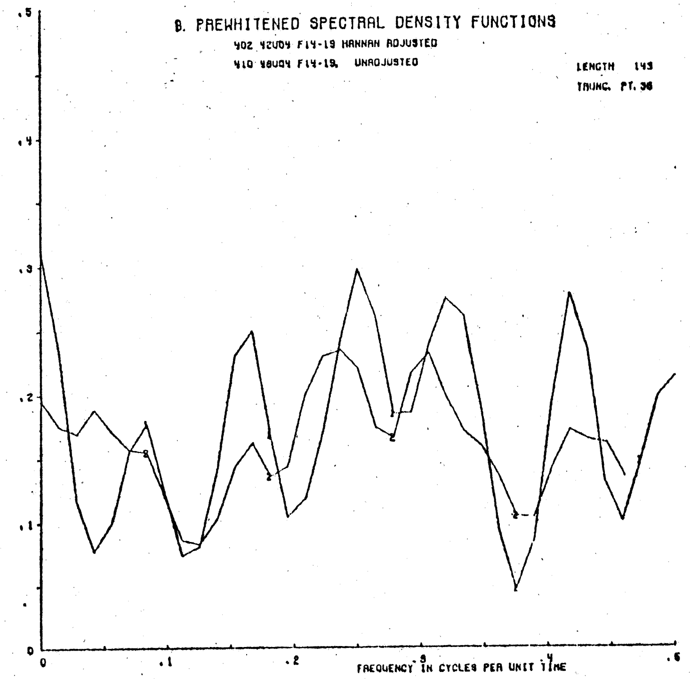


FIG. 4.1. Power spectra, coherence, phase and gains, U. S. unemployment, female, 14–19, original and BLS, seasonally adjusted.


A. ESTIMATED SPECTRAL DENSITY FUNCTIONS
HANNAN ADJUSTED
UNADJUSTED
LENGTH 143
TRUNC. PT. 36
FREQUENCY IN CYCLES PER UNIT TIME


B. PREWHITENED SPECTRAL DENSITY FUNCTIONS
HANNAN ADJUSTED
UNADJUSTED
LENGTH 143
TRUNC. PT. 36
FREQUENCY IN CYCLES PER UNIT TIME

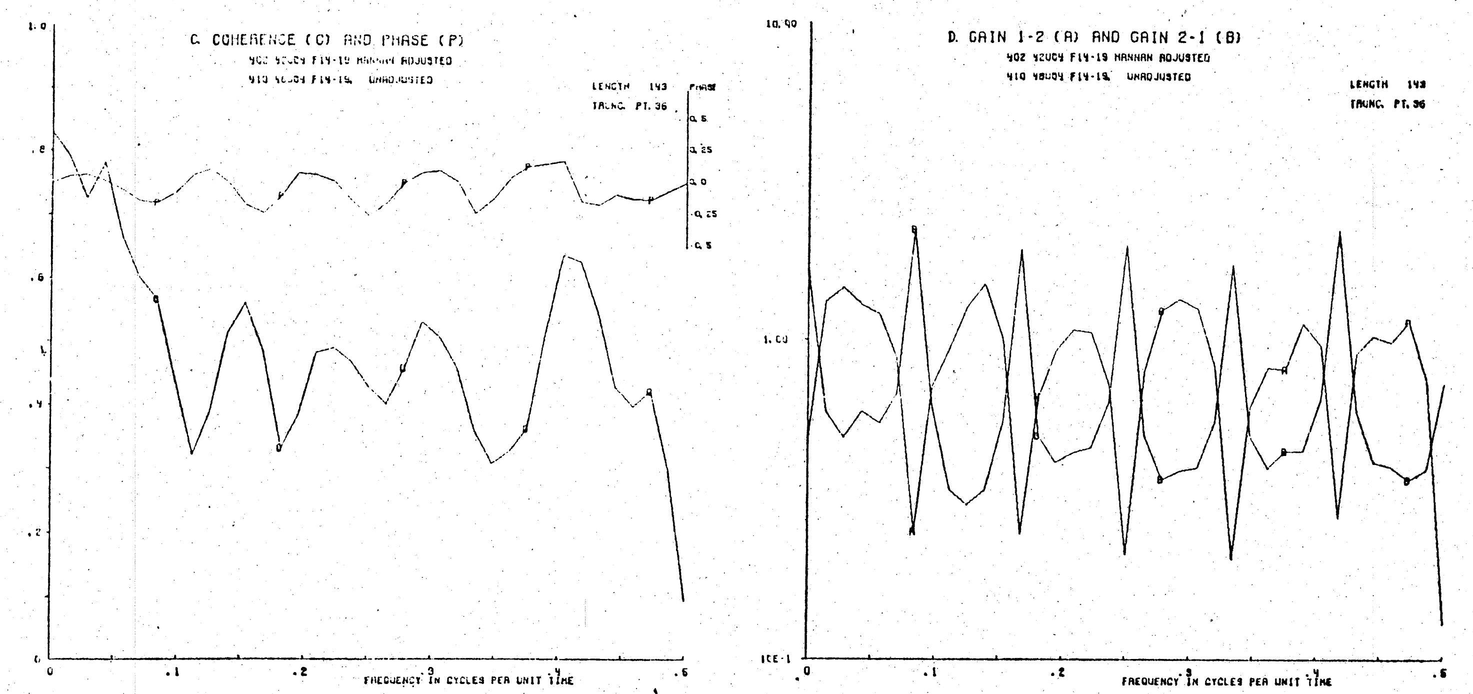
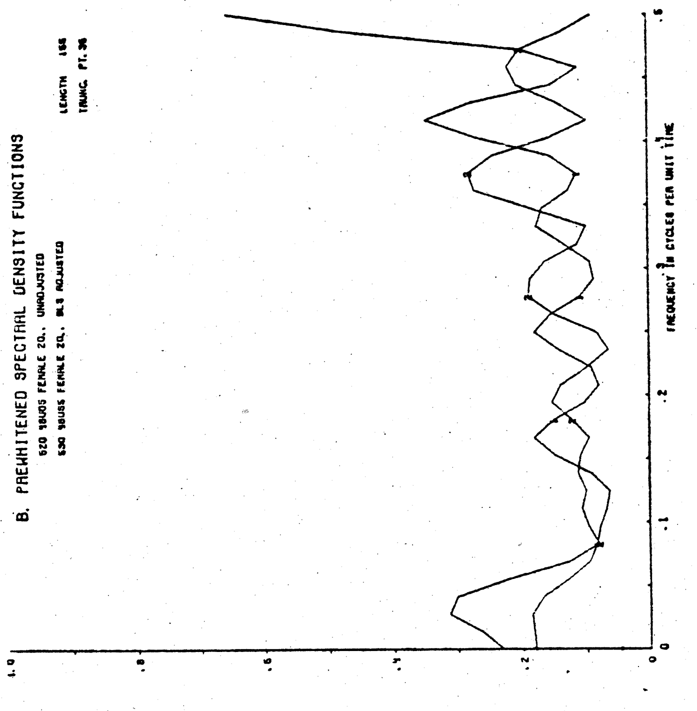
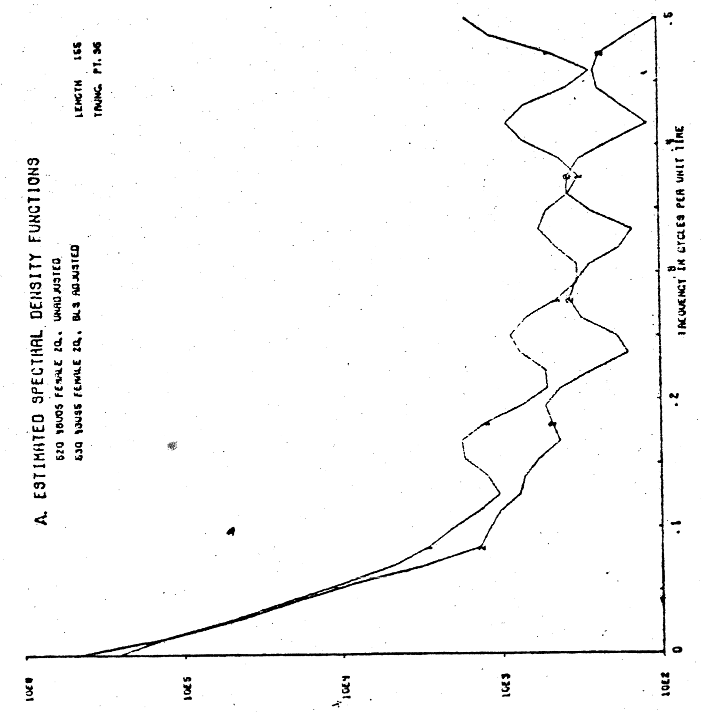


Fig. 4.2. Power spectra, coherence, phase and gains, U. S. unemployment, female, 14–19, original and "Hannan" adjusted.


B. PREWHITENED SPECTRAL DENSITY FUNCTIONS
620 16U05 FEMALE 20., UNADJUSTED
630 16U05 FEMALE 20., BLS ADJUSTED
LENGTH 155
TRUNC. PT. 36
FREQUENCY IN CYCLES PER UNIT TIME


A. ESTIMATED SPECTRAL DENSITY FUNCTIONS
620 16U05 FEMALE 20., UNADJUSTED
630 16U05 FEMALE 20., BLS ADJUSTED
LENGTH 155
TRUNC. PT. 36
FREQUENCY IN CYCLES PER UNIT TIME

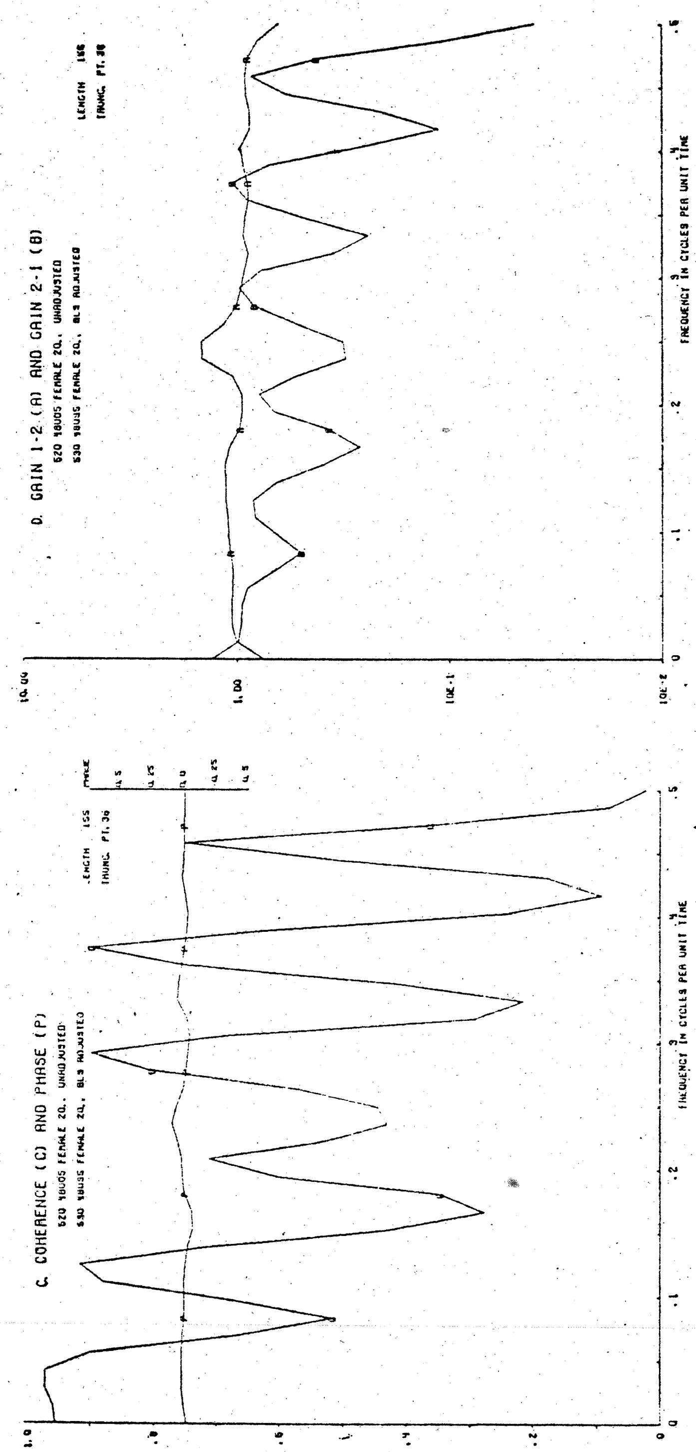
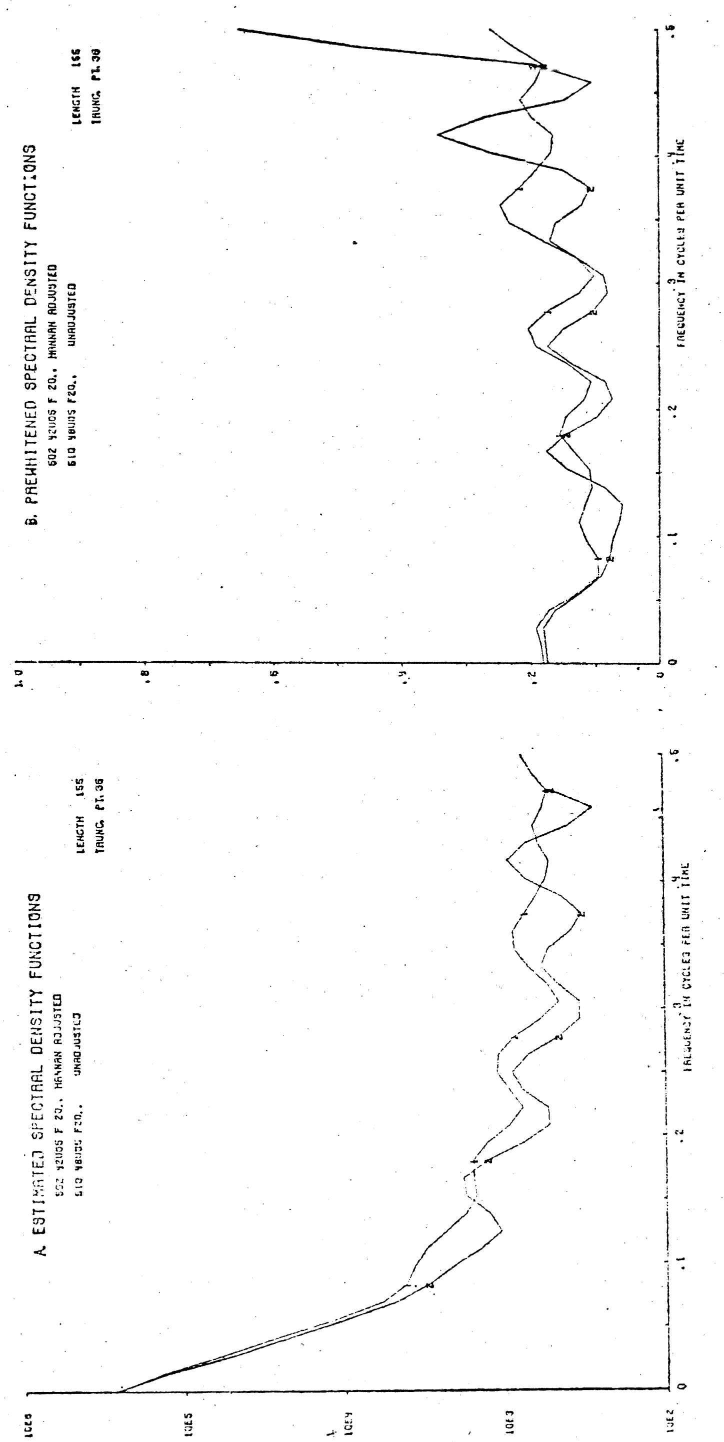


FIG. 5.1. Power spectra, coherence, phase and gains, U. S. unemployment, female, 20+, original and BLS, seasonally adjusted.


A. ESTIMATED SPECTRAL DENSITY FUNCTIONS
B. PREWHITENED SPECTRAL DENSITY FUNCTIONS
FREQUENCY IN CYCLES PER UNIT TIME

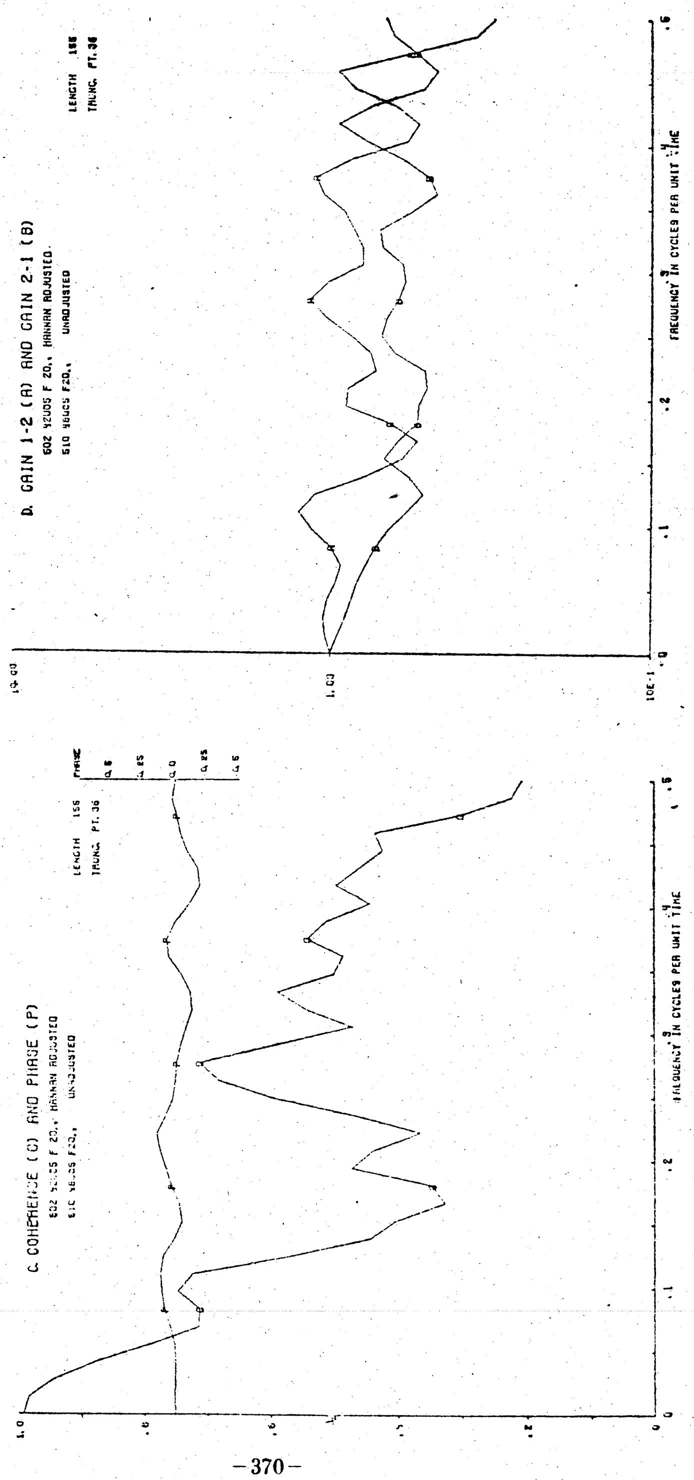


FIG. 5.2. Power spectra, coherence, phase and gains, U. S. unemployment, female, 20+, original and "Hannan" adjusted.

TABLE 1. AUTOREGRESSIVE COEFFICIENTS USED IN PREWHITENING

LAG	ANALYSIS														
	Original series: Total unemployment	Original series: Male, 14–19	Original series: Male, 20+	Original series: Female, 14–19	Original series: Female, 20+	BLS adjusted series: Total unemployment	BLS adjusted series: Male, 14–19	BLS adjusted series: Male, 20+	BLS adjusted series: Female, 14–19	BLS adjusted series: Female, 20+	Hannan adjusted series: total unemployment	Hannan adjusted series: Male, 14–19	Hannan adjusted series: Male, 20+	Hannan adjusted series: Female, 14–19	Hannan adjusted series: Female, 20+
1	−.926	−.746	−1.356	−.702	−.922	−1.129	−.939	−1.064	−.914	−.953	−.945	−.834	−.940	−.762	−.915
2	0.0	0.0	0.450	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
3															
4	0.0	0.0	0.0	0.0	0.0	0.173	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
6	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.126	0.0	0.0	0.0	0.0	0.0	0.0	0.0
7															
8															
9															
10															
11															
12	0.0	−.753	0.0	−.749	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
13	0.0	0.595	0.0	0.537	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
14+	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

curves lie farther below the spectra of the corresponding original series than do the Hannan curves. This makes the peaks appear sharper in the graphs for the Hannan method of adjustment.

The power spectrum of the BLS adjusted series lies farther below the spectrum of the original series than does the spectrum of the Hannan adjusted series. In one case (Fig. 5.2), however, the spectrum of the Hannan adjusted series lies *above* the original! While this would be impossible with the historical form of the adjustment procedure (which would essentially involve subtracting out a regression from the data), it can happen with the current form since the seasonally adjusted series does not then effectively represent a residual from a regression equation.[21] The difference between the two filters is particularly marked for the two younger age groups of unemployed; these are the groups with the most pronounced and stable seasonal patterns. It is most disturbing that this excessive loss of power occurs with the BLS filter even at relatively low frequencies, since it is these frequencies that tend to dominate the movements of the series. In a previous paper [12], I concluded from this that the BLS filter removed more from the series than could properly be considered as seasonal. In this respect the Hannan procedure, at least in part, appears to be superior but the conclusion cannot be drawn unambiguously because in certain cases the Hannan procedure leaves marked peaks in the spectrum at seasonal frequencies. That is to say, in certain cases the Hannan procedure does not perhaps, remove *all* that can properly be considered as seasonal.

Approximately the same sort of conclusion may be drawn from Panels C and D of the figures. In general, coherence and gain are low, not only at seasonal frequencies but also in between for the BLS filter. This is especially marked for the two younger age groups and suggests that the BLS procedure for determining seasonal factors is too flexible, and therefore attributes too much to seasonality and too little to inherent irregularity. In some cases, the Hannan procedure offers some improvement over the BLS filter in this respect, although, as already remarked, the coherence and gain are rather higher than we might like *at* seasonal frequencies, though better in between. In other cases, however, coherences between the original and Hannan adjusted series are substantially lower at non-seasonal frequencies than the corresponding coherences for the original and BLS adjusted series. In these situations, the excessive removal of information other than seasonal by the BLS procedure is compensated by the other distortions of the Hannan procedure. According to these criteria, the Hannan procedure, as modified for use in a current context, may be judged marginally superior to the BLS procedure, but hardly sufficiently so as to warrant exchanging the former for the latter.

Perhaps the most disturbing finding is the large phase shifts which occur with both methods of seasonal adjustment. If one procedure can be considered superior to the other in this respect—although I hardly think this is the case—the BLS procedure has, perhaps, a slight edge. But the phase shift occurring at 0.167 and nearby frequencies for the BLS filter is nonetheless especially disturbing since substantial power occurs in this frequency band in all the original

[21] George Hext pointed this out to me.

series considered here (and indeed in almost all economic time series with which I am familiar). The shift is nearly $\pi/2 = 90°$ for the two younger age groups and the total, and is absent for the two older age groups. At a frequency of 0.167 $(2\pi) = \pi/3$, a phase shift of $\pi/2$ corresponds to a time lag of a month and a half $(\pi/2 \div \pi/3)$. It is, of course, well known that one-sided filters must produce phase shifts of some sort. What is so remarkable here is that, despite its defects, the BLS procedure compares as favorably with the more sophisticated Hannan procedure as it does.[22]

It is much to be regretted that more definitive conclusions cannot be drawn from the results presented here. However, it does seem safe to say that the question of a good seasonal filter in a current context is by no means a settled one. Indeed, I am not altogether certain that, despite all its many problems, it will be easy to find an appropriate filter that is better than the BLS filter in every respect.

APPENDIX

Notes on Recoloring Spectra and Cross-Spectra and on Recovery of Seasonal Factors for the Trend-Contaminated Series

In the text of this paper it was stated that the stochastic increments appearing in the frequency representations of the covariance stationary input and output of a time-invariant linear filter are related by

$$dZ_y(\lambda) = l(\lambda) dZ_x(\lambda), \tag{A.1}$$

where $l(\lambda)$ is the frequency response function of the filter. From (A.1) we immediately obtain

$$\begin{aligned} dF_{yy}(\lambda) &= 2E\, dZ_y(\lambda)\, \overline{dZ_y(\lambda)} \\ &= 2\,|\, l(\lambda)\,|^2 E\, dZ_x(\lambda)\, \overline{dZ_x(\lambda)} \\ &= |\, l(\lambda)\,|^2\, dF_{xx}(\lambda). \end{aligned} \tag{A.2}$$

Thus, on the assumption that the singular and the jump components of $F_{yy}(\lambda)$ and $F_{xx}(\lambda)$ are identically zero, we have the following relation between the two power spectra:

$$f_{yy}(\lambda) = |\, l(\lambda)\,|^2 f_{xx}(\lambda). \tag{A.3}$$

The squared modulus of the frequency response function (or squared gain) is sometimes called the *transfer function* of the filter. It follows from (A.3) that if we have an estimate of the power spectrum for a filtered series, we can obtain an estimate for the unfiltered series by dividing at each frequency by the transfer function of the filter, $|l(\lambda)|^2$. Elsewhere [12] I have called this *recoloring*, since the purpose of filtering is usually to remove high power at low frequencies and render the spectrum to be estimated as flat as possible, i.e., *prewhitening*.

Now let us suppose we have two series, $x_1(t)$ and $x_2(t)$, which we filter using,

[22] In connection with a similar problem, E. J. Hannan pointed out that, for zero coherence, the phase angle between two jointly stationary processes is uniformly distributed on the internal $[-\pi/2, \pi/2]$. Thus, at least a portion of the phase shifts, namely those occurring at or near seasonal frequencies, are presumably not significant.

respectively, filters L_1 and L_2 with frequency response functions $l_1(\lambda)$ and $l_2(\lambda)$, to obtain, respectively, the output series $y_1(t)$ and $y_2(t)$. As before, assume that the jump and singular components of both spectra and the cross-spectrum of the input series and of the output series vanish identically. Write $f^x_{11}(\lambda)$, $f^x_{22}(\lambda)$, $f^y_{11}(\lambda)$, and $f^y_{22}(\lambda)$ for the spectra of the two input series and the two output series; and write $f^x_{12}(\lambda)$ and $f^y_{12}(\lambda)$ for the cross-spectra between the two series before and after filtering. (A.3) yields the relations between the various spectra:

$$\left.\begin{aligned} f^y_{11}(\lambda) &= |\, l_1(\lambda)\,|^2 f^x_{11}(\lambda), \\ f^y_{22}(\lambda) &= |\, l_2(\lambda)\,|^2 f^x_{22}(\lambda). \end{aligned}\right\} \tag{A.3'}$$

What we now seek to determine is the relation between $f^y_{12}(\lambda)$ and $f^x_{12}(\lambda)$.

Making use of (2.2′), (2.6), and (2.12), we find

$$\begin{aligned} dF^y_{12}(\lambda) &= 2E dZ^y_1(\lambda)\overline{dZ^y_2(\lambda)} \\ &= 2E l_1(\lambda) dZ^x_1(\lambda)\overline{l_2(\lambda)}\ \overline{dZ^x_2(\lambda)} \\ &= l_1(\lambda)\overline{l_2(\lambda)} dF^x_{12}(\lambda), \end{aligned}$$

so that

$$f^y_{12}(\lambda) = l_1(\lambda)\overline{l_2(\lambda)} f^x_{12}(\lambda). \tag{A.4}$$

The multiplier $l_1(\lambda)\overline{l_2(\lambda)}$ is a complex number, and, while it is perfectly possible to divide one complex number by another, it does not yield many useful insights. Instead, let us write all of our complex functions in polar form:

$$\left.\begin{aligned} l_1(\lambda) &= G_1(\lambda)e^{i\phi_1(\lambda)} \\ l_2(\lambda) &= G_2(\lambda)e^{i\phi_2(\lambda)} \\ f^x_{12}(\lambda) &= \Gamma_x(\lambda)e^{i\theta_x(\lambda)} \\ f^y_{12}(\lambda) &= \Gamma_y(\lambda)e^{i\theta_y(\lambda)} \end{aligned}\right\} \tag{A.5}$$

It follows that

$$\begin{aligned} f^y_{12}(\lambda) &= \Gamma_y(\lambda)e^{i\theta_y(\lambda)} = l_1(\lambda)\overline{l_2(\lambda)} f^x_{12}(\lambda) \\ &= G_1(\lambda)G_2(\lambda)e^{i[\phi_1(\lambda)-\phi_2(\lambda)]}\Gamma_x(\lambda)e^{i\theta_x(\lambda)} \\ &= \{G_1(\lambda)G_2(\lambda)\Gamma_x(\lambda)\}e^{i[\theta_x(\lambda)+\phi_1(\lambda)-\phi_2(\lambda)]} \end{aligned} \tag{A.6}$$

whence

$$\left.\begin{aligned} \Gamma_y(\lambda) &= \Gamma_x(\lambda)\cdot[G_1(\lambda)G_2(\lambda)] \\ \theta_y(\lambda) &= \theta_y(\lambda) + \phi_1(\lambda) - \phi_2(\lambda) \end{aligned}\right\}. \tag{A.7}$$

We see that the modulus of $f^y_{12}(\lambda)$ is multiplied by the product of the filter gains and the phase of $f^y_{12}(\lambda)$ is altered by the addition of the difference in phase angles of the two filters. Only if the two series whose cross-spectra we seek to determine are prewhitened by *the same filter*, so that $\phi_1(\lambda) = \phi_2(\lambda)$, will the phase relationship between the two series be unaltered. In most problems involving economic time series, such as the comparison of original and seasonally adjusted series, it is exceedingly important to determine phase relationships. Thus, some form of recoloring will surely be necessary to arrive at meaningful results.

Let us suppose now that $x_1(t)$ and $x_2(t)$ are, respectively, the input and the output of a filter, say one designed to eliminate seasonal fluctuations. Let the frequency response function of this filter be $l_s(\lambda)$, with gain $G_s(\lambda)$, and phase angle $\phi_s(\lambda)$. By our discussion in Section 2 of the text (equation 2.14):

$$\left.\begin{aligned} l_s(\lambda) &= \frac{f_{12}^x(\lambda)}{f_{11}^x(\lambda)} = \frac{c_{12}^x(\lambda)}{f_{11}^x(\lambda)} - i\frac{q_{12}^x(\lambda)}{f_{11}^x(\lambda)} \\ G_s(\lambda) &= \frac{\sqrt{[c_{12}^x(\lambda)]^2 + [q_{12}(\lambda)]^2}}{f_{11}^x(\lambda)} \\ &= \Gamma_x(\lambda)/f_{11}^x(\lambda) \\ \phi_s(\lambda) &= \arctan\left\{\frac{-q_{12}^x(\lambda)/f_{11}^x(\lambda)}{c_{12}^x(\lambda)/f_{11}^x(\lambda)}\right\} \\ &= -\theta_x(\lambda) \end{aligned}\right\} \tag{A.8}$$

It follows from (A.7) and (A.3′) that, in terms of the prewhitened series $y_1(t)$ and $y_2(t)$, the frequency response function is estimated by

$$l_s(\lambda) = \left[\frac{\Gamma_y(\lambda)}{f_{11}^y(\lambda)} \cdot \frac{G_1(\lambda)}{G_2(\lambda)}\right] e^{i[-\theta_y(\lambda)+\phi_1(\lambda)-\phi_2(\lambda)]} \tag{A.9}$$

where $f_{12}^y = \Gamma_y(\lambda)e^{i\theta_y(\lambda)}$. The multiplier in the first bracket is of course the gain of the filter and the expression in the exponent of e is the phase angle of the filter.

The coherence between the two series is in no way affected by filtering. To see this, write

$$\begin{aligned} \rho^2(\lambda) &= \frac{[c_{12}^x(\lambda)]^2 + [q_{12}^x(\lambda)]^2}{f_{11}^x(\lambda)f_{22}^x(\lambda)} \\ &= \frac{\Gamma_x^2(\lambda)}{f_{11}^x(\lambda)f_{22}^x(\lambda)} \\ &= \frac{\Gamma_y^2(\lambda)/[G_1^2(\lambda)G_2^2(\lambda)]}{[f_{11}^y(\lambda)/G_1^2(\lambda)][f_{22}^y(\lambda)/G_2^2(\lambda)]} \\ &= \frac{\Gamma_y^2(\lambda)}{f_{11}^y(\lambda)f_{22}^y(\lambda)} \\ &= \frac{[c_{12}^y(\lambda)]^2 + [q_{12}^y(\lambda)]^2}{f_{11}^y(\lambda)f_{22}^y(\lambda)}. \end{aligned} \tag{A.10}$$

In practice, one would not normally use (A.9) to determine the characteristics of the frequency response function, but would rather recolor the spectra and the co- and quadrature-spectra directly, then use these results to determine further properties of the relationship between the two series. Thus, write

$$l_1(\lambda)\overline{l_2(\lambda)} = u(\lambda) + iv(\lambda). \tag{A.11}$$

Then

$$\begin{aligned} \overset{y}{f}_{12}(\lambda) &= \overset{y}{c}_{12}(\lambda) - i\overset{y}{q}_{12}(\lambda) \\ &= [u(\lambda) + iv(\lambda)][\overset{x}{c}_{12}(\lambda) - i\overset{x}{q}_{12}(\lambda)] \\ &= [u(\lambda)\overset{x}{c}_{12}(\lambda) + v(\lambda)\overset{x}{q}_{12}(\lambda)] \\ &\quad - i[-v(\lambda)\overset{x}{c}_{12}(\lambda) + u(\lambda)\overset{x}{q}_{12}(\lambda)], \end{aligned} \tag{A.12}$$

so that

$$\begin{aligned} \hat{\overset{x}{c}}_{12}(\lambda) &= \frac{u(\lambda)\hat{\overset{y}{c}}_{12}(\lambda) - v(\lambda)\hat{\overset{y}{q}}_{12}(\lambda)}{u^2(\lambda) + v^2(\lambda)} \\ \hat{\overset{x}{q}}_{12}(\lambda) &= \frac{v(\lambda)\hat{\overset{y}{c}}_{12}(\lambda) + u(\lambda)\hat{\overset{y}{q}}_{12}(\lambda)}{u^2(\lambda) + v^2(\lambda)}, \end{aligned} \tag{A.13}$$

where the hats denote estimated values.

The frequency response function for the pth order quasi-difference filter with parameter γ has been determined in [12, footnote 33]; it is

$$\begin{aligned} l(\lambda) &= [1 - \gamma e^{i\lambda}]^p \\ &= \sum_{r=0}^{p} \binom{p}{r} (-\gamma)^{p-r} [e^{i\lambda}]^{p-r} \\ &= A(\lambda) + iB(\lambda), \end{aligned} \tag{A.14}$$

where

$$\begin{aligned} A(\lambda) &= \sum_{r=0}^{p} \binom{p}{r} (-\gamma)^{p-r} \cos \lambda(p - r) \\ B(\lambda) &= \sum_{r=0}^{p} \binom{p}{r} (-\gamma)^{p-r} \sin \lambda(p - r). \end{aligned} \tag{A.15}$$

[Formula (3.11) of the text follows at once from (A.15) if $A_k = A(\lambda_k)$ and $B_k = B(\lambda_k)$, where $\lambda_k = 2\pi k/12$.] The transfer function of this filter is easily seen to be

$$\begin{aligned} | l(\lambda) |^2 = l(\lambda)\overline{l(\lambda)} &= \{[1 - \gamma e^{i\lambda}][1 - \gamma e^{-i\lambda}]\}^p \\ &= [1 - 2\gamma \cos \lambda + \gamma^2]^p. \end{aligned} \tag{A.16}$$

If two identical quasi-difference filters are used on the two series, we have

$$\begin{cases} u(\lambda) = |\, l(\lambda)\,|^2 = [1 - 2\gamma\cos\lambda + \gamma^2]^p \\ v(\lambda) = 0 \end{cases} \tag{A.17}$$

in (A.11), (A.12), and (A.13).

The problem of recoloring in practice becomes somewhat more complicated when Parzen's autoregressive prewhitening is used. To derive the corresponding results for general autoregressive filters, we proceed as follows: Suppose $x_1(t)$ and $x_2(t)$ have been transformed according to

$$\left.\begin{aligned} y_1(t) &= \sum_{k=0}^{n_1} a_k x_1(t-k) \\ y_2(t) &= \sum_{k=0}^{n_2} b_k x_2(t-k) \end{aligned}\right\}. \tag{A.18}$$

Without loss of generality, let $m=\max\{n_1, n_2\}$, and let $a_k=0$ for $k=n_1+1, \cdots, m$, or $b_k=0$ for $k=n_2+1, \cdots, m$. Then

$$\left.\begin{aligned} l_1(\lambda) &= \sum_{k=0}^{m} a_k e^{-i\lambda k} \\ l_2(\lambda) &= \sum_{k=0}^{m} b_k e^{-i\lambda k} \end{aligned}\right\}. \tag{A.19}$$

Thus,

$$\begin{aligned} l_1(\lambda)\overline{l_2(\lambda)} &= \left(\sum_{j=0}^{m} a_j e^{-i\lambda j}\right)\left(\sum_{k=0}^{m} b_k e^{i\lambda k}\right) \\ &= \sum_{j,k=0}^{m} a_j b_k e^{-i\lambda(j-k)} \\ &= \sum_{j,k=0}^{m} a_j b_k \cos\lambda(j-k) \\ &\quad - i \sum_{j,k=0}^{m} a_j b_k \sin\lambda(j-k) \end{aligned} \tag{A.20}$$

Hence,

$$\left.\begin{aligned} u(\lambda) &= \sum_{j,k=0}^{m} a_j b_k \cos\lambda(j-k) \\ v(\lambda) &= -\sum_{j,k=0}^{m} a_j b_k \sin\lambda(j-k) \end{aligned}\right\} \tag{A.21}$$

in (A.13). The numbers $u(\lambda)$ and $v(\lambda)$ may be computed for any frequency and given autoregressive transformation. The results may then be used to recolor the cross-spectrum of two series according to (A.13).

As a final item in this Appendix we consider the problem of recovering seasonal factors for a trend-contaminated series when the coefficients of the trigonometric representation are themselves trending linearly with time. In this formulation, the seasonal factors may be written

$$\begin{aligned} s(t) &= \sum_{k=1}^{6} \{[\alpha_{1k} + \alpha_{2k}t] \cos \lambda_k t + [\beta_{1k} + \beta_{2k}t] \sin \lambda_k t\} \\ &= \sum_{k=1}^{6} [\alpha_{1k} \cos \lambda_k t + \beta_{1k} \sin \lambda_k t] + t \sum_{k=1}^{6} [\alpha_{2k} \cos \lambda_k t + \beta_{2k} \sin \lambda_k t] \end{aligned} \tag{A.22}$$

where, as before,

$$\lambda_k = \frac{2\pi k}{12}$$

and

$$\beta_{16}, \beta_{26} \equiv 0.$$

Equation (A.22) may be written in more convenient, complex form by introducing the complex coefficients σ_{1k} and σ_{2k} defined according to

$$\left.\begin{aligned} \sigma_{1k} &= \tfrac{1}{2}[\alpha_{1k} + i\beta_{1k}] \\ \sigma_{1,-k} &= \bar{\sigma}_{1k} \\ \sigma_{2k} &= \tfrac{1}{2}[\alpha_{2k} + i\beta_{2k}] \\ \sigma_{2,-k} &= \bar{\sigma}_{2k} \end{aligned}\right\}. \tag{A.23}$$

Note that

$$\lambda_{-k} = \frac{2\pi(-k)}{12} = -\lambda_k.$$

Then, denoting a summation which omits the term $k=0$ by $\sum'$, we may rewrite (A.22) as

$$s(t) = \sum_{-6}^{6}{}' \sigma_{1k} e^{-i\lambda_k t} + t \sum_{-6}^{6}{}' \sigma_{2k} e^{-i\lambda_k t}. \tag{A.24}$$

Determination of the effects of a general linear filter in this scheme has been worked out by Hannan [9, pp. 13–14]. However, derivation of Hannan's general result is extremely tedious. Here we shall give only the simpler derivation for the formula for the pth order quasi-difference with parameter γ. To do so we introduce the lag operator u such that $u^k x(t) = x(t-k)$; then the effect of the pth order quasi-difference on $s(t)$ is given by

$$(1 - \gamma u)^p s(t) = \sum_{k=-6}^{6}{}' \sigma_{1k}(1 - \gamma u)^p e^{-i\lambda_k t} + \sum_{k=-6}^{6}{}' \sigma_{2k}(1 - \gamma u)^p t e^{-i\lambda_k t}. \tag{A.25}$$

We may expand $(1-\gamma u)^p$ according to the binomial theorem to obtain

$$(1 - \gamma u)^p = \sum_{r=0}^{p} \binom{p}{r} (-\gamma)^{p-r} u^{p-r}. \tag{A.26}$$

Inserting this result in (A.25):

$$
\begin{aligned}
(1-\gamma u)^p s(t) &= \sum_{k=-6}^{6}{}' \sigma_{1k}\left\{\sum_{r=0}^{p}\binom{p}{r}(-\gamma)^{p-r}e^{i\lambda_k(p-r)}\right\}e^{-i\lambda_k t} \\
&\quad + \sum_{k=-6}^{6}{}' \sigma_{2k}\left\{\sum_{r=0}^{p}\binom{p}{r}(-\gamma)^{p-r}e^{i\lambda_k(p-r)}[t-(p-r)]\right\}e^{-i\lambda_k t} \\
&= \sum_{k=-6}^{6}{}' \left\{\sigma_{1k}(1-\gamma e^{i\lambda_k})^p\right\}e^{-i\lambda_k t} + \sum_{k=-6}^{6}{}' \left\{\sigma_{2k}(1-\gamma e^{i\lambda_k})^p\right\}te^{-i\lambda_k t} \\
&\quad - \sum_{k=-6}^{6}{}' \left\{\sigma_{2k}\sum_{r=0}^{p}\binom{p}{r}(-\gamma)^{p-r}e^{i\lambda_k(p-r)}(p-r)\right\}e^{-i\lambda_k t} \\
&= \sum_{k=-6}^{6}{}' \left\{\sigma_{1k}(1-\gamma e^{i\lambda_k})^p - \sigma_{2k}\sum_{r=0}^{p}\binom{p}{r}(-\gamma)^{p-r}e^{i\lambda_k(p-r)}(p-r)\right\} \\
&\quad \cdot e^{-i\lambda_k t} + \sum_{k=-6}^{6}{}' \left\{\sigma_{2k}(1-\gamma e^{i\lambda_k})^p\right\}te^{-i\lambda_k t}.
\end{aligned}
\tag{A.27}
$$

Thus, if the seasonal factors appropriate to the trend contaminated series are defined as

$$
s^*(t) = \sum_{k=-6}^{6}{}' \sigma^*_{1k}e^{-i\lambda_k t} + t\sum_{k=-6}^{6}{}' \sigma^*_{2k}e^{-i\lambda_k t}, \tag{A.28}
$$

we have

$$
\left.
\begin{aligned}
\sigma^*_{1k} &= \sigma_{1k}(1-\gamma e^{i\lambda_k})^p - \sigma_{2k}\sum_{r=0}^{p}\binom{p}{r}(-\gamma)^{p-r}e^{i\lambda_k(p-r)}(p-r) \\
\sigma^*_{2k} &= \sigma_{2k}(1-\gamma e^{i\lambda_k})^p
\end{aligned}
\right\}^{23}. \tag{A.29}
$$

Using the definitions of σ_{1k}, σ_{2k}, and

$$
\left.
\begin{aligned}
\sigma^*_{1k} &\equiv \tfrac{1}{2}[\alpha^*_{1k} + i\beta^*_{1k}] \\
\sigma^*_{2k} &\equiv \tfrac{1}{2}[\alpha^*_{2k} + i\beta^*_{2k}]
\end{aligned}
\right\}, \tag{A.30}
$$

we find, after some manipulation, that (A.30) implies

$$
\left.
\begin{aligned}
\alpha_{2k} &= \frac{\alpha^*_{2k}A_{2k} + \beta^*_{2k}B_{2k}}{[1-2\gamma\cos\lambda_k+\gamma^2]^p} \\
\beta_{2k} &= \frac{-\alpha^*_{2k}B_{2k} + \beta^*_{2k}A_{2k}}{[1-2\gamma\cos\lambda_k+\gamma^2]^p}
\end{aligned}
\right\} \tag{A.31}
$$

---

[23] One of the referees pointed out that some further simplifications in (A.27) and (A.29) are obtainable by making use of the fact that

$$
\sum_{r=0}^{p}\binom{p}{r}(-\gamma)^{p-r}e^{i\lambda_k(p-r)}(p-r) = -p\gamma e^{i\lambda_k}(1-\gamma e^{i\lambda_k})^{p-1}.
$$

where

$$A_{2k} = \sum_{r=0}^{p} \binom{p}{r} (-\gamma)^{p-r} \cos \lambda_k (p - r)$$

$$B_{2k} = \sum_{r=0}^{p} \binom{p}{r} (-\gamma)^{p-r} \sin \lambda_k (p - r),$$

and

$$\left.\begin{aligned} \alpha_{1k} &= \frac{[\overset{*}{\alpha}_{1k} + \alpha_{2k} A_{1k} - \beta_{2k} B_{1k}] A_{2k} + [\overset{*}{\beta}_{1k} + \alpha_{2k} B_{1k} + \beta_{2k} A_{1k}] B_{2k}}{[1 - 2\gamma \cos \lambda_k + \gamma^2]^p} \\ \beta_{1k} &= \frac{-[\overset{*}{\alpha}_{1k} + \alpha_{2k} A_{1k} - \beta_{2k} B_{1k}] B_{2k} + [\overset{*}{\beta}_{1k} + \alpha_{2k} B_{1k} + \beta_{2k} A_{1k}] A_{2k}}{[1 - 2\gamma \cos \lambda_k + \gamma^2]^p} \end{aligned}\right\} \quad (A.32)$$

TABLE A.1

SEASONAL COEFFICIENTS FOR DIFFERENCED SERIES

I	ALPHA 1(I)	BETA 1(I)	VARIANCE	STD. ERROR
0	−10.09616	0.	2.90136	1.70334
1	1.39688	1.92998	0.74983	0.86592
2	10.77231	0.38686	2.92283	1.70963
3	−5.59730	8.69477	4.72428	2.17354
4	8.21701	3.29561	4.03622	2.00903
5	−12.09962	−14.79886	4.52325	2.12679
6	−5.70284	0.	7.63530	2.76321

I	ALPHA 2(I)	BETA 2(I)	VARIANCE	STD. ERROR
0	0.08787	0.	0.00006	0.00754
1	0.05763	−0.00044	0.00001	0.00383
2	0.08270	−0.04166	0.00006	0.00756
3	−0.04732	0.04532	0.00009	0.00962
4	0.02567	−0.01849	0.00008	0.00889
5	−0.07902	0.13525	0.00009	0.00941
6	0.02800	0.	0.00015	0.01223

COVARIANCES OF COEFFICIENTS

0	0.01816
1	0.00469
2	0.01829
3	0.02956
4	0.02526
5	0.02830
6	0.04778

TABLE A.2

SEASONAL COEFFICIENTS FOR UNDIFFERENCED SERIES

I	A1(I)	B1(I)	VARIANCE	STD. ERROR
1	−7.03648	4.68838	10.80251	3.28672
2	−0.79810	13.20229	4.42748	2.10416
3	−6.39137	−1.86109	1.93506	1.39107
4	1.16874	3.64513	0.75476	0.86877
5	−1.60726	−6.45354	0.55240	0.74323
6	−1.85432	−0.00000	0.81409	0.90227
I	**A2(I)**	**B2(I)**	**VARIANCE**	**STD. ERROR**
1	−0.01311	0.21836	0.00021	0.01454
2	0.04733	0.10369	0.00009	0.00931
3	−0.03632	−0.02095	0.00004	0.00616
4	0.01323	0.00350	0.00001	0.00384
5	−0.04533	0.03068	0.00001	0.00329
6	0.00914	0.00000	0.00002	0.00399

COVARIANCES OF COEFFICIENTS

1	0.06760
2	0.02771
3	0.01211
4	0.00472
5	0.00346
6	0.00509

where α_{2k} and β_{2k} are given by equations (A.31), and where

$$A_{1k} = \sum_{r=0}^{p} \binom{p}{r} (-\gamma)^{p-r} [p - r] \cos \lambda_k (p - r)$$

$$B_{1k} = \sum_{r=0}^{p} \binom{p}{r} (-\gamma)^{p-r} [p - r] \sin \lambda_k (p - r).$$

As an example of seasonals computed from a trigonometric representation with time trending coefficients, the following coefficients were estimated for total U. S. unemployment after filtering by second-order quasi-differences with $\gamma = 0.75$. Table A.1 gives the coefficients and their asymptotic variances and covariances as estimated for the filtered series.

The notations ALPHA 1(I), ALPHA 2(I), etc., should be obvious. The zero order coefficients are constant terms of linear trend. The asymptotic variances of corresponding alpha and beta coefficients are equal and may be determined from the spectrum of the residuals from the regression by a formula given by Hannan [9, p. 15]. The covariances between coefficients are zero except between the constant and the time-trending part of each coefficient, then they are equal for corresponding alphas and betas. These covariances are given at the bottom of the table.

TABLE A.3

SEASONAL FACTORS FOR UNDIFFERENCED SERIES

JAN	FEB	MAR	APR	MAY	JUNE	JULY	AUG	SEPT	OCT	NOV	DEC	YEAR
8.25	18.43	4.60	−2.55	−11.81	14.54	22.69	− 6.31	0.91	−20.53	−17.44	−16.82	1943
11.03	21.25	7.32	−1.98	−11.92	16.51	22.32	− 7.56	− 2.85	−23.69	−19.37	−17.12	1944
13.82	24.07	10.04	−1.41	−12.03	18.49	21.95	− 8.81	− 6.61	−26.85	−21.30	−17.42	1945
16.61	26.90	12.76	−0.84	−12.14	20.46	21.58	−10.06	−10.37	−30.01	−23.22	−17.72	1946
19.39	29.72	15.48	−0.27	−12.25	22.43	21.22	−11.31	−14.13	−33.17	−25.15	−18.02	1947
22.18	32.54	18.20	0.31	−12.36	24.41	20.85	−12.55	−17.89	−36.33	−27.08	−18.32	1948
24.97	35.37	20.92	0.88	−12.47	26.38	20.48	−13.80	−21.65	−39.49	−29.01	−18.62	1949
27.75	38.19	23.64	1.45	−12.59	28.35	20.11	−15.05	−25.41	−42.65	−30.93	−18.93	1950
30.54	41.01	26.36	2.02	−12.70	30.33	19.74	−16.30	−29.16	−45.81	−32.86	−19.23	1951
33.33	43.84	29.09	2.59	−12.81	32.30	19.37	−17.55	−32.92	−48.97	−34.79	−19.53	1952
36.11	46.66	31.81	3.16	−12.92	34.27	19.00	−18.79	−36.68	−52.13	−36.71	−19.83	1953
38.90	49.48	34.53	3.73	−13.03	36.25	18.63	−20.04	−40.44	−55.29	−38.64	−20.13	1954
41.69	52.31	37.25	4.30	−13.14	38.22	18.26	−21.29	−44.20	−58.45	−40.57	−20.43	1955
44.47	55.13	39.97	4.87	−13.25	40.19	17.89	−22.54	−47.96	−61.61	−42.49	−20.73	1956
47.26	57.95	42.69	5.44	−13.36	42.17	17.53	−23.79	−51.72	−64.77	−44.42	−21.03	1957
50.04	60.77	45.41	6.01	−13.47	44.14	17.16	−25.03	−55.48	−67.93	−46.35	−21.33	1958
52.83	63.60	48.13	6.58	−13.59	46.11	16.79	−26.28	−59.23	−71.09	−48.27	−21.63	1959
55.62	66.42	50.85	7.15	−13.70	48.09	16.42	−27.53	−62.99	−74.25	−50.20	−21.93	1960
58.40	69.24	53.57	7.72	−13.81	50.06	16.05	−28.78	−66.75	−77.41	−52.13	−22.23	1961

TABLE A.4

THE SEASONALLY ADJUSTED SERIES

JAN	FEB	MAR	APR	MAY	JUNE	JULY	AUG	SEPT	OCT	NOV	DEC	YEAR
139.8	123.6	107.4	103.5	106.8	115.5	116.3	111.3	86.1	98.5	88.4	85.8	1943
70.0	47.8	61.7	65.0	84.9	71.5	66.7	75.6	62.9	67.7	69.4	67.1	1944
49.2	39.9	49.0	54.4	65.0	70.5	73.0	91.8	171.6	182.8	195.3	214.4	1945
213.4	238.1	257.2	233.8	243.1	236.5	205.4	216.1	217.4	226.0	216.2	229.7	1946
232.6	230.3	238.5	268.3	232.3	264.6	266.8	252.3	229.1	214.2	203.2	201.0	1947
214.8	252.5	249.8	243.7	215.4	232.6	234.2	240.6	225.9	213.3	234.1	239.3	1948
278.0	313.6	319.1	328.1	369.5	382.6	427.5	416.8	386.6	421.5	391.0	390.6	1949
442.2	444.8	410.4	369.6	343.6	345.6	327.9	282.0	279.4	250.6	268.9	262.9	1950
237.5	216.0	208.6	194.0	193.7	197.7	196.3	200.3	220.2	223.8	234.9	202.2	1951
194.7	192.2	174.9	184.4	196.8	183.7	200.6	214.5	209.9	199.0	196.8	175.5	1952
180.9	151.3	153.2	175.8	169.9	148.7	158.0	172.8	199.7	210.1	241.7	287.8	1953
320.1	349.5	370.5	383.3	382.0	343.8	359.4	374.0	386.4	351.3	353.6	324.1	1954
328.3	305.7	295.8	314.7	284.1	263.8	259.7	282.3	280.2	290.4	299.6	286.4	1955
264.5	258.9	273.0	271.1	303.3	299.8	295.1	275.5	278.0	274.6	307.5	292.7	1956
276.7	255.0	245.3	263.6	285.4	291.8	283.5	284.8	306.7	315.8	263.4	358.0	1957
399.0	456.2	474.6	506.0	503.5	499.9	511.8	495.0	466.5	448.9	429.3	432.3	1958
419.2	411.4	387.9	356.4	352.6	351.9	357.2	369.3	382.2	398.1	415.3	379.6	1959
359.4	326.6	370.1	358.9	359.7	393.9	385.6	406.5	402.0	432.2	453.2	475.0	1960
480.6	509.8	496.4	488.3	490.8	507.9	497.9	482.8	475.8	470.4	451.1	431.2	1961

Table A.2 gives the coefficients appropriate for the trend-contaminated series as recovered from the coefficients given in Table A.1 by formulae (A.31) and (A.32). These are denoted by A1(I), B1(I), etc., corresponding to the previous notation. Since the A's and B's are known linear combinations of the alphas and betas, and since we have an estimate of the asymptotic variance-covariance matrix of the alphas and betas, it is a simple matter to determine the asymptotic variance-covariance matrix of the A's and B's. Furthermore, it is easily seen that it must have the same properties as before.

Table A.3 shows the seasonal factors computed for each month since January, 1943, according to equation (A.22) from the coefficients appearing in Table A.2. Finally, Table A.4 gives the seasonally adjusted series. Note that the seasonal adjustment has been assumed to be additive throughout in the actual values of unemployment.

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THE ESTIMATION OF SEASONAL VARIATION IN ECONOMIC TIME SERIES

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The problem of estimating the seasonal component in an economic time series is discussed and it is pointed out that the effects of any moving average operator on the seasonal component may be easily reversed so that one may use any suitable operator to remove the trend. The computational procedure is to estimate the seasonal index for the trend free series and to convert this index into a seasonal index for the original series by taking 12 term moving averages of this series (continued periodically) with weights depending on the operator used to remove trend. Weights for some commonly used operators are tabulated. The problem of estimating a slowly evolving seasonal is considered.

1. INTRODUCTION

WE CONSIDER here the problem of estimating the seasonal component of an economic time series. Any reasonable model will have to allow for a trend component in addition to the seasonal component itself. The remainder over and above these two components will usually be serially correlated. Our basic model is therefore of the form

$$y_t = p_t + s_t + x_t$$

where p_t is the trend component, s_t is the seasonal component and x_t is the serially correlated residual. We may assume that x_t has zero mean, since a non zero mean may be incorporated in p_t. We also assume that x_t is stationary, i.e. that

$$\mathcal{E}(x_s x_{s+t}) = \gamma_t$$

which depends on t but not on s [3]. This amounts to saying that the covariance structure of the x_t sequence does not change with time. We assume that the observations are taken monthly but it should be easy to see how to modify the results for other cases.

Some comments are called for in relation to this model.

(1) The assumption of an additive model does not accord with experience with economic data for which a model multiplicative in the various components is more realistic. Such a model can be reduced to the additive form by taking logarithms, provided all observations are positive (which is the standard situation). It is the writer's opinion that this should be done. The labour involved is not great and the log transformation has other virtues also (in reducing the importance of very discrepant observations, for example).

(2) The "trend plus seasonal plus random" model seems to repel many people who have been concerned with the use of modern spectral methods of time series analysis. They point out that, from a series of finite length, what we have called a trend is indistinguishable from what they would prefer to call a "low frequency component," which they would include in x_t. The methods to

be used below can be looked at from this point of view. However, the writer's point of view (the more conventional one) seems preferable (a) because it is known that economies are evolving so that an evolutive (non stationary) component has to be included somewhere, (b) because the assumption that p_t is really only a substantial low frequency component implies that as the number of observations becomes greater its influence can be neglected and that the seasonal can be estimated efficiently ignoring it. Experience suggests that this is not so and that the passage of time leads rather to an increasingly complex "trend" which can never be ignored. Of course the passage of time also shows that the seasonal is evolving so that if periods over which the seasonal component is unchanging are used the number of observations is never really large.

(3) The assumption of stationarity for x_t is, of course, the assumption of a fiction. Nevertheless the observed x_t components (especially after a log transformation) *look* stationary in most cases, apart perhaps from occasional excursions by the series, for which the reason is usually known (e.g. a strike). Spectral methods (which will be used below) appear to be robust against reasonable departures from stationarity.

(4) Through most of what is said below the seasonal component will be assumed to be unchanging (though some reference to a changing seasonal will be made in section 6). It can then be represented in the form

$$s_t = \sum_{j=1}^{12} a_j s_{j,t} \tag{1}$$

where $s_{j,t}$ is unity for $t-j$ divisible by 12 and is zero otherwise. Thus a_j is the additive seasonal component for the jth month of the year and if logarithms of the original data have been taken then antilog a_j will be the seasonal factor by which the figure for the jth month of the year must be divided to give the seasonally corrected series. Of course the a_j are unknown and have to be estimated.

We may assume that

$$\sum_{1}^{12} a_j = 0 \tag{2}$$

since we may achieve this, if it is not so, by subtracting a constant from s_t and adding it to p_t.

Though the formula (1) is most easily understood, and is the relevant one from the point of view of the application of the end results of the estimation procedure, a more relevant formula from the point of view of this estimation procedure is the equivalent formula

$$s_t = \sum_{k=1}^{6} (\alpha_k \cos \lambda_k t + \beta_k \sin \lambda_k t), \qquad \lambda_k = \frac{2\pi k}{12}. \tag{3}$$

In this formula $\beta_6 \sin \lambda_6 t$ is identically zero and has been included only because its omission makes the notation more complex. The α_k and β_k are related to the a_j by

$$\left.\begin{aligned}\alpha_k &= \frac{1}{6}\sum_{j=1}^{12} a_j \cos j\lambda_k \\ \beta_k &= \frac{1}{6}\sum_{j=1}^{12} a_j \sin j\lambda_k\end{aligned}\right\} \quad k \neq 6$$
$$\alpha_6 = \frac{1}{12}\sum_{j=1}^{12} a_j \cos j\lambda_6. \tag{4}$$

The methods to be discussed below have been presented earlier, [4], in somewhat the same form, though here some extensions and modifications will be made. Reference must also be made to two important papers by Durbin ([1], [2]), along the same lines.

2. THE USE OF ITERATIVE METHODS

We shall not discuss here the use of regression methods to estimate s_t and p_t. The problem we are considering is the estimation of s_t and it is the *elimination* of p_t which we are faced with (so as to avoid its interference with the estimation of s_t). As will be seen from what will be said below almost any operator which removes p_t will be acceptable for the estimation of s_t (though there are some second order effects involved) so that efficiency in the *estimation* of p_t is not an issue. In practice the methods used always involve the use of moving average operators to eliminate p_t and we shall discuss only these. The standard operators can be written in the form

$$A = \sum_{-p}^{q} \delta_s U_s$$

where U is the operator which moves t forward one step: $Uy_t = y_{t+1}$. The most commonly used formula, "a centred moving average," has $p=q=6$ and $\delta_j = 1/12$, except for $j = \pm 6$ when these are 1/24. This operator (A_0 say) has the property that $A_0 s_t \equiv 0$. Unfortunately, as everyone who has used these methods knows [6, p. 149], A_0 tends to give a series, $A_0 y_t$, which cuts across the waves in the trend component (which means that $A_0 p_t \neq p_t$) so that after $A_0 y_t$ is subtracted from y_t to form $(I - A_0)y_t$, some of the trend is left in with the seasonal. (Here I is the identity operator, for which $Iy_t = y_t$.) There is then some bias in the seasonal estimate. Other operators which might be used follow the trend more closely. Some examples follow, in the notation of Kendall [5, Ch. 29], which will be explained after the formulae are given.

$$A_1\text{(Spencer's 15 pt. formula)} = \frac{1}{320}[4]^2[5][-3, 3, 4, 3, -3]$$

$$A_2\text{(Spencer's 21 pt. formula)} = \frac{1}{350}[5]^2[7][-1, 0, 1, 2, 1, 0, -1]$$

$$(I - A_3)^r (r\text{th differences}) = \Delta^r y_t$$

$$A_4\text{(Leong [6, p. 152])} = \frac{1}{36}[6]^2.$$

In each case the formula involves a moving average of the type of A described above, with $p=q$ and the result of this moving average is always written opposite the middle term of the sequence averaged. Thus by $A_1y_{t_0}$ we mean the average of the 15 y_t values for t going from t_0-7 to t_0+7 (inclusive). The operator A_j is in each case compounded of simpler operators which are applied successively *and in any order*. The operation $[k]$ is that of taking sums of successive sets of k of the y_t. In carrying out the computations the end result of this operation should be written opposite the centre term of those summed. If k is even this will involve interlining but this can be avoided by a simple convention since summations involving an even number of terms always occur an even number of times in the composite formula. The operation Δ is that of taking first differences. Similar remarks to that for $[k]$ when k is even, apply here. The remaining operation is defined by

$$[\delta_{-p}, \delta_{-p+1}, \cdots, \delta_0, \cdots, \delta_p] = \sum_{-p}^{p} \delta_s U^s$$

so that we are merely using an alternative notation to that used for A above.

All of the operators described above, save A_3, are designed to reproduce the trend so that the extraction of the trend from the data is attained by $y_t - A_jy_t$ which, as before, we write as $(I-A_j)y_t$. It is this series which is further processed to estimate the seasonal. As the notation used indicates the formula for $(I-A_3)^r$ is already in the form appropriate to the removal of the trend.

All of these formulae affect the seasonal to some extent so that $(I-A_j)s_t \neq s_t$ for $j\neq 0$.[1] The point in their favour is, as already mentioned, that it is much nearer true for them that $(I-A_j)p_t\equiv 0$. While A_0 satisfies $(I-A_0)s_t=s_t$ the deviation of $(I-A_0)p_t$ from zero may be much larger than for other A_j. The point in favour of the other A_j is the crucial one however for, as will be explained, the effect of A_j on s_t is known and may be allowed for. On the other hand it may be essential that p_t be removed as the use of techniques for estimating the a_j, based on the assumption that it has been removed, may lead to large biases if this is not so. Before going on to describe our procedures we will first discuss the effect of the operator on p_t in relation to certain iterative techniques which have been recommended.

The use of these iterative techniques has arisen from the recognition that such operators as those listed above do affect the seasonal. They have been developed in particular by Shiskin and Eisenpress [7]. In formally describing these techniques[2] we shall write Ay, As, Ap, Ax for the vector (column of num-

[1] It appears to be claimed by Leong [6, p. 154] that the operator $I-A_4$ will not affect the seasonal component but this is not so. Indeed as we shall show below it reduces the amplitude of the (first harmonic) sinusoidal component of the seasonal, having a fundamental period of 12 months, by over 40 per cent. Other components are either not affected at all or not affected greatly (see Table 1). The constructed series used by Leong is misleading in this regard [6, p. 155, footnote] since only 3 per cent of the corrected sum of squares of the constructed seasonal index is due to this first harmonic. This does not seem to be a standard situation and one would normally expect the first harmonic to explain more of the seasonal variation than any other component. (An example where over 80 per cent is due to this component is given by Hannan [4, p. 6].)

[2] The description of this procedure [4, pp. 3, 4] has been found difficult by some readers partly because of errors in the printing which were not corrected by the writer at the proof stage. The notation used here differs from that previously used. The reader should also note that the procedure developed by Eisenpress and Shiskin differs from those considered here in that the former combines trend removal by operators linear in the data with seasonal adjustments linear in the logarithms.

ters) resulting from the application of the operator A to the vector of observations y_t, s_t, p_t, x_t, respectively. We assume that there are other observations at the ends of the series to be moved in so that the moving average operators are well defined on these vectors and that the dimension of the space on which A operates (i.e. the number of entries down the column) is a multiple of 12, say $12m$. This amounts to saying that for A_1 (say) we shall use 14 more observations than we finish up with, after the use of this operator, and that we shall use just so many observations that we shall finish with a multiple of twelve after applying A_1. Strictly this need not be done but some formulae to be used below will not hold exactly unless this rule is observed.

The technique is as follows. First $(I-A_0)y$ is formed. In this the influence of p should be much reduced, and a first estimate of the seasonal may be made by the simple and obvious device of calculating the mean for each month and then adjusting these twelve figures to add to zero by subtracting their mean. We may represent the vector of these 12 seasonal indices, repeated m times down the column, by

$$E(I - A_0)y.$$

It is fairly clear that $Es=s$ and it follows easily from this that $E^2=E$, and that $EAE=AE$.[3] Now

$$y - E(I - A_0)y = \{I - E(I - A_0)\}y$$

is formed, which should be relatively free of s. Thus the p component can be estimated from this, by some A more suited to this purpose than A_0, without the danger of the resulting estimate being contaminated so badly by s. Thus

$$A\{I - E(I - A_0)\}y$$

is formed. This is subtracted from y to get a trend free series and the seasonal is re-estimated by the same method as before, i.e. by the use of E. Thus the column of seasonal values is estimated as

$$E[I - A\{I - E(I - A_0)\}]y.$$

If we take expectations and use the rules concerning E set down above this becomes

$$E[I - A\{I - E(I - A_0)\}](s + p) = s + E(I - A)p + AE(I - A_0)p.$$

Let us assume that $Ap=p$ so that the use of $(I-A)$ completely eliminates the trend. Then the last expression becomes

$$s + AE(I - A_0)p.$$

The last term is the bias involved in the procedure. It will be shown below that under the assumptions used here (and in particular the assumption that $Ap=p$) a computationally simpler method is available which involves no bias.[4]

[3] E applied to any vector of $12m$ members produces a new column which is periodic with period 12. Since E leaves any such column unchanged $E^2=E$. A will change such a column into a new column (in general) but one which is still periodic. Thus $EAE=AE$.

[4] All of the methods discussed give the same asymptotic variance for the estimates so that the consideration of bias alone gives a valid criterion.

In practice, of course, the model will not be that set down here so that these results have only suggestive value. This is characteristic of methods of applied mathematics. It is believed that the results derived have a very real relevance to the problem considered.

The bias term in the last expression will be a column of $12m$ numbers which form a periodic sequence with period 12 months. If A were not applied the bias would be $E(I-A_0)p$ which is again periodic of course. It will be shown below that the imposition of A will reduce the amplitude of the oscillation formed by this sequence of $12m$ numbers. This suggests that the sequence of operations involving A be iterated, say r times. The sequence of operations to be iterated is that following after the formation of $\{I-E(I-A_0)\}y$ and the commencing vector for each iteration would be the seasonally corrected series using the end results from the previous iteration to make the corrections. Then the bias term will become

$$A^rE(I-A_0)p. \tag{5}$$

3. THE EFFECT OF A_j ON THE SEASONAL COMPONENT

We consider s_t in the form (3). Then as shown by Hannan [4]

$$(I-A)s_t = \sum_{k=1}^{6}\{\alpha_k' \cos\lambda_k t + \beta_k' \sin\lambda_k t\}$$

where

$$\alpha_k' = \alpha_k\left\{1-\sum_{-p}^{q}\delta_s\cos s\lambda_k\right\} - \beta_k\sum_{-p}^{q}\delta_s\sin s\lambda_k$$

$$\beta_k' = \alpha_k\sum_{-p}^{q}\delta_s\sin s\lambda_k + \beta_k\left\{1-\sum_{-p}^{q}\delta_s\cos s\lambda_k\right\}.$$

If $p=q$ and $\delta_s=\delta_{-s}$ (the symmetric case) then these reduce to

$$\alpha_k' = \{1-h(\lambda_k)\}\alpha_k, \qquad \beta_k' = \{1-h(\lambda_k)\}\beta_k,$$

$$h(\lambda) = \sum_{-p}^{p}\delta_s\cos s\lambda.$$

The $h(\lambda_k)$ for A of the form of some of the A_j described above are shown in Table 1.

TABLE 1. VALUES OF $h(\lambda_k)$

	1	2	3	4	5	6
A_1 = Spencer's 15 pt. formula	.809	.094	.000	−.013	−.003	.000
A_2 = Spencer's 21 pt. formula	.554	−.014	−.006	−.003	.000	.000
A_4 = Leong	.415	.000	.056	.000	.030	.000

Thus the effect of the operator upon s_t is known and can be accounted for. There is thus no need to use iterative methods to prevent a sufficiently flexible operator (from the point of view of trend removal) from affecting s_t. This avoidance of iterative methods also avoids the addition of the bias, (5), to the seasonal of course. Explicit formulae for the estimates, $\hat{\alpha}_j$ and $\hat{\beta}_j$, are given below though, as we presently show, the use of the form (3) may be avoided and, equivalent, computationally simpler methods used. The formulae are:

Put

$$y_t' = (I - A)y_t$$

$$u_j' = \frac{1}{m}\sum_{t=1}^{m} y_{12t+j}' \qquad j = 1, \cdots, 12.$$

Then

$$\left.\begin{aligned} \hat{\alpha}_k &= \frac{\sum u_j' \cos j\lambda_k}{6\{1 - h(\lambda_k)\}}, \qquad k \neq 6,\ \hat{\alpha}_6 = \frac{\sum u_j' \cos j\lambda_6}{12\{1 - h(\lambda_6)\}} \\ \hat{\beta}_k &= \frac{\sum u_j' \sin j\lambda_k}{6\{1 - h(\lambda_k)\}}, \qquad k = 1, \cdots, 5 \end{aligned}\right\}. \tag{6}$$

Formulae have been given only for the symmetric case.

These formulae follow from formulae (4) given above wherein the α_k, β_k, and a_j are to be primed since $(I-A)$ has been applied. Since u_j' estimates a_j' we recover the estimate of α_k' by the use of formulae (4) using u_j' in place of a_j'. We then obtain $\hat{\alpha}_k$ and $\hat{\beta}_k$ from the estimates of the corresponding primed quantities by dividing by the known factors $\{1-h(\lambda_k)\}$.

The asymptotic variance of these estimates may be simply written down in terms of the spectral density $f(\lambda_k)$ of the x_t sequence [3, p. 13]. The variance is in fact

$$\text{var}(\hat{\alpha}_k) = \text{var}(\hat{\beta}_k) = \frac{4\pi}{n} f(\lambda_k) \qquad k \neq 6$$

$$\text{var}(\hat{\alpha}_6) = \frac{2\pi}{n} f(\lambda_6)$$

here $n = 12m$.

The spectral density of the sequence $x_t' = (I-A)x_t$ is $\{1-h(\lambda)\}^2 f(\lambda)$ and this (and hence $f(\lambda_k)$ itself) may be estimated [3, Ch. III], [4, p. 9] from the sequence obtained by seasonally correcting y_t' by means of the u_j'. The different $\hat{\alpha}_k$ and $\hat{\beta}_k$ will be nearly uncorrelated. These asymptotic formulae will be good approximations even for m quite small. It may be shown that they are the asymptotic variances of the best linear unbiased estimates obtainable when p is not present. The simple covariance properties of these estimates $\hat{\alpha}_k$, $\hat{\beta}_k$, give them some importance, even though as we have already said, they may be replaced by computationally simpler formulae. They are also of value in showing the influence of A on the seasonal component. For example, for A_2 we have

$$\{1 - h(\lambda)\} = 1 - \left(\frac{\sin 3\lambda}{6 \sin \frac{1}{2}\lambda}\right)^2$$

which for $\lambda = \lambda_1$ is 0.585, so that this component is reduced by over 40 per cent as mentioned above. For other λ_j it is unity or near to unity. Unless the effects of A_2 is inverted as in (6) above a very bad estimate of the seasonal will be obtained for any case where the first harmonic component of the seasonal is not relatively small.

Finally it is also possible to investigate formula (5) using Table 1. $E(I-A_0)p$ is a column of components which is periodic with period 12. It is the bias effect which will result if a simple, centred, 12 months moving average is used and which the iterative technique is designed to reduce or eliminate. It can fairly certainly be said that most of this oscillation will be constituted by the component with frequency $\pi/6$, i.e. with fundamental period 12 months. This is the same as saying that the seasonal component most strongly correlated with the trend is this one with a 12 months fundamental period. The components with shorter periods will oscillate so quickly as to have a much lower correlation with the slowly varying trend. For $r=1$ and $A=A_1$ (Shiskin and Eisenpress' technique) we see that $h(\lambda_1)=.809$. This is the factor by which the amplitude of the component with period 12 months will be multiplied in forming $AE(I-A_0)p$ from $E(I-A_0)p$. Insofar as this component constitutes most of the latter, as is likely to be the case, the iterative technique will have reduced the bias by only 19 per cent. If r iterations are carried out the reduction will rise to $\{1-(.809)^r\}$ which does not fall below .5 until r is 4.

4. A SIMPLE COMPUTATIONAL PROCEDURE

In fact it is not necessary to use the formulae (3) and (6). We describe a preferable computational procedure here. The proof of the validity of the formulae we give in an appendix. The procedure is a simple one.

(a) Form the u_j' and adjust these to add to zero by subtracting their mean. Call the mean corrected set u_j. The u_j, to repeat, are just the monthly means for the trend reduced series adjusted to add to zero, the trend reduction having been obtained by forming $(I-A)y_t$.

(b) For each operator A a sequence of 12 constants v_k can be determined once and for all. Formulae for certain A are given in Table 2 below and we shall also presently quote a formula which enables them to be computed for any A. The additive seasonal adjustments $\hat{a}_j$ for the 12 months then are estimated by taking a moving average, with weights v_k, of the u_k, continued periodically

$$\hat{a}_j = \sum_{k=1}^{12} u_k v_{k-j} \tag{7}$$

where we define $v_k = v_{12+k}$ for $k \leq 0$.

A simple procedure would be to write the numbers in Table 2 down the side of a card, repeating the whole set twice. The number for $k=0$ at the beginning of the second set down the card is then brought opposite the u_j for the required month and products are formed and added. The 12 results obtained in this way constitute the additive seasonal index for the initial data.

The v_k are defined by

$$v_k = \frac{1}{12} \sum_{s=1}^{11} \frac{1}{1-h(\lambda_s)} e^{-is\lambda_k} \qquad k = 1, \cdots, 12. \tag{8}$$

This is always real since $h(\lambda_{12-s}) = h(\lambda_s)$ and $h(\lambda_6)$ is real. When A is symmetric the formula becomes

$$v_k = \frac{1}{6}\sum_{s=1}^{5}\frac{\cos s\lambda_k}{1-h(\lambda_s)} + \frac{1}{12}\,\frac{(-)^k}{1-h(\lambda_6)}\,.$$

The v_k for various A are shown in Table 2.

TABLE 2. COEFFICIENTS v_k FOR INVERTING EFFECT OF OPERATOR ON SEASONAL

	0	1	2	3	4	5	6	7	8	9	10	11
Spencer's 15 pt. formula	1.638	.539	.262	−.103	−.444	−.686	−.774	−.686	−.774	−.103	.262	.539
Spencer's 21 pt. formula	1.120	.095	.023	−.082	−.187	−.264	−.293	−.264	−.187	−.082	.023	.095
$I-\{\frac{1}{6}[6]\}^2$	1.050	.015	−.032	−.083	−.135	−.181	−.216	−.181	−.135	−.083	−.032	.015

We have not dealt with the operator A_3 in Table 2 basically because this is more easily dealt with in another way. This is the only operator, among those studied, for which $(I-A)$ is simply expressed as a product of operators. In this case it is simpler to invert the effect of the operator taking the product factor by factor. We now assume that A has been applied in the form $\Delta y_t = y_{t+1} - y_t$. To invert the effect of Δ^r use the following rule r times, at the jth stage ($j=2, \cdots, r$) beginning with the end result of the previous stage in place of the u_k named below.

Form u_{12}, $u_{12}+u_1$, $u_{12}+u_1+u_2$, $\cdots$, $u_{12}+u_1+\cdots+u_{11}$ and adjust these to add to zero by subtracting their mean.

The relevance of this method will be recognized by a comparison with the commonly employed link relatives method.

5. A NUMERICAL EXAMPLE

To illustrate the computational procedure we use data for the advances of the major Australian Trading Banks over the period August, 1958 to February, 1962 inclusive. Table 3 shows the common logarithms of this data (the units for the original data being thousands of millions of Australian pounds). The table also shows the trend estimate obtained by Spencer's 15 pt. formula.

The quantities u_j' are shown in Table 4. To the order of accuracy being used these may be also regarded as the u_j as they add nearly to zero. The a_j are also shown in this table together with their antilogarithms, which are the factors by which the original series must be divided to eliminate the seasonal fluctuation.

6. EVOLVING SEASONALS

It is a common experience that the seasonal component is not unchanging but is varying with time. Of course the variation must be slow otherwise we are hardly entitled to speak of a seasonal component at all. Techniques for dealing with a seasonal component changing linearly with time were considered by Hannan [4] which are extensions of those described in the first part of Section 2.

TABLE 3. CALCULATION OF TREND

y_t	Ay_t	y_t	Ay_t	y_t	Ay_t
0.9348		0.9616	0.9589	1.0383	1.0359
0.9330		0.9620	0.9629	1.0372	1.0346
0.9359		0.9713	0.9668	1.0335	1.0301
0.9556		0.9702	0.9701	1.0247	1.0235
0.9698		0.9683	0.9723	1.0130	1.0167
0.9757		0.9755	0.9730	1.0050	1.0113
0.9841		0.9771	0.9727	1.0072	1.0081
0.9805	0.9813	0.9741	0.9719	1.0045	1.0071
0.9772	0.9808	0.9705	0.9716	1.0086	1.0072
0.9787	0.9779	0.9686	0.9733	1.0149	1.0073
0.9763	0.9733	0.9706	0.9781	1.0063	
0.9695	0.9677	0.9880	0.9860	1.0003	
0.9640	0.9620	1.0002	0.9964	1.0031	
0.9539	0.9576	1.0007	1.0079	0.9996	
0.9472	0.9556	1.0254	1.0186	0.9964	
0.9567	0.9561	1.0278	1.0275	0.9898	
		1.0324	1.0335	0.9846	

TABLE 4. COMPUTATION OF SEASONAL INDEX

	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sept	Oct	Nov	Dec
$u_j = u_j'$	.0007	−.0040	−.0074	.0002	.0013	−.0022	.0063	−.0001	−.0029	.0019	.0033	.0025
a_j	−.0037	−.0122	−.0171	−.0082	−.0034	−.0021	.0109	.0079	.0063	.0101	.0085	.0031
anti-log a_j	.991	.972	.961	.981	.992	.995	1.025	1.019	1.015	1.024	1.020	1.007

These procedures are computationally arduous and their usefulness also seems dubious to the writer because they are founded upon a very special model. A standard technique [7, section 2] used to cope with an evolving seasonal is to graph the trend reduced values, y_t', separately for each month of the year and to smooth these twelve sets of figures to obtain an estimate of the seasonal component. If it can be assumed that the evolution of the seasonal pattern is substantially due only to a change in its amplitude from month to month, a technique due to Wald [8, p. 227] is appropriate. All of these techniques estimate only the seasonal in the series y_t' and this is not the same as the seasonal in the original series, if the seasonal is evolving, even if $y_t' = (I - A_0)y_t$. The effect for A_0 will be small if the evolution is slow as can be judged by considering the case of a seasonal component of the form $s_t = a_t b_t$ where b_t is a stable seasonal constant from year to year and a_t is itself periodic repeating itself every five years. Each of the six components of b_t, with frequencies $\pi j/6$, when combined with a_t give rise to two components (in general of unequal amplitude) with frequencies differing from $\pi j/6$ by $\pm\pi/30$. Only those for $j = 1$ are at all badly distorted by the operation of $(I - A_0)$, that at $(\pi/6 - \pi/30)$ being reduced in amplitude by 23 per cent while that at $(\pi/6 + \pi/30)$ is increased in amplitude by 15 per cent. The form of the s_t is unlikely to be so simple as this nor is the effect of A_0 likely to be so large since a_t will be compounded of all frequencies from arbitrarily low ones up to frequencies, perhaps, around $\pi/30$, with greatest weight given to the very low frequencies. These very low frequencies when combined with b_t will produce components hardly affected by A_0 at all.

It will be important to eliminate p_t adequately if an evolving seasonal is to be allowed for and if some operator other than A_0 is used the estimate obtained from y_t' will certainly have to be corrected. Thus if $\hat{s}_t'$ is the estimate of s_t obtained from y_t' we may form

$$\hat{s}_t = \sum_{-6}^{6} \hat{s}_{t-k}' v_k' \tag{9}$$

where v_k is v_k save for $k = \pm 6$ when it is $\frac{1}{2}v_6$. (Remember that $v_{-k} = v_{12-k}$.)

The formula (9) results in the loss of 6 observations at either end of the series. However this loss can (adequately) be eliminated by using the technique described in formula (7) for the first and last 12 observations of the series, or equivalently by continuing on the first and last 12 observations $\hat{s}_t'$ periodically to six further time points.

The operation (9) will produce a seasonal estimate having the same disadvantages as those discussed for A_0, though here the problem will be aggravated. For the particular case discussed for the operator A_1 the amplitudes of the components at $(\pi/6-\pi/30)$ will now be reduced by 65 per cent, for example, and that at $(\pi/6+\pi/30)$ will be increased by 80 per cent. Also the effects for other j will not be so small.

Various devices could be used to improve the estimate. One would be to replace the moving average used in (9) by a longer average. An example is given in the appendix but we shall not discuss the matter further here as more work needs to be done before any such procedure could be recommended.

APPENDIX

Let us denote the vector of seasonal indices for the original series by a. Then the subtraction of Ay_t from y_t alters this to μ, let us say. We may assume that the components of a add to zero so that those of μ do so also, i.e.

$$1'a = 1'\mu = 0$$

where 1 is the vector composed entirely of units. We may write the relation between a and μ in the form

$$[I - \sum \delta_s U^s]a = \mu \tag{10}$$

where now U is the elementary circulant matrix whose elements are zero save for units in the first superdiagonal and the bottom left-hand corner. It follows that, also,

$$Va = \left[I - \sum \delta_s U^s + \frac{1}{12} 11'\right] a = \mu.$$

We assume now that V is nonsingular so that

$$a = V^{-1}\mu.$$

Since V is a circulant so is V^{-1} and its eigenvalues are θ_t $t=1, \cdots, 12$

$$\theta_s = \sum_{k=0}^{11} v_k'' e^{ik\lambda_s}$$

where v_k'' is the element in the $(k+1)$th column of the first row of V^{-1}. Thus

$$v_k'' = \frac{1}{12} \sum_{0}^{11} \theta_s e^{-ik\lambda_s} \qquad k = 0, \cdots, 11.$$

But θ_s is the reciprocal of the eigenvalue of V corresponding to the same eigenvector so that

$$\begin{aligned} \theta_s &= [1 - h(\lambda_s)]^{-1} \qquad & s \neq 0 \\ &= 1 & s = 0. \end{aligned}$$

Thus

$$v_k'' = \frac{1}{12}\left[1 + \sum_{s=1}^{11} \frac{1}{1 - h(\lambda_s)} e^{-ik\lambda_s}\right] \qquad k = 0, \cdots, 11$$

and a is obtained from μ by the formula

$$a_j = \sum_{k=1}^{12} \mu_k v_{k-j}''.$$

However, since the μ_j add to zero we may replace, in this formula, the v_k'' by

$$v_k = \frac{1}{12} \sum_{s=1}^{11} \frac{e^{-ik\lambda_s}}{1 - h(\lambda_s)}$$

as indicated in formula (8).

Since the u_k are the (asymptotically) best linear unbiased estimates of the μ_k it follows that the $\hat{a}_j$ are the best linear unbiased estimates of the a_j.

These formulae may also be obtained from (3) and (6).

If the seasonal s_t is evolving the action of $(I-A)$ upon it is not described by (10). Instead we have

$$[I - \sum \delta_s U^s] s_t = s_t'$$

where now U is no longer a circulant matrix but once again the operator which moves s_t on one time point. To approximately invert $(I-A)$ we may proceed as above however. If we consider U as the elementary circulant, but now with m rows, we find that the matrix V^{-1} now has first row

$$v_k'' = \frac{1}{m}\left[\sum_{s=1}^{m-1}\left\{1 - h\left(\frac{\pi s}{m}\right)\right\}^{-1} e^{-ik(\pi s/m)}\right] + \frac{1}{m} \cdot \qquad k = 0, \cdots, m-1.$$

We again put $v_k = v_k'' - 1/m$.

We may now form the moving average operator with weights

$$\left.\begin{array}{ll} w_{-k} = w_k = v_k, & k = 0, \cdots, \dfrac{m-2}{2} \\ w_{-m/2} = w_{m/2} = \tfrac{1}{2}v_k & \end{array}\right\} \quad m \text{ even}$$

$$w_{-k} = w_k = v_k, \qquad k = 0, \cdots, \frac{m-1}{2}, \quad m \text{ odd}.$$

The effect of applying this to the evolving seasonal, $\hat{s}_t'$, estimated from the y_t' can be judged by considering its effect when applied to s_t'. This in turn can be judged by examining

$$\sum w_s U^s [I - \sum \delta_s U^s].$$

This has the "frequency response function" or "transfer function" [9, p. 12] (which describes the way the component with frequency λ in s_t is modified)

$$(\sum w_s e^{is\lambda})(1 - h(\lambda)) = \sum_{s=1}^{m-1} \left[\frac{h_m\left(\lambda - \dfrac{2\pi s}{m}\right)(1 - h(\lambda))}{1 - h\left(\dfrac{2\pi s}{m}\right)} \right]. \tag{11}$$

Here $h_m(\lambda)$ is the transfer function corresponding to a simple centred m months moving average. Since $h_m(\lambda)$ is unity at the origin and zero at the points $2\pi s/m$, $s \neq 0$, it is easily seen that (11) is unity at all points $2\pi s/m$, $s \neq 0$ and zero at the origin. If m is chosen as 24 then this function is unity at all points $\pi j/6$, $j \neq 0$ and also at all points differing from these by $\pi/12$. There would then be very little distortion of $(\sum w_s U^s) s_t'$ away from s_t. This suggests that we apply to these $\hat{s}_t'$ the 25-term moving average with weights

$$w_k = \left\{ \frac{1}{12} \sum_{s=1}^{11} \frac{\cos s \dfrac{\pi k}{12}}{1 - h\left(\dfrac{\pi s}{12}\right)} \right\} + \frac{1}{24} \frac{(-)^k}{1 - h(\pi)}, \qquad k = 0, 1, \cdots, 11$$

$$w_{12} = \left\{ \frac{1}{24} \sum_{s=1}^{11} \frac{(-)^s}{1 - h\left(\dfrac{\pi s}{12}\right)} \right\} + \frac{1}{48} \frac{1}{1 - h(\pi)}, \qquad w_k = w_{-k}.$$

The loss of 24 terms (12 at each end of the series) could be avoided by continuing on the first and last 12 $\hat{s}_t'$ periodically.

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Seasonal Adjustment and Relations Between Variables

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This article studies the effect of official seasonal adjustment procedures on the relations between variables. By considering time-invariant linear filters, and in particular a linear approximation to the Census Method II adjustment program, the effect of adjusting one or both of the variables in a distributed lag relation is examined, and the distortions which can arise are described. Applying the actual (nonlinear) adjustment procedure to artificial data indicates that at least for the particular x-series used, the results of the linear filter analysis provide a good guide to the behavior of estimates obtained from data adjusted by the official method.

1. INTRODUCTION

Discussion of seasonal adjustment procedures has generally proceeded in terms of their effect on a single economic time series. Official statisticians (Brown, *et al.* [1], Burman [2, 3], Shiskin, *et al.* [22], U.S. Bureau of Labor Statistics [27]) have been concerned with designing seasonal adjustment procedures which satisfy various criteria, and they, together with others (Godfrey and Karreman [7], Nerlove [17, 18], Rosenblatt [20, 21]), have evaluated by various means the extent to which these and other criteria are met. Underlying much of this work is the "classical" additive or multiplicative components model, where the time series is taken to comprise trend-cycle, seasonal, and irregular components. Seasonality is seldom defined rigorously; one of the more explicit statements is that of Nerlove [17], who defines seasonality as "that characteristic of a time series that gives rise to spectral peaks at seasonal frequencies." In the time domain, following Thomas and Wallis [26], "by seasonal variation we understand those systematic, though not necessarily regular intra-year movements in economic time series which are often caused by non-economic phenomena, such as climatic changes and the regular timing of religious festivals." Broadly speaking, the objective of seasonal adjustment is to remove the seasonal component without distorting the remainder, which perhaps provides an ex-post definition of the seasonal component as the difference between the original and adjusted series. The predominant uses are those of short-term forecasting and policy analysis, where the implicit view seems to be that the seasonal component is of little interest, being not only exogenous to the economic system but also uncontrollable, yet predictable. Thus, most macroeconomic aggregates are appraised in their adjusted forms. However, there are exceptions to this, such as total unemployment, which is taken to be politically sensitive irrespective of season, and at a less aggregated level, various stock-flow relationships, where the existence of a seasonal peak in demand is explicitly acknowledged (as with banks' reserve ratios and retailers' inventories).

In general, little attention has been paid to the effect that seasonal adjustment of separate time series has on the relations between them, although changes in dynamic specification when moving between adjusted and unadjusted data have been observed.[1] Perhaps this neglect results from the view that the seasonal component of a given series is noise, and even if correlated with the seasonal component of another series, it is still noise. Also, the nonlinear nature of the official adjustment procedures, largely based on ratio-to-moving-average methods, makes theoretical investigation difficult, although at the simplest level it is clear that they preserve neither sums nor ratios, so that an adjusted aggregate is not generally equal to the total of the adjusted components, and an adjusted unemployment rate is not generally equal to the ratio of the adjusted number unemployed to the adjusted labor force. Regression methods of seasonal adjustment are easier to investigate, and Lovell [14] has developed a rationale for their use; a number of their implications for the regression analysis of the relations between variables in adjusted or original form are described by Thomas and Wallis [26]. However, such techniques have found little use, "since no regression models have yet been demonstrated empirically to provide sufficiently accurate estimates of the trend-cycle and the seasonal, particularly in the current period" (Shiskin, *et al.* [22]).

[1] For example, in the London Business School quarterly model of the U.K. economy, on switching from unadjusted to adjusted data, the consumption function for nondurables moved from an equation containing current and one- and two-quarter lagged values of income, estimated in four-quarter differences, to a more conventional form with current income and the one-quarter lagged dependent variable, the implied adjustment of consumption to income becoming much slower. At the same time the consumption function for durable goods became a static equation, the stock of durable goods dropping out, being highly collinear with income in the adjusted data. (Source: various discussion papers of the LBS Econometric Forecasting Unit.)

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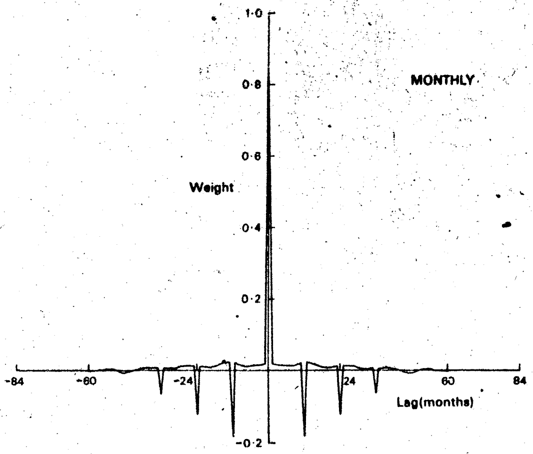
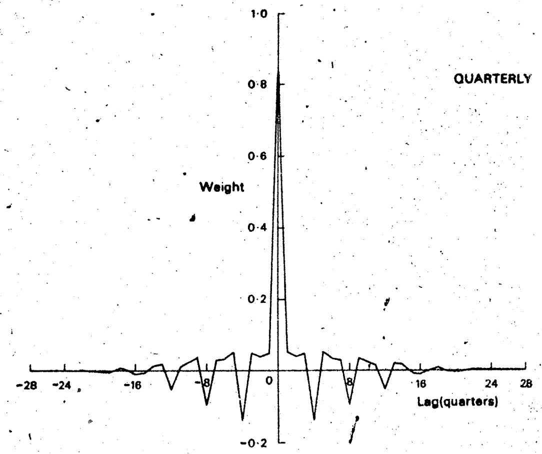
Applications Section

A. Weights for Linear Adjustment Filters





This article studies the effect of official seasonal adjustment procedures on the relations between series. For the present purposes, "official" means the U.S. Bureau of the Census Method II, Variant X-11 [22], as modified by the British Central Statistical Office. The view taken is that seasonality in one economic variable is not necessarily an isolated phenomenon, but may be related to the seasonality in other economic variables with which that variable interacts. Thus the seasonal components themselves may contain information about the relationships between series. Various possibilities are considered, and the effects of separate seasonal adjustments on the underlying relationship between two series and on the statistical estimation procedures employed to detect that relationship are investigated. The investigation proceeds by means of a linear filter approximation to the official procedure in Section 2, and by the analysis of artificial data in Section 3, where the actual official method is applied. Some concluding remarks are presented in Section 4.

2. LINEAR FILTERS AND RELATIONS BETWEEN VARIABLES

2.1 An Approximation to the Official Adjustment Procedure

A linear filter approximation to the official adjustment procedure is first presented, and then the characteristics of the actual procedure which are neglected in the approximation are described. Given an original series $\{x_t\}$, the adjusted series is obtained as

$$x_t^a = \sum_{-m}^{m} a_j x_{t-j} \quad (a_j = a_{-j}) \tag{2.1}$$

and the linear filter or adjustment coefficients $\{a_j\}$ summarize the following steps (a monthly series is assumed, and "moving average" is abbreviated m.a.):

a. Compute the differences between the original series and a centered 12-term m.a. (a 2 × 12 m.a., that is, a 2-term average of a 12-term average), as a first estimate of the seasonal and irregular components.
b. Apply a weighted 5-term m.a. to each month separately (a 3 × 3 m.a.), to obtain an estimate of the seasonal component.
c. Adjust these seasonal components to sum to zero (approximately) over any 12-month period by subtracting a centered 12-term m.a. from them.
d. Subtract the adjusted seasonal component from the original series, to give a preliminary seasonally adjusted series.
e. Apply a 9-, 13- or 23-term Henderson m.a. to the seasonally adjusted values, and subtract the resulting trend-cycle series from the original series to give a second estimate of the seasonal and irregular components.
f. Apply a weighted 7-term m.a. (a 3 × 5 m.a.) to each month separately, to obtain a second estimate of the seasonal component.
g. Repeat step (c).
h. Subtract these final estimates of the seasonal component from the original series, giving the seasonally adjusted series.

The net effect of these eight steps is represented as the $2m + 1$ term m.a. given in (2.1), where the "half-length" m is the sum of the half-lengths of the component m.a.'s, namely, 82, 84, or 89, depending on the choice made at (e). In actual practice this choice depends on the relative contributions of the trend-cycle and irregular components to the variability of the preliminary seasonally adjusted series obtained at (d)—the greater the irregular contribution, the longer the moving average used. With quarterly data this choice is not available, and step (e) comprises a 5-term Henderson m.a.; otherwise, replacing centered 12-month m.a.'s by centered 4-quarter m.a.'s where appropriate gives the linear filter approximation for the adjustment of quarterly data ($m = 28$). The coefficients for the monthly ($m = 84$) and quarterly adjustment filters are shown in Figure A.[2] In both illustrations a seasonally adjusted observation is obtained as a moving average of original observations

[2] These coefficients are analogous to the weights for seasonal factor, trend-cycle, and irregular estimates in the present method and the MLS method presented by Young [30].

up to seven years before and after, although the weights attached to the more distant observations are very small.

There are four important features of the official adjustment procedure which are not captured in this representation.

1. *Multiplicative models* are often employed in place of the additive model implicit above. Thus, seasonal components are estimated as average ratios to, rather than average differences from, the trend-cycle, and the result is a set of seasonal adjustment factors which average 100.0% over a year.
2. *Graduation of extreme values* is undertaken in order to improve the estimation of seasonal and trend-cycle components by preventing the moving averages from responding "too much" to a single outlier. Each value of a preliminary estimated irregular component is compared to the standard deviation, σ, computed over a moving 5-year period. Values between $1\frac{1}{2}\sigma$ and $2\frac{1}{2}\sigma$ distant from 0.0 (100.0 for multiplicative models) are weighted, decreasing linearly from full weight at $1\frac{1}{2}\sigma$ to zero weight at $2\frac{1}{2}\sigma$, and values outside $(-2\frac{1}{2}\sigma, +2\frac{1}{2}\sigma)$ are discarded as extreme. The original series is then modified by adding this graduated irregular component back to the other two components, and trend-cycle and seasonal components are reestimated. The new irregular component is studied for extreme values once again, and after this second modification of the original series, final estimates of the trend-cycle and seasonal are developed. Thus steps (a)—(h) or their multiplicative equivalents are followed three times, each time beginning with a slightly different input series; the choice of $1\frac{1}{2}$ and $2\frac{1}{2}$ as σ limits is in practice optional, and if these limits were set sufficiently wide, no irregular component values would be considered extreme, and identical calculations would result at each of the three iterations.
3. *Calendar-year totals* of the adjusted series are constrained to equal the calendar-year totals of the original series by making a further adjustment to the output from step (h). This could be done by simply adding one-twelfth of the discrepancy to each month's figure, but the actual corrections are smoothed by a piecewise cubic to avoid discontinuities at year-ends.
4. *End-corrections* are necessary, since seven years of data on either side of an observation to be adjusted are never available; hence, asymmetrical "equivalent" moving averages are constructed. This general heading covers the important problem of the adjustment of current observations, and the possible need to revise estimated or extrapolated seasonal factors as subsequent observations become available. That problem is not considered in this article, where the focus of interest is the econometric analysis of historical time series. Nevertheless, in Section 3 it is assumed that a limited series of original observations is available over, say, 15 years, and a corresponding adjusted series is required over the whole period.

These four features will be of no further concern in Section 2 of this paper, but are described here to emphasize those elements of the official procedure to be used in Section 3 that the linear filter approximation does not capture.[3]

2.2 Filtering a Single Time Series

The properties of the linear filter (2.1) can be described in a number of ways, and it is convenient to introduce some further time series concepts and notation. The generating function or z-transform of a sequence $\{a_j\}$ is defined as

$$A(z) = \sum a_j z^j.$$

The backward shift or lag operator L is defined by $L^j x_t = x_{t-j}$, hence (2.1) may be written

$$x_t^a = A(L)x_t.$$

The effect of the filter on particular frequencies ω of the input series is given by the frequency response function

$$A(\omega) = \sum a_j e^{-i\omega j} = |A(\omega)| e^{i\theta(\omega)}.$$

$|A(\omega)|$ represents the gain of the filter and $\theta(\omega)$ the phase shift; the latter is zero for the symmetric moving averages considered here, as $A(\omega)$ is real:

$$A(\omega) = a_0 + 2\sum_{j=1}^{m} a_j \cos \omega j.$$

The autocovariances of a zero-mean stationary time series are given by

$$\gamma_k = E(x_t x_{t-k}), \quad \gamma_k = \gamma_{-k},$$

with generating function

$$\Gamma(z) = \sum \gamma_k z^k.$$

The autocorrelation coefficients γ_k/γ_0, $k = 0, 1, \cdots$ give the correlogram of the series. The autocovariance generating function of a filtered series is obtained from that of the input series and the coefficients of the filter as

$$\Gamma(z)^a = A(z)\Gamma(z)A(z^{-1}). \tag{2.2}$$

The spectral density function is given by

$$f(\omega) = \frac{1}{2\pi}\sum \gamma_k e^{-ik\omega} = \frac{1}{2\pi}\Gamma(e^{-i\omega}),$$

thus the spectra of the original and filtered series are related by

$$f^a(\omega) = |A(\omega)|^2 f(\omega).$$

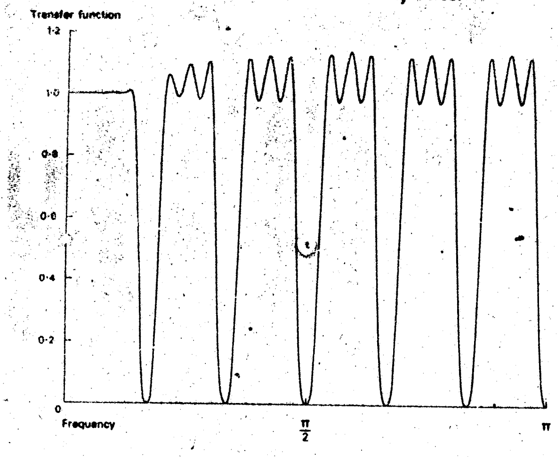
The squared gain $|A(\omega)|^2$, or [illegible] function of the filter, represents the extent [illegible] contribution of the [illegible] total variance of the series [illegible] the filter, and the [illegible] filter is plotted in Figure B.[4] It is seen that the filter completely removes the seasonal frequencies $\pi k/6$, $k = 1, \cdots, 6$, all of them being treated equally. Of course if a particular seasonal pattern can be adequately represented by sine and cosine waves at fewer than six seasonal frequencies,[5] then the filter is in effect overadjusting by unnecessarily modifying certain frequencies.

[3] "Trading-day" corrections also feature in practical seasonal adjustment programs, variations in the number of working days per month having an impact on certain 'flow' variables, and changes in the day of the week on which accounts are closed being relevant for certain 'stock' variables. In the Census Bureau method, the necessary corrections can be either imposed *a priori* or estimated by regression; for the present purposes it is assumed that any required corrections have been made.

[4] Similar functions are calculated for two other seasonal adjustment procedures by Hext [13].

[5] For an example, see Brown, *et al.* [1].

B. Transfer Function of Monthly Filter



In the time domain the effect of the quarterly version of the filter is illustrated in Figure C, which compares the correlograms of original and filtered series, computed as in (2.2), for three simple examples. The first is a "white noise" or independent input series, with $\gamma_k = 0$ for $k \neq 0$; the resulting "adjusted" series has small positive autocorrelation coefficients at lags of 1–3, 5–7, $\cdots$ quarters, and somewhat larger negative correlations between observations 4, 8, $\cdots$ quarters apart. The second illustration uses as input series the familiar first-order autoregression or AR(1) process, with autocorrelation coefficients $\rho^{|k|}$, and the correlograms when $\rho = 0.7$ are shown in the central panel of Figure C. The moving average increases the autocorrelations overall, inducing little seasonal effect. As ρ decreases from this value, the picture moves towards that of the first illustration, thus when $\rho = 0.5$, the autocorrelations at multiples of 4 lags are negative, though small. The final example takes the simple AR(4) process

$$x_t = \rho x_{t-4} + \epsilon_t,$$

for which the autocorrelation coefficient is zero unless k is an integer multiple of 4, in which case it is $\rho^{k/4}$. It can be seen from the final panel of Figure C (where $\rho = 0.9$) that the autocorrelation at lags 4, 8, $\cdots$ is reduced, but substantial autocorrelation at all other lags is introduced by the moving average procedure. The first-order autocorrelation coefficient is almost equal to the fourth; indeed, for $\rho \leq 0.65$, the largest coefficient is the first, which might result in a filtered fourth-order scheme being identified as a first-order scheme.

The correlogram values at $k = 1$ are the (asymptotic) expected values of least squares coefficients when fitting a first-order autoregesssive model. Thus if the correct model, as used in the second illustration, is

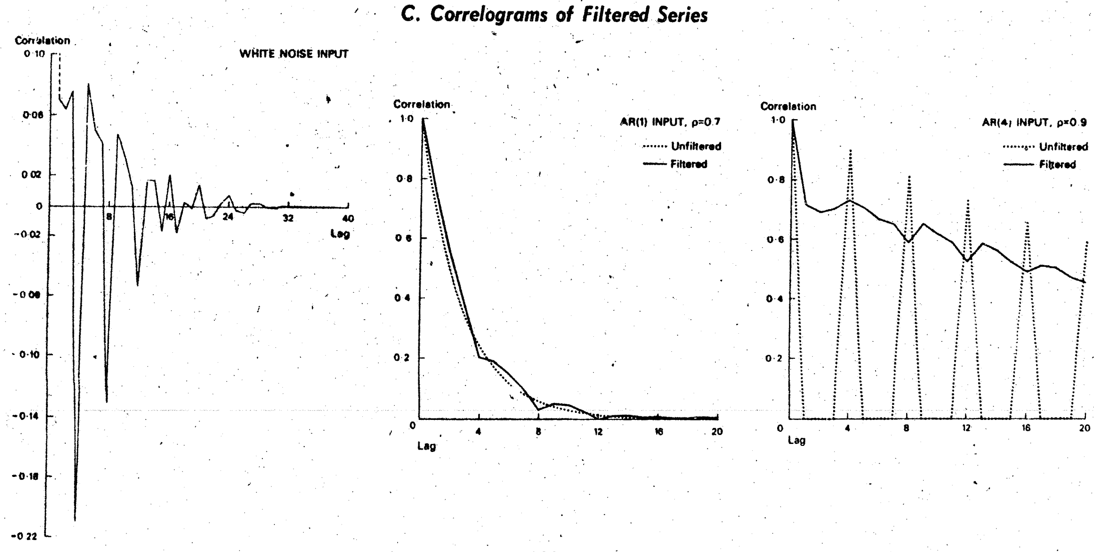
$$x_t = \rho x_{t-1} + \epsilon_t,$$

then the filtered data give

$$x_t^a = \rho x_{t-1}^a + \epsilon_t^a,$$

and the least squares coefficient $\sum x_t^a x_{t-1}^a / \sum x_{t-1}^{a^2}$ provides an asymptotically biased estimate of ρ, for although x_{t-1} and ϵ_t are independent, the same is not true for the moving averages x_{t-1}^a and ϵ_t^a. The extent of the asymptotic bias is given by the difference between the two correlogram values at $k = 1$, and is reported for a range of values of ρ and both monthly and quarterly versions in Table 1. This indicates that the parameter ρ will always tend to be overestimated (asymptotically). However, the opposite is true when the simple fourth-order autoregression

C. Correlograms of Filtered Series



is considered, as suggested by the comparison at $k = 4$ in the final panel of Figure C; the parameter estimate calculated from filtered data is downward biased, and this applies for all values of ρ.

1. Asymptotic Bias in Estimates of First Order Autoregression from Filtered Data

Filter	ρ									
	−.9	−.7	−.5	−.3	−.1	.1	.3	.5	.7	.9
Monthly ($m = 84$)	.033	.011	.013	.018	.024	.029	.033	.037	.036	.019
Quarterly	.184	.110	.075	.062	.066	.076	.080	.070	.047	.017

The same effects as those described in the preceding paragraphs are obtained with the monthly filter, provided that 4 is replaced by 12 at appropriate points in the discussion.

2.3 Relations Between Variables

The relation between two variables is represented by the distributed lag model, familiar in econometrics, of the general form

$$y_t = \sum_0^\infty \beta_j x_{t-j} + u_t. \tag{2.3}$$

It is assumed that x_t and u_s are independent for all t and s, and the relationship between y and x is generally represented as a *one-sided* time invariant filter, as indicated. The sequence of distributed lag coefficients $\{\beta_j\}$ is required to converge to zero at some suitable rate for theoretical reasons, and further restrictions are often imposed for practical estimation purposes, such as requiring the coefficients to be functions of a small number of parameters. If the coefficients are all positive, an "average lag" is given by $\sum j\beta_j / \sum \beta_j$, and estimates of this quantity are often reported in empirical work.

Following previous notational conventions, the model can be written

$$y_t = \sum \beta_j L^j x_t + u_t = B(L)x_t + u_t$$

and the distributed lag frequency response function is

$$B(\omega) = \sum \beta_j e^{-ij\omega}.$$

Introducing the cross-covariance function

$$\gamma_{yx}(k) = E(y_t x_{t-k}) \quad \text{(independent of } t\text{)}$$

and the spectral density functions

$$f_{yx}(\omega) = \frac{1}{2\pi} \sum \gamma_{yx}(k) e^{-ik\omega}, \qquad f_{xx}(\omega) = \frac{1}{2\pi} \sum \gamma_{xx}(k) e^{-ik\omega},$$

then

$$f_{yx}(\omega) = B(\omega) f_{xx}(\omega).$$

Assuming now that the two series are adjusted or filtered, possibly using different filters

$$y_t^a = A_y(L) y_t, \quad x_t^a = A_x(L) x_t,$$

then the relation between the adjusted variables is

$$y_t^a = \frac{A_y(L)B(L)}{A_x(L)} x_t^a + A_y(L)u_t,$$

i.e.,

$$y_t^a = \sum \beta_j^* x_{t-j}^a + u_t^a,$$

where

$$B^*(L) = \frac{A_y(L)B(L)}{A_x(L)}. \tag{2.4}$$

Although $B(L)$ is one-sided, $B^*(L)$ is in general doubly infinite. The spectral functions for filtered data are

$$f_{yx}^a(\omega) = A_y(\omega)\overline{A_x(\omega)} f_{yx}(\omega), \quad f_{xx}^a(\omega) = |A_x(\omega)|^2 f_{xx}(\omega),$$

and the frequency domain expression corresponding to (2.4) is

$$B^*(\omega) = \frac{A_y(\omega)B(\omega)}{A_x(\omega)}, \tag{2.5}$$

which is not defined at seasonal frequencies. It is seen that the effect of the filtering operations is to change the lag function to $B^*(L)$ and the error term to u_t^a. Of course if the same linear filter adjustment procedure is applied to both series ($A_y = A_x$), then the relationship between them is not changed, and the only effect is on the error term—if u_t is nonautocorrelated, then u_t^a is a high-order moving average process, and least squares estimates with adjusted data will not be fully efficient.[6] This is virtually the situation which applies in this section, for the adjustment procedure under consideration amounts to the application of the same linear filter to both series.[7]

A further consequence of converting an independent u-series to an autocorrelated u^a-series is that the usual formula for calculating the covariance matrix of the estimated coefficients is invalid when applied to adjusted data. As indicated by Malinvaud [15, Sect. 13.5], whether the application of the standard least squares formula leads to an underestimate or overestimate of the actual variances depends on the product of the autocorrelation coefficients of the error term and the explanatory variable. The first panel of Figure C indicates that if u is an independent series, then u^a has small positive autocorrelation coefficients at lags of 1–3, 5–7, $\cdots$

[6] This corresponds to the case discussed by Thomas and Wallis [26, Sect. 3], where the loss in efficiency due to the unnecessary inclusion of seasonal variables in a regression equation is evaluated. In the framework introduced by Watson [29], the lower bound to the efficiency of least squares estimates with adjusted data is zero, being attained when x is composed only of seasonal harmonics, so that the filter not only produces an autocorrelated error term but also annihilates the explanatory variable!

[7] The only departure from this lies in the choice of a 9-, 13-, or [illegible] moving average at Step (e), with monthly data, and different choices [illegible] y series would be made if the relative contributions of the irregular [illegible] the two series differed substantially. The distributed lag functions found [illegible] work are generally very smooth, hence the irregular component of x contributes relatively little to irregularities in y, the main source being the random error term u. Thus the longer options will tend to be selected the greater the relative variance of u, or the proportion of variance in y unexplained by x (assuming u to be white noise). Nevertheless the overall differences between the three options at this stage are very small, this being only one of many steps in the procedure. The main effect is on the rate of decline of the transfer function to the value of 0 at the seasonal frequencies from the value of 1 at frequencies $\pi/30$ on either side (see Figure B), but even here there is little variation, and consequently B^* is very close to B even when one filter has $m = 82$ and the other $m = 89$.

quarters, offset by rather larger negative coefficients at lags of 4, 8, ··· quarters. Since an adjusted x-series will typically exhibit positive autocorrelation, with low-order coefficients dominating, in determining the net effect on the variance matrix some canceling will occur. Thus the overall effect may be positive or negative but is likely to be small, so that in this situation the standard inference procedures are unlikely to go seriously awry when applied to adjusted data.

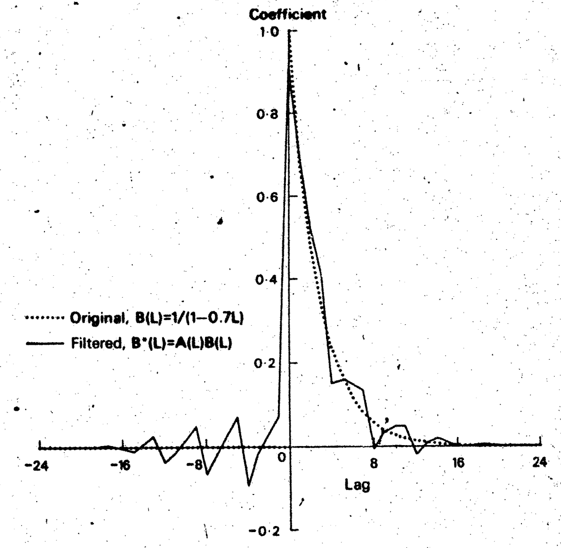
A simple case in which B^* differs from B, corresponding to one which is often found in empirical work, occurs when the explanatory variable is nonseasonal and hence is not adjusted. Examples are found with prices or interest rates as explanatory variables, displaying no seasonality, while the dependent variable is used in its adjusted form, seasonality arising from the error term. Thus $A_x(L) = 1$ and (2.4) becomes

$$B^*(L) = A_y(L)B(L).$$

The estimated relationship differs from the true relationship, and an illustration of the distortion is given in Figure D, where the original distributed lag function is the familiar geometrically declining function, $B(L) = 1/(1 - \lambda L)$, with $\lambda = 0.7$, and $A_y(L)$ is the quarterly adjustment filter. While $B^*(L)$ appears to be longer and flatter than $B(L)$ for positive lags, the most striking features are the pronounced seasonal dips in the new lag function, and the coefficients which appear at negative lags and which might suggest to the unwary that x_{t+j} influences y_t. A simpler example is obtained if the original model is static [$B(L) = 1$], whereupon adjusting y produces a "dynamic" model. As already seen, these distortions can be avoided by applying the same filter to both or neither of the series, and which course is adopted depends on the postulated nature of the error term. It is assumed in this case that u is seasonal, this being the source of the seasonality in y, and so direct estimation (say by ordinary least squares) using the original data will not be fully efficient. The efficient estimator is a generalized least squares-type estimator, which is usually implemented in the time domain by applying OLS to transformed data, the transformation being that required to convert $\{u_t\}$ to an independent series. Thus to the extent that the adjustment filter deseasonalizes u, adjusted data provide more efficient estimates.[8] Note that in order to accomplish these objectives, the same filter is applied to both series irrespective of their nature. Although the success of this procedure requires successful adjustment of the unobservable error term, in the present context of linear models and methods this is achieved whenever the y series is successfully adjusted.

[8] It is interesting that in this case there does not appear to be a regression counterpart, for if the true regression model is $y = X\beta + D\alpha + u$, where D is a matrix of seasonal variables, but these are erroneously excluded and y is simply regressed on the X-variables, then the resulting coefficient estimates are not only unbiased but also efficient when D and X are orthogonal, that is when X is nonseasonal.

D. Distributed Lag Function for Adjusted Y-Series



2.4 Seasonal and Nonseasonal Components of the x-Variable

To analyze further the possible effects of seasonal adjustment on the various relations between series, the notion that the different components of the two series may be related differently is introduced, following Nerlove [19]: "It is plausible, for example, that a manufacturer deciding on inventory levels will react somewhat differently to a change in sales he regards as being purely seasonal in character than he will to one he regards as more permanent or longer lasting or one he regards as exceptionally ephemeral." For present purposes the x-variable is divided into two unobservable components, the seasonal and other-than-seasonal components,

$$x_t = x_t^o + x_t^s,$$

and the distributed lag model is written

$$y_t = \sum \beta_{1j}x_{t-j}^o + \sum \beta_{2j}x_{t-j}^s + u_t. \qquad (2.6)$$

(It might be assumed that the second term gives the seasonal component of y, and the first term the rest, but little is gained by separating these, which would ignore the role of the error term.) Assuming that x^o and x^s are uncorrelated, the spectral density function of x can be similarly split up,

$$f_{xx}(\omega) = f_{xx}^o(\omega) + f_{xx}^s(\omega),$$

whereupon

$$f_{yx}(\omega) = B_1(\omega)f_{xx}^o(\omega) + B_2(\omega)f_{xx}^s(\omega).$$

As before, the filtered series obey the relation

$$y_t^a = \sum \beta_j^* x_{t-j}^a + u_t^a,$$

it now

$$B^*(\omega) = \frac{A_y(\omega)}{A_x(\omega)}\left(\frac{f^o_{xx}(\omega)}{f_{xx}(\omega)} B_1(\omega) + \frac{f^s_{xx}(\omega)}{f_{xx}(\omega)} B_2(\omega)\right). \quad (2.7)$$

This provides an extension of (2.5), which represents the special case $B_1 = B_2$. The situation discussed in the preceding paragraph arises when $f^s_{xx}(\omega) = 0$. One further special case is now discussed.

If $B_2 = 0$, then the seasonal component of x is truly noise, being unrelated to y. In terms of observable variables, (2.6) now becomes

$$y_t = \sum \beta_{1j}x_{t-j} + (u_t - \sum \beta_{1j}x^s_{t-j}), \quad (2.8)$$

which is of the standard errors-in-variables form. Whether y is seasonal (which depends on u), setting $B_2 = 0$ in (2.7) indicates that the relation between the observed variables provides underestimates of B_1. If no adjustment is made, then the observed variables give the relation

$$B(\omega) = \frac{f_{yx}(\omega)}{f_{xx}(\omega)} = B_1(\omega)\left(\frac{1}{1 + f^s_{xx}(\omega)/f^o_{xx}(\omega)}\right)$$

where f^s_{xx}/f^o_{xx} is the noise-to-signal ratio for x. Thus estimates of B_1 obtained by replacing f_{yx} and f_{xx} by their sample equivalents are inconsistent. An obvious possibility is the adjustment of x so that the actual regressor is "closer" to the true explanatory variable; however, this does not entirely work. From (2.7), the observed relation between y and x^a is then

$$B^*(\omega) = B_1(\omega) \frac{f^o_{xx}(\omega)}{A_x(\omega) f_{xx}(\omega)}, \quad (2.9)$$

and since $f^a_{xx}(\omega) = |A_x(\omega)|^2 f_{xx}(\omega)$, whether the ratio on the right side is close to 1 depends on (a) whether $A_x(\omega) \simeq A^2_x(\omega)$, which is approximately true since A is in general close to either 0 or 1, and (b) whether $f^a_{xx}(\omega) \simeq f^o_{xx}(\omega)$. This raises the general question of how "good" the seasonal adjustment is, and although not much can be said in the absence of specific time series models, it is clear that the relationship will not hold exactly, for $x^a = Ax^o + Ax^s \neq x^o$.

In (2.6), the distributed lag function relating the observable variables is a hybrid, whether or not the adjustment assumed in (2.7) is applied. While it might be convenient to assume that B^* is given by B_2 at seasonal frequencies and by B_1 elsewhere, i.e., that the ratios f^o_{xx}/f_{xx} and f^s_{xx}/f_{xx} are accordingly zero or one, this can only be an approximation. In practice, there is not a single seasonal frequency but a narrow band of frequencies around $k\pi/6$ at which seasonal effects are manifested. This also applies to the models of seasonal components introduced by Hannan [12] and Grether and Nerlove [9], which moreover have non-zero power at all frequencies; on the other hand the typical spectral shape (see [8]) which might represent x^o certainly has power at seasonal frequencies. Nevertheless an analysis at separate frequencies with unadjusted data might indicate whether B_1 and B_2 differ substantially, although a sample of the size common in applied econometrics might not offer sufficient resolution.

For adjusted data, the lag function is given by (2.7), with attendant difficulties and distortions already discussed. As before, some distortion can be avoided by using either the same adjustment filter ($A_x = A_y$) or none at all, and which course is adopted depends, for reasons of efficiency, on the nature of the error term. This can be assessed by tests such as that described by Wallis [28] or, more generally, by means of the cumulated periodogram of regression residuals (see Durbin [6]). The observation that were the unobservable x^o and x^s suddenly to become available then (2.6) could be estimated directly (either as it stands or taking x^o and x^s separately, for they are assumed independent) might suggest that the components x^o and x^s be replaced by their estimates x^a and $(x - x^a)$. However, not only are these estimates inexact, as indicated above, but the assumed independence of the components, used in deriving (2.7), is not reproduced in the estimates.[9] The adjusted series has spectral density function $|A(\omega)|^2 f_{xx}(\omega)$, that of the estimated seasonal component is $|1 - A(\omega)|^2 f_{xx}(\omega)$, and their cross-spectrum is $A(\omega)\{1 - A(\omega)\}f_{xx}(\omega)$,[10] which is, however, small except within $\pi/30$ on either side of the six seasonal frequencies (monthly data).

Implicit in (2.6) is the assumption that, while the components x^o and x^s are unobservable by the investigator, they are nevertheless known to the economic agent, and form the basis of his separate reactions. However, in some situations it might be more plausible to assume that the components are equally not observed by the economic agent, who consequently has to form his own estimates; such an assumption seems more in keeping with the previously cited quotation from Nerlove [19]. Thus actual decisions are based on estimates of the seasonal and nonseasonal components, calculated from the observed past of the series in a manner which can be represented as a *one-sided* filtering operation, current estimates being based on current and past data. Writing $C(L)$ for the agent's "adjustment" filter, so that

$$\hat{x}^o_t = C(L)x_t = \textstyle\sum_0^\infty c_j x_{t-j}, \quad \hat{x}^s_t = \{1 - C(L)\}x_t,$$

then replacing the unobserved components in (2.6) by these estimates yields the following relation between observable variables:

$$y_t = [B_1(L)C(L) + B_2(L)\{1 - C(L)\}]x_t + u_t.$$

This can be regarded as a generalization of the simple adaptive expectations model, which arises when $B_1(L) = \beta$, $B_2(L) = 0$, and $C(L)$ is the exponentially

[9] This precludes the use of x^a as an instrumental variable in the errors-in-variables problem (2.8), for although it is clearly correlated with x, it is not independent of x^s.

[10] Note from Figure B that erroneously assuming independence and calculating the spectrum of the estimated seasonal component as $\{1 - A^2(\omega)\}f_{xx}(\omega)$ would result in some negative values.

weighted moving average operator. The overall effect is a one-sided lag function of the initial form (2.2), though rather complex. Separate estimation of B_1 and B_2 would require knowledge of C, or an assumption about the prediction method applied. For the "official" adjusted series x^a to be of use, it would be necessary to assume that the two filters produced similar results, or that the economic agent reacted to the current value of the official seasonally adjusted series, re-introducing considerations of end-corrections and one-sided m.a.'s previously discussed.

In Section 3 the linear filter approximation is replaced by the official adjustment procedure, and the various cases discussed in the preceding paragraphs are constructed from artificial time series.

3. A SIMULATION STUDY

In this section we describe simulation experiments carried out by generating data according to the various models discussed in Sections 2.3 and 2.4, adjusting the series where appropriate by the official procedure described in Section 2.1, and comparing the results of estimation using adjusted and unadjusted data.

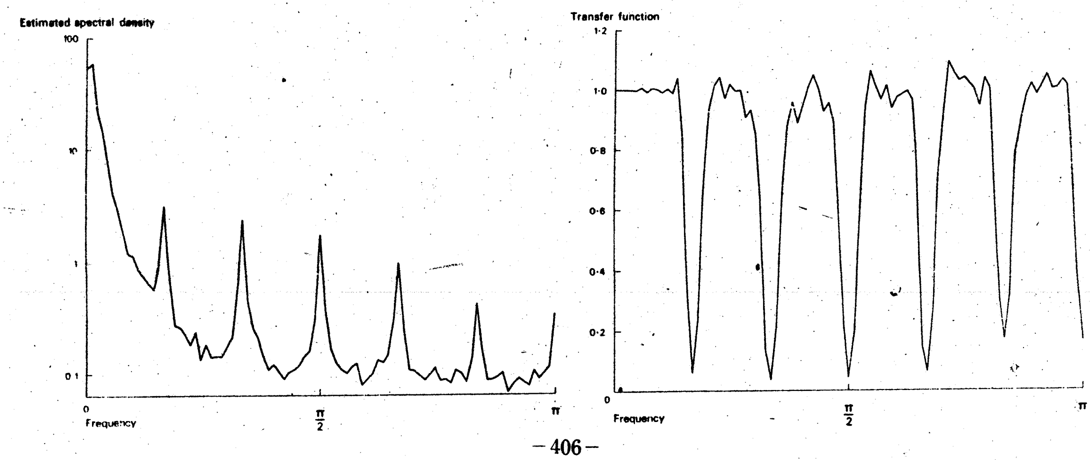
A single x-process is used, namely that designed by Grether and Nerlove [9] and employed by Stephenson and Farr [25]:

$$x_t = \frac{(1 + 0.8L)}{(1 - 0.95L)(1 - 0.75L)} \zeta_t + \frac{(1 + 0.6L)}{(1 - 0.9L^{12})} \eta_t + \xi_t,$$

where $\{\zeta_t\}$, $\{\eta_t\}$, $\{\xi_t\}$ are mutually uncorrelated normally distributed random variables, with variances chosen so that the three terms have variances 8.5, 1.0, and 0.5, respectively. Where separate components are required, the second term is taken as the seasonal component x_t^s, and the first and third terms as x_t^o. While it is conceptually helpful to keep the three components separate, by giving the rational lag operators a common denominator $(1 - 0.95L)(1 - 0.75L)(1 - 0.9L^{12})$ it can be seen that the x-process has a standard autoregressive-moving average representation, of order (14, 14), although one subject to considerable restrictions on its parameters. The effective sample size is 180, corresponding to 15 years of monthly data, and each experiment consists of 50 replications, in each of which a new x-series is generated. On inspecting the x-series, it is immediately clear that the seasonal pattern changes much more rapidly than is observed in practice. This feature could be removed by making the coefficient of 0.9 in the x_t^s lag operator much closer to 1, but it presents no difficulty in the present circumstances since the adjustment procedure is designed to cope with changing seasonal patterns; nevertheless it would render useless any comparison with simple dummy variable methods, which assume a fixed seasonal pattern. The adjustment procedure is applied incorporating all the features omitted from the linear approximation as described earlier. Thus the filter is truly symmetric only in the eighth of our fifteen years, for only then are there seven years of data on either side of an observation, and the adjustment procedure is entirely one-sided for the first and last observations. Since the multiplicative form is used, a constant term is added to the model presented above, in order to ensure that the series is positive-valued. The estimated spectrum of the x-process, calculated by averaging the periodograms of the fifty series, and the implied transfer function of the seasonal adjustment procedure are presented in Figure E. The latter is calculated as the ratio of the "before" and "after" estimated spectral density functions, thus the spectrum presented on the left of Figure E is the denominator of this ratio. The actual transfer function differs from the linear filter approximation (Figure B) in not reaching zero at seasonal frequencies, although the correspondence is close.

The distributed lag functions used are the simple geometric $1/(1 - \lambda L)$, and an inverted V (see [5]), beginning with $\beta_1 = \frac{1}{12}$, increasing linearly to $\beta_6 = 6/12$,

E. Estimated Spectral Density and Adjustment Transfer Function for Grether-Nerlove Model



and decreasing to $\beta_{11} = \frac{1}{12}$. The independent random error term ϵ_t has unit variance throughout. For a parameter θ, the mean and standard deviation of the 50 estimates, $\bar{\hat{\theta}}$ and $SD(\hat{\theta})$ are reported, thus $\bar{\hat{\theta}} - \theta$ is an estimate of the bias $E(\hat{\theta}) - \theta$, with standard error $SD(\hat{\theta})/\sqrt{50}$. In some cases the inconsistency plim $(\hat{\theta}) - \theta$ has been evaluated to provide a yardstick for the finite sample results. The estimation methods employed are ordinary least squares (OLS) and the spectral distributed lag estimator (see [11]). In all cases an intercept term is estimated, although its value is not reported.

3.1 Experiment A

$$y_t = B(L)x_t + \epsilon_t$$

We begin with the general case of Section 2.3, and consider the adjustment of both x and y series.

(i) $B(L) = \beta/(1 - \lambda L)$, $\beta = 1.0$, $\lambda = 0.7$.

Equation estimated: $y_t = \lambda y_{t-1} + \beta x_t + u_t$.

This is the familiar geometric lag function for which OLS provides inconsistent estimates, the estimated equation having a first-order moving average error $u_t = \epsilon_t - \lambda\epsilon_{t-1}$. The probability limits of OLS estimates are 1.093 and .666, and the results with the original data (sample size 180) correspond very closely, as shown in Table 2. The effect of the seasonal adjustment is a small increase in these biases, although the observed biases are smaller than the asymptotic biases calculated assuming that the linear filter approximation is employed.

2. Experiment A(i), Geometric Lag Function

Statistic	Parameter			
	β	λ	β	λ
	Original data		Adjusted data	
Mean estimate	1.093	.663	1.108	.658
Bias	.093	−.037	.108	−.042
SD	.051	.019	.057	.020
Inconsistency	.093	−.034	.146[a]	−.049[a]

[a] Calculated for linear filter.

(ii) $B(L) = \sum_{j=1}^{11}\beta_j L^j$, $\{\beta_j\} = \frac{1}{12}\{1, 2, \cdots, 6, 5, \cdots, 1\}$.

The equation is estimated by OLS ignoring information about the shape of the distribution, and the results are presented in Table 3. The original data produce unbiased estimates, as expected. Although the estimated biases with adjusted data are slightly greater, the estimates remain unbiased overall. The variances increase, thus the linear filter result that adjustment in this case reduces efficiency without inducing bias is reproduced by the official adjustment procedure.[11] The final row of Table 3 presents the means of the coefficient standard errors calculated by applying the usual OLS formula in each replication. A comparison with the standard deviation of the estimated coefficients indicates a slight tendency for the standard errors to underestimate, but overall the correspondence is close, as suggested in Section 2.3.

3.2 Experiment B

$$y_t = B(L)x_t^0 + x_t^s$$

This gives an example of a nonseasonal explanatory variable and a seasonal dependent variable, being the special case discussed in Section 2.3. The seasonal component of the Grether-Nerlove model serves as the "unobservable" error term, while x_t^0 is the observed explanatory variable.

(i) $B(L) = \beta/(1 - \lambda L)$, $\beta = 1.0$, $\lambda = 0.7$.

Equations estimated: (a) $y_t = \lambda y_{t-1} + \beta x_t^0 + u_{1t}$
(b) $y_t^a = \lambda y_{t-1}^a + \beta x_t^0 + u_{2t}$
(c) $y_t^a = \lambda y_{t-1}^a + \beta (x_t^0)^a + u_{3t}$.

Again OLS estimates of Equation (a) are inconsistent, and from the results given in Table 4 we see that the probability limits once more provide a good guide to the finite sample results. Adjusting only the dependent variable, the sole seasonal variable, retains substantial biases, although the signs are reversed and the variance is somewhat reduced. Finally, adjusting both variables, despite the nonseasonal nature of the explanatory variable, substantially reduces the biases and further reduces the variances to about one quarter of the original variances.

(ii) $B(L) = \sum_{j=1}^{11}\beta_j L^j$, $\{\beta_j\} = \frac{1}{12}\{1, 2, \cdots, 6, 5, \cdots, 1\}$.

The results presented in Table 5 show that here the original data produce unbiased estimates, as in Experiment A(ii). If just the dependent variable is adjusted, conforming to a commonly-observed practice, significant biases result, the general tendency being a flattening of the inverted V shape. Adjusting both variables not only restores unbiasedness but also reduces the variances, in accordance with the linear filter results, noting that as the error term is seasonal, the efficient estimator is a generalized least squares-type estimator.

3.3 Experiment C

Errors-in-variables:

$$y_t = B_1(L)x_t^0 + u_t.$$

Turning to the situations discussed in Section 2.4, the case $B_2 = 0$ is first considered. The observed explanatory variable exhibits seasonal variation, but the seasonal

[11] A substantially greater increase in the variances is observed when simple dummy variables are used in conjunction with the original data. However, the present x-series exhibits a changing seasonal pattern, as noted above, and so simple dummy variable adjustment is not appropriate. Studies with more extensive sets of seasonal variables (such as those employed by Stephenson and Farr [25]) have not been performed, for the main point of interest is the behavior of the official procedure, nor has an x-series with a constant pattern been constructed, for such a series is somewhat unrealistic.

3. Experiment A(ii), Inverted V Lag Function

Statistic	Parameter											
	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	β_9	β_{10}	β_{11}	$\Sigma\beta_j$
Original data												
Mean estimate	.082	.173	.235	.348	.430	.480	.417	.3[illegible]4	.258	.166	.084	2.997
Bias	−.001	.006	−.015	.015	.013	−.020	.000	−.009	.008	−.001	.001	
SD	.063	.078	.076	.081	.079	.069	.049	.080	.069	.065	.061	
Adjusted data												
Mean estimate	.089	.166	.254	.325	.434	.473	.400	.350	.249	.155	.103	2.998
Bias	.006	−.001	.004	−.008	.017	−.027	−.017	.017	−.001	−.012	.020	
SD	.072	.095	.100	.113	.102	.085	.081	.110	.092	.094	.087	
Mean std error	.078	.090	.092	.093	.093	.093	.093	.093	.092	.090	.078	

component is unrelated to the dependent variable, which is nonseasonal given the nature of the error term, thus the relative seasonality of x and y is the reverse of that in Experiment B.

(i) $B_1(L)=\beta/(1-\lambda L)\quad \beta=1.0,\quad \lambda=0.7,\quad u_t=B_1(L)\epsilon_t.$

Estimated equation (original data):

$$y_t = \lambda y_{t-1} + \beta x_t + (\epsilon_t - \beta x_t^s)$$

(adjusted data):

$$y_t = \lambda y_{t-1} + \beta x_t^a + v_t.$$

To focus attention on the errors-in-variables problem, this experiment is run with a first-order autoregressive error term $u_t = \lambda u_{t-1} + \epsilon_t$, to give an estimated equation of the partial adjustment form with independent error, and the results are presented in Table 6. The estimates with the original x-series are badly biased as expected, with $\hat{\beta}$ coming off worst. The estimated average lag $\hat{\lambda}/(1-\hat{\lambda})$ has a mean of 4.3 months, compared with the true value of $2\frac{1}{3}$ months. The use of the adjusted x-series improves matters considerably, reducing the biases to less than one-third of their former values, although they are still by no means negligible. An improvement also occurs in the variance of the estimates. Using x^a as an instrumental variable in estimating the original equation achieves results which show slightly greater biases than those obtained when x^a is used directly, although the correspondence is close. The essential elements in determining the biases, namely the covariances of y_{-1} and x^a with the equation's error term, are very similar in the two approaches, given the relatively small seasonal variance, thus this particular instrumental variable has little to commend it (see Footnote 9).

4. Experiment B(i), Geometric Lag Function, Nonseasonal x

Statistic	Parameter					
	β	λ	β	λ	β	λ
	Original data		*Adjusted y-series*		*Adjusted y and x^a*	
Mean estimate	1.067	.675	.928	.720	.991	.702
Bias	.067	−.025	−.072	.020	−.009	.002
SD	.053	.018	.034	.011	.026	.009
Inconsistency	.065	−.022				

(ii) $B_1(L) = \sum_{j=1}^{11} \beta_j L^j,$

$$\{\beta_j\} = \tfrac{1}{12}\{1, 2, \cdots, 6, 5, \cdots, 1\} \quad u_t = \epsilon_t$$

[illegible] Table 7, the original estimates are again [illegible] inverted V being considerably flattened,

5. Experiment B(ii), Inverted V [illegible] tion, [illegible] sonal x

Statistic	β_1	β_2	β_3	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	β_9	β_{10}	β_{11}	$\Sigma\beta_j$
[illegible]												
Mean estimate	.078	.173	.273	.333	.402	[illegible]	.421	[illegible]	.247	.177	.088	3.000
Bias	−.005	[illegible]	.023	.000	−.015	[illegible]	.004	[illegible]	−.003	.010	.005	
SD	.076	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	.087	[illegible]	.078	.067	.078	
Adjusted y-series: y^a on x^o												
Mean estimate	.113	.190	.263	.321	.391	.462	.395	.317	.251	.184	.121	3.008
Bias	.030	.023	.013	−.012	−.026	−.038	−.022	−.016	.001	.017	.038	
SD	.04[illegible]	.034	.028	.033	.039	.042	.031	.036	.037	.033	.043	
Adjusted data: y^a on $(x^o)^a$												
Mean estimate	.085	.175	.259	.329	.413	.487	.411	.330	.253	.164	.093	2.999
Bias	.002	.008	.009	−.004	−.004	−.013	−.006	−.003	.003	−.003	.010	
SD	.034	.034	.031	.036	.030	.042	.032	.043	.040	.031	.045	

6. Experiment C(i), Geometric Lag Function, Nonseasonal y

Statistic	Parameter					
	β	λ	β	λ	β	λ
	Original data		Adjusted data		x^a as instrumental variable	
Mean estimate	.624	.807	.885	.732	.879	.734
Bias	−.376	.107	−.115	.032	−.121	.034
SD	.100	.030	.072	.023	.070	.021

although the total multiplier is surprisingly well-estimated on average. The adjusted x-series (note that the y-series is nonseasonal) achieves a great improvement: taking coefficients singly, only $\hat{\beta}_6$ is significantly biased, although the variances have increased. Thus the ratio on the right side of (2.9) is apparently close to 1 when the official adjustment procedure is applied to this particular x-series.

3.4 Spectral Distributed Lag Estimation

The results to be reported here, based on runs with unadjusted data, are less conclusive. Spectra and cross-spectra are estimated by applying a modified Daniell window to the corresponding periodograms, calculated by a fast Fourier transform algorithm.

First, the "Hannan inefficient" estimator is applied, calculating coefficients as in Hannan [11, Sec. 7] for lags of 0–19 months. The results given in Table 8 are based on the original data of Experiment A(ii), with $B(L) = \sum_{j=1}^{11} \beta_j L^j$, but are not directly comparable to those given in Table 3, where an efficient method and a correct specification were employed.

In general, the method performs moderately well. The overall pattern of coefficients emerges somewhat smoothed, with relatively little noise introduced at lags where the true coefficient is zero. However a number of the individual coefficients appear to be biased, and the sum of the coefficients β_1 to β_{11} of 2.872 underestimates the true value of 3, even neglecting the negative estimates on either side of the inverted V.

The method is then applied to (2.6),

$$y_t = B_1(L)x_t^o + B_2(L)x_t^s + \epsilon_t,$$

where $B_1(L)$ is the geometric lag function ($\beta = 1.0$, $\lambda = 0.7$) and $B_2(L)$ is the inverted V. Estimation of B_1 is attempted by omitting seasonal frequencies from the calculation of distributed lag coefficients using unadjusted data.[12] The periodograms are calculated at frequencies $\pi k/90$, $k = 0, \cdots, 90$, and in smoothing these to obtain spectral estimates prior to calculating $\hat{\beta}$'s, the (seasonal) points $k = 15, 30, \cdots$ together with one point on either side are omitted. Of course this method does not provide efficient estimates of β and λ; the main objective is to see how well the general form of B_1 is estimated.

The results in Table 9 indicate that the general shape of B_1 is reproduced, although the initial values after the first are underestimated and the subsequent geometric decline is correspondingly too slow. The relatively large downward bias at lag 12 suggests that deleting the seasonal frequencies in an attempt to pick out B_1 alone may nevertheless produce seasonal dips in the estimated distribution analogous to those obtained in Figure D. The possibility remains, however, that the Grether-Nerlove x-process is not well suited to an investigation of this particular point, for the seasonal component has spectral peaks not only at seasonal frequencies but also at the origin, and some contamination of the estimate of B_1 may result. In the context of this model an alternative approach is to use adjusted data and regress y on x^a and $x - x^a$ by least squares without constraining the lag functions. The resulting estimate of B_2 is very erratic, the high variances of the estimates corresponding to the relatively small contribution of the seasonal component, but as indicated in Table 10 the estimate of B_1 corresponds a little more closely to the true geometric lag function and the variances are slightly smaller than those of the spectral estimator using unadjusted data, although a direct comparison is not possible since fewer coefficients could be estimated in the OLS approach.

On turning to the "Hannan efficient" estimator (see [11, Sec. 1]) further difficulties emerge. Calculation of spectra and cross-spectra for each individual explanatory variable, in this case each lagged x-value, is required. When these are obtained by the short-cut method of

[12] Frequency-band regression analysis is described by Groves and Hannan [10]. An example of spectral distributed lag estimation in which "seasonal adjustment" is accomplished by omitting bands of seasonal frequencies is presented by Sims [23].

7. Experiment C(ii), Inverted V Lag Function, Nonseasonal y

Statistic	Parameter										
	β_1	β_2	β_3	β_4	β_5	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
	Original data											
Mean estimate	.207	.195	.250	.317	.338	.381	.343	.295	.266	.255	.193	3.010
Bias	.124	.028	.000	−.016	−.079	−.119	−.074	−.038	.016	.058	.110	
SD	.088	.076	.079	.072	.082	.100	.102	.082	.076	.091	.094	
	Adjusted x-series											
Mean estimate	.100	.166	.250	.354	.396	.452	.399	.317	.251	.201	.101	2.987
Bias	.017	−.001	.000	.021	−.021	−.048	−.018	−.016	.001	.034	.018	
SD	.104	.080	.110	.091	.096	.121	.135	.107	.099	.105	.111	

8. Hannan Inefficient Estimates of Inverted V Lag Function

Statistic	Lag 0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
Mean estimate	−.005	.118	.196	.255	.346	.428	.465	.385	.294	.215	.115	.055	−.007	−.006	−.023	−.009	−.015	−.014	−.026	.006
Bias	−.005	.035	.029	.005	.012	.011	−.035	−.031	−.039	−.035	−.052	−.028	−.007	−.006	−.023	−.009	−.015	−.014	−.026	.006
SD	.099	.104	.075	.096	.094	.092	.081	.108	.087	.086	.093	.086	.085	.082	.088	.087	.089	.089	.078	.084

9. Hannan Inefficient Estimates of "Seasonally Adjusted" Geometric Lag Function

Statistic	Lag 0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
Mean estimate	1.047	.598	.384	.287	.186	.141	.128	.096	.091	.078	.060	−.006	−.089	−.033	−.003	.019	−.008	.001	.008	.011
Bias	.047	−.102	−.106	−.056	−.054	−.027	.010	.014	.034	.038	.032	−.026	−.103	−.043	−.009	.015	−.012	−.001	.006	.010
SD	.258	.146	.121	.166	.151	.134	.148	.143	.155	.130	.116	.132	.138	.107	.108	.124	.099	.101	.106	.094

10. Regression Estimates of "Seasonally Adjusted" Geometric Lag Function

Statistic	Lag 0	1	2	3	4	5	6	7	8	9	10	11	12
Mean estimate	.942	.642	.485	.349	.246	.230	.146	.115	.074	.071	.036	.002	−.029
Bias	−.058	−.058	−.005	.006	.006	.062	.029	.032	.016	.031	.008	−.018	−.043
SD	.130	.108	.109	.097	.121	.099	.113	.104	.106	.116	.122	.111	.105

11. Hannan Efficient Estimates of Inverted V Lag Function

Statistic	Lag 0	1	2	3	4	5	6	7	8	9	10	11	12
Mean estimate	.010	.117	.173	.214	.324	.364	.431	.420	.343	.289	.206	.108	−.029
Bias	.010	.033	.006	−.036	−.009	−.053	−.069	.003	.009	.039	.039	.025	−.029
SD	.497	.233	.234	.188	.178	.218	.203	.213	.202	.209	.211	.175	.375

estimating $f_{yx}(\omega)$ and $f_{xx}(\omega)$, and then calculating the cross-spectrum between y_t and x_{t-k} as $e^{ik\omega}f_{yx}(\omega)$ and that between x_{t-j} and x_{t-k} as $e^{i(k-j)\omega}f_{xx}(\omega)$, a significant spurious contemporaneous coefficient ($\hat{\beta}_0$) appears in the estimated lag function. This disappears when separate spectral and cross-spectral estimates for $x_t, x_{t-1}, \ldots, x_{t-12}$ are calculated,[13] as indicated in the results presented in Table 11, which again are based on the original data of Experiment A(ii). Overall, the biases are little different from those given by the Hannan inefficient estimator, and the inverted V is again rather smoothed.[14] But the standard deviations are most striking. Not only are they substantially greater than those obtained in Experiment A(ii) (OLS estimates are best linear unbiased in finite samples in this model), they are also greater than those of the inefficient estimator,[15] notwithstanding differences in the number of coefficients estimated. This feature, together with the high computational burden of this estimator, has precluded further investigation of its behavior.

4. CONCLUSION

The foregoing discussion has considered the problems which arise in estimating a distributed lag relation using seasonally adjusted data. In Section 2 the argument was constructed in terms of linear filters, and various distortions which might result from filtering one or both of the series were described. While most of the results

[12] The author is indebted to Christopher A. Sims for the suggestion that the short-cut method biases $\hat{\beta}_0$ via its biased spectral estimates. Frequency-domain methods treat a series as wrapping around back on itself, and the short-cut method of calculating the cross spectrum between y_t and x_{t-k} implicitly aligns the sequence $y_1, \cdots, y_T$ with the sequency $x_{T-k+1}, \cdots, x_T, x_1, \cdots, x_{T-k}$, creating a bias if the values at the end of the series are substantially different from those at the beginning (as is often the case). This problem does not arise in the direct estimate of the cross-spectrum, for the offending observations are deleted, and the effective sample size becomes $T - k$.

In computing the coefficient estimates, a direct estimate of the OLS residual spectrum is used.

[14] This smoothing of lag functions in the Hannan estimates is present in the simulation results of Cargill and Meyer [4]. They also find that in cases with an autocorrelated error term, the efficient procedure gives little improvement over the inefficient procedure in terms of bias in samples of 100 observations.

[15] E.J. Hannan has pointed out that the number of frequency bands used in computing the estimates is rather larger than would normally be recommended for samples of this size, but recomputing with a much smaller number did not lead to any improvement. However, the present combination of a white noise error and a sharp-peaked regressor spectrum is the worst possible case for the Hannan efficient estimator.

are perfectly general, and not specifically concerned with filters designed for seasonal adjustment, the introduction of spurious "future" coefficients as illustrated in Figure D is a direct consequence of the two-sided nature of the filter under consideration. A one-sided adjustment filter would not produce this particular distortion, and would also enable one to answer the question, "For what autoregressive-moving average input series is the filtered series white noise?" The "optimal" seasonal adjustment of Grether and Nerlove [9], based on the theory of minimum mean-square error extraction and prediction, can indeed produce a one-sided adjustment filter, though the optimal prediction theory argument is not so compelling when parameters have to be estimated from a finite sample of data and, moreover, the correct autoregressive-moving average representation of the series is not known. So to retain practical relevance we concentrated on the official adjustment procedure. In Section 3 the actual official method was applied, in contrast to the linear filter approximation, but the conclusion in the cases studied is that the approximation is a good guide to the performance of the actual nonlinear method. In particular the nonlinearities were not sufficiently pronounced to negate the argument of Section 2 that applying the same linear filter to both series prevents distortion of the lag relationship. The problem of detecting different (non-zero) relations for the seasonal and nonseasonal components requires further investigation; while some success was achieved with the nonseasonal relationship, the particular seasonal component used presented difficulties.

The practical investigator typically has little prior knowledge of the nature of the seasonal relationships, and which of our experimental results is most representative or relevant must be determined in the specific context. The economist's usual *a priori* theorizing, if specifically focussed on short-run adjustment problems, would be helpful in some applications, such as the theory of the behavior of firms. More generally, which of our cases is likely to apply can be determined by examining the x and y series themselves, to see what seasonality they exhibit—this would allow one to discriminate between the cases considered in experiments A, B, and C, for example. Second-stage diagnostic devices such as examination of the residuals of the first estimates, or reestimation subject to high-order autoregressive error terms, then permit further discrimination. As we have seen, filtering or the use of adjusted data is appropriate in some circumstances, although it should be seen more as an adjustment to the model than an adjustment to the data. The indiscriminate use of filters, or the non-availability of unadjusted data, will inevitably lead to mistaken inferences about the strength and dynamic pattern of relationships. Naturally, other techniques such as the use of dummy variables should be included in the list of possible approaches, and it should be noted finally that such variables can be used to relax one particular assumption of the foregoing analysis, namely that the distributed lag relationship is time-invariant. Cases in which the seasonality in y is caused by coefficients changing from season to season remain to be investigated, although some empirical instances have already been noted.[16]

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Seasonal Adjustment and Revision of Current Data: Linear Filters for the X-11 Method

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Summary

Procedures for the seasonal adjustment and revision of current data are studied by linear filter methods. This contrasts with previous research based on small samples of actual series. The properties of the *X*-11 method are described, and it is argued that the practice of calculating seasonal factors for use up to 1 year ahead should be replaced by running the adjustment program every month. It is seen how the relative performance of forecast-augmented methods depends on the structure of the time series.

Keywords: TIME SERIES; SEASONAL ADJUSTMENT; LINEAR FILTER; MOVING AVERAGE; X-11 METHOD; ANNUAL REVISIONS; FORECASTING

1. Introduction

Seasonally adjusted data are widely employed in the analysis of current economic conditions. Users of such data wish to perceive the short-term trend and cyclical developments of the economy undistorted by the more-or-less regular month-to-month changes that constitute seasonal variation. The general view is that while seasonality in one variable may be related to the seasonality in other economic variables with which that variable interacts, ultimately the seasonal component represents the effect of non-economic factors that are exogenous to the economic system and uncontrollable.

Public attention usually focusses on the *first-announced* or *preliminary* seasonally adjusted figure, that is, the seasonally adjusted value of the current month's unemployment or money supply, for example. As time goes by, and more data become available, the particular month's figure is re-adjusted, since the additional data permit a more accurate estimate of the slowly evolving seasonal component, and one that is less distorted by possible outliers. Eventually the re-adjustment process stops with a *final* or *historical* seasonally adjusted figure when it is felt that new data contain no additional information that is useful in adjusting the original figure. In practice there are many reasons why preliminary data are subsequently revised, but this paper is concerned only with those revisions that result from repeated seasonal adjustment as more data become available, and the terms *revision* and *re-adjustment* are used synonymously.

Typically, revisions in adjusted data are uncontroversial and receive little attention, although counterexamples do exist, such as the June 1975 unemployment rate in the US (Klein, 1978, p. 31; Shiskin, 1978, pp. 99–100) and the early 1978 money stock figures in the UK (Bank of England, 1978). Nevertheless, it is important that policy makers should not base their actions on data that are inaccurate and liable to substantial change subsequently. Of course an alternative possibility is to heed the arguments against the indiscriminate use of adjusted data for economic modelling in Wallis (1974, 1978) and elsewhere, but the modelling of seasonal variation is still in its infancy. Thus research effort has recently been devoted, by official statisticians and others, to improvements in existing procedures for the adjustment of current data. The most widely used procedure is the US Bureau of the Cenus Method II, Variant *X*-11 (Shiskin *et al.*, 1967), which is generally held to give satisfactory results in the adjustment of historical data. For discussion of amendments and extensions see, for example, Dagum (1975, 1979), and associated papers by the Seasonal Adjustment and Times Series Staff of Statistics Canada, Geweke (1978), Burman (1980), Pierce (1980), and Kenny and Durbin (1982) and associated Research Exercise Notes of the Central Statistical Office.

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In general the existing procedures and their extensions have been evaluated by applying them to a few selected series (see, for example, Kuiper, 1978), and there has been little systematic study of the properties of these procedures away from a specific context. This paper provides an approach to such a systematic study, using linear filter methods. Elsewhere (Wallis, 1974; Cleveland and Tiao, 1976) linear filter approximations have been found to provide a good guide to the performance of actual adjustment procedures, although in those papers attention was restricted to the symmetric filters used for the adjustment of historical data.

The general framework of a set of "time-varying" linear filters is presented in Section 2, and the relations with forecasting procedures are considered in Section 3. The properties of the linear filter approximation to X-11 are discussed in Section 4. The use of year-ahead seasonal factors is considered in Section 5, and X-11-forecast methods in Section 6. Section 7 contains some concluding remarks.

2. SETS OF "TIME-VARYING" LINEAR FILTERS

We denote the original observable series as x_t. This is the input to a filtering operation, of which the output is denoted y_t. In this paper y is the seasonally adjusted value of x, but the framework of this section can be used for any problem of signal extraction, interpolation, extrapolation, smoothing and so forth, by linear methods. Given an input record x_t, $t = 1, \ldots, T$, then for t sufficiently far removed from the ends of the series $(m+1 \leqslant t \leqslant T-m)$ the adjusted value is obtained by application of the symmetric filter $a_m(L)$,

$$y_t = a_m(L)\,x_t = \sum_{j=-m}^{m} a_{m,j}\,x_{t-j},$$

where L is the lag operator and $a_{m,j} = a_{m,-j}$. We describe this filter equivalently as a $2m+1$-term moving average (abbreviated m.a.) or as a symmetric filter of half-length m. However, for current and recent data $(T-m < t \leqslant T)$ this filter cannot be applied, and truncated, asymmetric filters $a_i(L)$ are employed:

$$y_T^{(0)} = a_0(L)\,x_T = \sum_{j=0}^{m} a_{0,j}\,x_{T-j}$$

$$\vdots$$

$$y_{T-i}^{(i)} = a_i(L)\,x_{T-i} = \sum_{j=-i}^{m} a_{i,j}\,x_{T-i-j}$$

$$\vdots$$

$$y_{T-m}^{(m)} = a_m(L)\,x_{T-m} = \sum_{j=-m}^{m} a_{m,j}\,x_{T-m-j}.$$

For the filter $a_i(L)$, $i = 0, 1, \ldots, m$, the subscript i indicates the number of "future" values of x entering the moving average, that is, the number of negative powers of L that appear, or the negative of the lower limit of summation in the expression $\Sigma\, a_{i,j}\,x_{t-j}$. In seasonal adjustment programs the upper limit of summation is usually kept at m for all filters, as written above, although other possibilities exist: for example, in certain problems one might wish to keep the number of terms in the moving averages constant, and so have an upper limit of $2m-i$. At different points in time different seasonally adjusted values corresponding to a given unadjusted value can be calculated, and the superscript on y keeps track of this. Thus

$$y_t^{(i)} = a_i(L)\,x_t = \sum_{j=-i}^{m} a_{i,j}\,x_{t-j}$$

is the adjusted value of x_t calculated from observations at times $t-m, t-m+1, \ldots, t, \ldots, t+i$.

The filters are "time-varying" in the sense that, on running such a seasonal adjustment program at time T with original data $x_1, \ldots, x_T$, each of the first and last $m+1$ elements of the adjusted series is a different linear filter of the input. The most recent values are

$$y_t^{(T-t)} = a_{T-t}(L)\,x_t, \quad t = T-m, T-m+1, \ldots, T,$$

each y-value being obtained as a moving average of values on either side of the corresponding x-value, with *different* coefficients. A series of the form $y_t^{(T-t)}$ corresponds to what is published as a seasonally adjusted series in a given issue of an official statistical publication, but its properties are difficult to analyse. For example, if x_t is stationary, this adjusted series is not; in particular

$$E(y_t^{(T-t)} y_{t-l}^{(T-t+l)}) \neq E(y_s^{(T-s)} y_{s-l}^{(T-s+l)}), \quad t \neq s, \quad T-m \leqslant t \leqslant T.$$

However this problem is not our concern in the present paper, and we side-step the difficulty by considering series of the form $y_t^{(i)}$, where t varies but i is constant; for example $y_t^{(0)}$ is the series of preliminary adjusted values. While such series are not usually employed by those who choose to carry out econometric analysis using published adjusted data, they can of course be constructed from a sequence of issues of the official statistical publication, and nevertheless enable us to describe the properties of the adjustment procedure using conventional time series ideas.

At times $t+i$ and $t+k$, two seasonally adjusted values $y_t^{(i)}$ and $y_t^{(k)}$ can be calculated, corresponding to the unadjusted value x_t, and the *revision* is defined as

$$r_t^{(i,k)} = y_t^{(k)} - y_t^{(i)}, \quad i<k.$$

The final seasonally adjusted value is $y_t^{(m)}$, and the *total revision* from a given point in time $t+i$ is $r_t^{(i,m)}$. The revision reflects the information in the new data $x_{t+i+1}, x_{t+i+2}, \ldots, x_{t+k}$, but is in general a moving average of $m+k+1$ terms:

$$r_t^{(i,k)} = [a_k(L) - a_i(L)]\,x_t.$$

Thus the time series properties of the revisions can be deduced from those of the original series and the coefficients of the linear filters. For example, one might ask whether the successive revisions $r_t^{(i,i+1)}$ constitute, for fixed i, a white noise process: if so, $y_t^{(i)}$ is the "best" estimate of $y_t^{(i+1)}$, in the usual sense. Often statistical agencies only run seasonal adjustment programs once a year, so that the revisions then made are $r_t^{(i,i+12)}$. Below we pay particular attention to the first annual revision $r_t^{(0,12)}$.

3. FORECAST-AUGMENTED PROCEDURES

Recent proposals for amendments to established procedures have included the suggestion that current data be adjusted not by the one-sided filter $a_0(L)$ but by later filters applied to a series extended by forecasts. If 12 such forecasts are used, then the suggestion is that the preliminary adjusted value should be calculated not as $a_0(L)\,x_T$ but as $a_{12}(L)\,\tilde{x}_T$, where the tilde (˜) indicates that the input to the filter is the series $x_1, \ldots, x_T, \hat{x}_{T+1}, \ldots, \hat{x}_{T+12}$. The various proposals (Dagum, 1975, 1979; Geweke, 1978; Kenny and Durbin, 1982) differ according to the forecasting method employed. However, since the forecasts are calculated as linear combinations of observed values, the result is still a one-sided filter of the original data. Writing the forecasts as

$$\hat{x}_{T+k} = \sum_{j=0}^{p} f_{kj}\,x_{T-j}, \quad k = 1, \ldots, 12$$

the new one-sided moving average is

$$\tilde{a}_0(L)\,x_T = \sum_{j=0}^{\max(m,p)} \left(a_{12,j} + \sum_{k=1}^{12} a_{12,-k} f_{kj} \right) x_{T-j},$$

whose properties can be studied in the usual way.

Given a symmetric moving average this expression suggests a way of constructing asymmetric moving averages for the adjustment of recent data, based on appropriate forecasting equations. Support for this approach is provided by a result given by Geweke (1978) and Pierce (1980), namely that the asymmetric filter $a_i(L)$, $0 \leq i \leq m-1$, for which the total revision $r_t^{(i,m)}$ has smallest mean square is given by the application of the symmetric filter $a_m(L)$ to a series extended to the extent necessary by *optimal* linear forecasts. (This result applies to the complete set of filters, whereas in the practical forecast-augmented procedures only 1 year of forecasts is computed, and so only the first 12 filters are modified.) For a complete set of filters obtained in this way, Pierce shows that the revisions $r_t^{(i,k)}$ follow a moving average process of order $k-i-1$. Thus the single-period revisions, $r_t^{(i,i+1)}$, t varying and i fixed, form a white noise process. Also the successive revisions in a given value, $r_t^{(i,k)}$ and $r_t^{(k,l)}$ for t fixed and any $i<k<l$, are uncorrelated. These results hold not only for stationary x-series but also for series that are non-stationary in original form but stationary after a suitable differencing transformation.

Given a complete set of filters one can study its properties by likewise asking, what forecast function is implied if $a_i(L)$ is equivalent to the application of $a_m(L)$ to a series extended by $m-i$ forecasts. For example, the 5-term Henderson m.a.'s used for estimation of the trend-cycle component in the quarterly version of the X-11 program (Shiskin *et al.*, 1967, Appendix B, Table 3A are

$$y_T^{(0)} = 0{\cdot}670x_T + 0{\cdot}403x_{T-1} - 0{\cdot}073x_{T-2},$$

$$y_{T-1}^{(1)} = 0{\cdot}257x_T + 0{\cdot}522x_{T-1} + 0{\cdot}294x_{T-2} - 0{\cdot}073x_{T-3},$$

$$y_{T-2}^{(2)} = -0{\cdot}073x_T + 0{\cdot}294x_{T-1} + 0{\cdot}558x_{T-2} + 0{\cdot}294x_{T-3} - 0{\cdot}073x_{T-4}.$$

If the filter $a_1(L)$ is assumed to be the result of applying the symmetric filter $a_2(L)$ to a series extended by the forecast $\hat{x}_{T+1}$, then this forecast must only depend on x_T and x_{T-1} since the last two coefficients of each filter coincide. Further, each filter's coefficients sum to 1, and so the appropriate forecasting equation is

$$\hat{x}_{T+1} = (1-\lambda)x_T + \lambda x_{T-1},$$

which is optimal if the first differences of x follow a first-order autoregression:

$$(1-L)(1+\lambda L)x_t = \varepsilon_t.$$

Simple calculation yields $\lambda = 0{\cdot}49$. To obtain $a_0(L)$ from $a_2(L)$ the forecast $\hat{x}_{T+2}$ is also required, again a function only of x_T and x_{T-1} due to the coincidence of the last coefficient in each filter. Assuming that this is the optimal forecast

$$\hat{x}_{T+2} = (1-\lambda)\hat{x}_{T+1} + \lambda x_T,$$

where $\hat{x}_{T+1}$ is the same form as above yields a different values of λ, namely 0·43. Curiously, the previous value of 0·49 is obtained if the forecasts are assumed to be identical:

$$\hat{x}_{T+2} = \hat{x}_{T+1} = (1-\lambda)x_T + \lambda x_{T-1},$$

which reflects an inconsistency in the optimal-forecast interpretation of these filters.

4. THE X-11 LINEAR FILTERS

The linear filters considered here represent the operation of the X-11 procedure (Shiskin *et al.*, 1967) assuming that an additive model is employed, that is, that seasonal components are estimated as average differences from, not average ratios to, the trend-cycle. The multiplicative version of the program is comparable to the application of the additive model to the logarithms of the series. Our calculations neglect the option of graduating extreme irregular

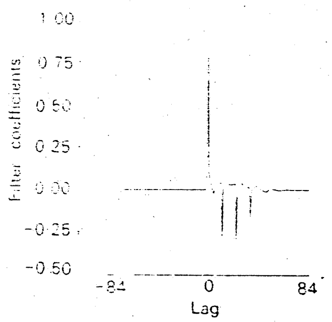

Fig. 1. Panel A.

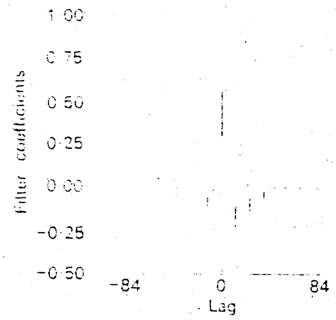

Fig. 2. Panel A.

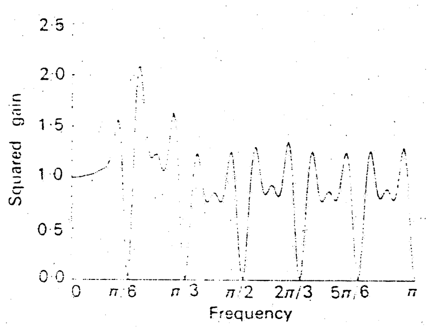

Fig. 1. Panel B.

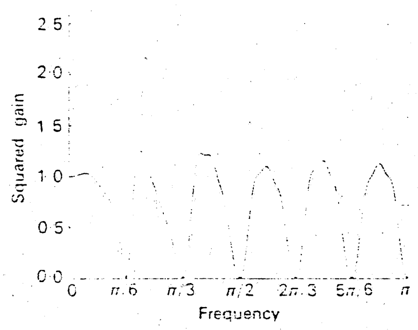

Fig. 2. Panel B.

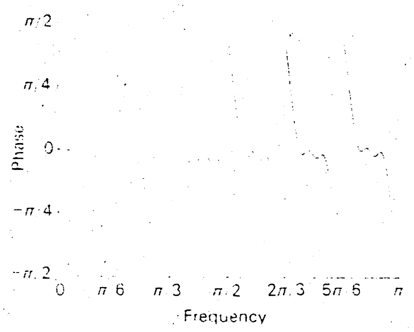

Fig. 1. Panel C.

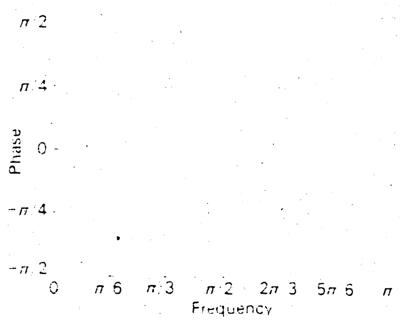

Fig. 2. Panel C.

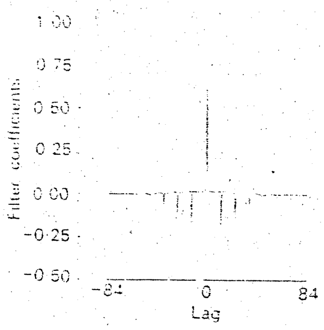

FIG. 3. Panel A.

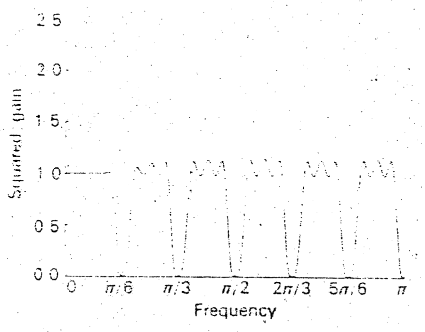

FIG. 3. Panel B

values that is available in the X-11 program. The linear filters summarize the following steps, the description of which extends that given in Wallis (1974, p. 19) for the symmetric case.

(a) Compute the differences between the original series and a centred 12-term m.a., as a first estimate of the seasonal and irregular components. Six values at each end of the series are lost.

(b) Apply a weighted 5-term m.a. to each month separately, to obtain an estimate of the seasonal component. For the next-to-last 12 values use an asymmetric 4-term m.a., and for the last 12 values available use a one-sided 3-term m.a. (Shiskin *et al.*, 1967, Appendix B, Table 1B).

(c) Adjust these seasonal components to sum to zero (approximately) over any 12-month period by subtracting a centred 12-term m.a. from them. To obtain the six missing values at the end of this average, repeat the last available moving average value six times.

(d) Subtract the adjusted seasonal component from the original series, to give a preliminary seasonally adjusted series. The seasonal component for the last six values is missing as a result of step (a); for these last 6 months use the estimated seasonal component for the corresponding month of the preceding year.

(e) Apply a 9-, 13- or 23-term Henderson m.a. to the seasonally adjusted values, and subtract the resulting trend-cycle series from the original series to give a second estimate of the seasonal and irregular components. (The calculations reported in this paper use the 13-term m.a.) At the ends of the series use "equivalent" asymmetric m.a.'s (Table 3C).

(f) Apply a weighted 7-term m.a. to each month separately, to obtain a second estimate of the seasonal component. At the ends of the series use "equivalent" asymmetric m.a.'s (Table 1C).

(g) Repeat step (c).

(h) Subtract these final estimates of the seasonal component from the original series, giving the seasonally adjusted series.

In general, if a particular step involves the use of a symmetric m.a. given sufficient data, the corresponding asymmetric m.a.'s applied to recent data still involve the same number of past values. There is no attempt to compensate for the truncation of the weights applied to future data by increasing the number of past values entering the m.a., future and past being interpreted relative to the value being adjusted. Thus the net effect of steps (a)–(h) is a set of $m+1$ linear filters, with respectively $0, 1, \ldots, m$ coefficients of future values, and m coefficients of past values, as in Section 2; the value of m is 84. The only exception to this occurs as a result of the substitution of missing values in step (d); bringing forward the estimated seasonal component from the corresponding month of the preceding year implies that the first six filters

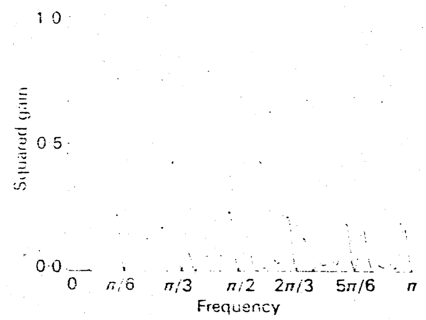

FIG. 4.

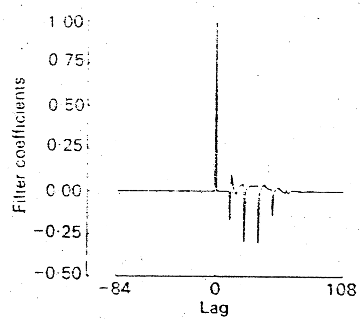

FIG. 5. Panel A.

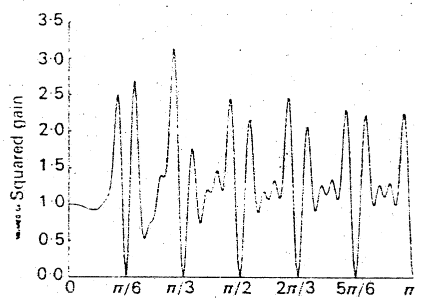

FIG. 5. Panel B.

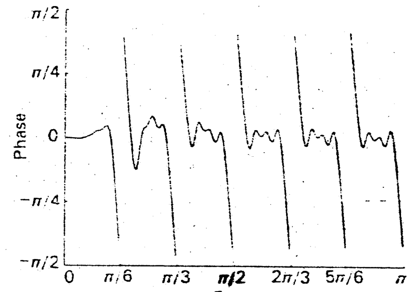

FIG. 5. Panel C.

involve 90, 89, ..., 85 past values respectively. In all cases the remote weights are very small, however. The weights are calculated by repeated application of the algorithm described in the Appendix.

The properties of the filters can be studied by considering their effect on the component of frequency ω of the input series. For the filter $a(L)$ this is given by the frequency response function

$$a(\omega) = \sum a_j e^{-i\omega j} = |a(\omega)|\, e^{i\theta(\omega)},$$

where $|a(\omega)|$ represents the gain of the filter and $\theta(\omega)$ the phase shift. The phase shift is zero for a symmetric filter, and is not defined at frequencies at which an asymmetric filter has zero gain. The spectral densities $f_x(\omega)$ and $f_y(\omega)$ of the input and output series respectively are related by

$$f_y(\omega) = |a(\omega)|^2 f_x(\omega),$$

the squared gain $|a(\omega)|^2$, or transfer function of the filter, representing the extent to which the contribution of the component of frequency ω to the total variance of the series is modified by the action of the filter. The weights, transfer function and phase diagram of the filters $a_0(L)$, $a_{12}(L)$ and $a_{84}(L)$ are presented in Figs 1–3. These weights are analogous to the central and end-year weights for seasonal factor, trend-cycle and irregular estimates in X-11 and the BLS method presented by Young (1968). Corresponding transfer functions and phase diagrams for the quarterly version of X-11 are presented by Laroque (1977). The two panels of Fig. 3, for the symmetric filter, appear in Figs A and B of Wallis (1974); for this filter the phase shift is zero.

From these figures we see that the phase shifts of the asymmetric filters are small except, as expected, at frequencies close to the seasonal frequencies $\pi k/6$, $k = 1, \ldots, 6$, and in particular there is little distortion in the business cycle frequency range. The transfer function of the symmetric filter is very close to 1 over the low frequency range, indicating that the contribution of these frequencies to the series variance is unaltered. The filter completely removes the seasonal components, as expected, and substantially reduces the contribution of a narrow band of frequencies centred on the seasonal frequencies, thus allowing for seasonal patterns that are slowly changing. If the objective of the asymmetric filters is to approximate the transfer function of the symmetric filter as closely as possible, then we see that there is only moderate success. The transfer function of $a_0(L)$, in Fig. 1, exceeds 1 immediately before the first seasonal frequency, and has greater oscillations around 1 between the higher seasonal frequencies, nevertheless it has the same characteristic shape, and its seasonal dips cover a band of frequencies of approximately the same width. Whereas one might expect the transfer function of $a_{12}(L)$ to be closer to that of the symmetric filter than $a_0(L)$, we see in Fig. 2 that this is not so: the characteristic shape does not appear, the seasonal dips cover a wider band of frequencies, and the initial departure from 1 occurs at a much lower frequency.

As noted in Section 2, the revisions in seasonally adjusted data are again linear filters of the original data. For example, the first annual revisions are

$$r_t^{(0,12)} = [a_{12}(L) - a_0(L)]\, x_t.$$

The characteristics of the revisions depend on those of the x-process as well as the calculated filters, and we again summarize the contribution of the filter by presenting the transfer function of the first annual revision filter in Fig. 4. The main effect noticeable is that the revision series possesses no low-frequency component.

5. YEAR-AHEAD SEASONAL FACTORS

In the practical application of X-11 by many statistical agencies the program is run only once a year, and at that time seasonal factors or seasonal components for the adjustment of the next 12 months' data are projected, to be used as the data become available. If computations are based on observations x_t, $t = 1, \ldots, T$ and the seasonal components resulting from steps (a)–(g) are denoted s_t, $t = 1, \ldots, T$, then step (h) above reads $y_t = x_t - s_t$. Using the "year-ahead" seasonal components, the first-announced adjusted figure is given as

$$y_t^* = x_t - (1{\cdot}5 s_{t-12} - 0{\cdot}5 s_{t-24}), \quad t = T+1, \ldots, T+12.$$

The implied linear filters are one-sided, with leading coefficient equal to 1·0, and although the remote weights are again very small, strictly speaking the weights extend back over 108 previous observations. In Fig. 5 we present the weights, transfer function and phase shift of the 12-months-ahead filter. It is seen that the distortions are rather greater than those of the filter $a_0(L)$ on comparing with Fig. 1. The transfer function moves away from 1 in the business cycle frequency range and exhibits much wilder fluctuations at higher frequencies, while the phase diagram shows greater shifts at frequencies close to the annual cycle, which is the dominant cycle in most seasonal patterns.

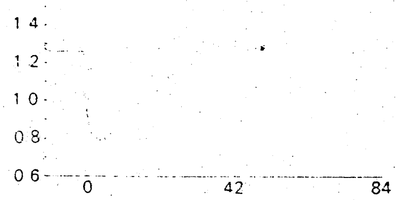

FIG. 6. Sum of squared filter coefficients: year-ahead filters and $a_i(L)$, $i = 0, 1, \ldots, 84$.

Following Bongard (1960) we calculate the sums of squares of the m.a. coefficients, as the "ratio of reduction of the irregular variance", and the results are plotted in Fig. 6. The 12 year-ahead filters are given first, beginning with that plotted in Fig. 5 and extending to the "1-month-ahead" filter, followed by the filters $a_i(L)$, $i = 0, 1, \ldots, 84$. Since the year-ahead filters have their leading coefficients equal to 1, their sums of squares all exceed 1, and there is thus a sharp break between the year-ahead filters and $a_0(L)$, whose sum of squared coefficients is [illegible]. The sum of squares of the later filters settles down rapidly, variation being concentrated at the beginning of the sequence.

Given modern computing capability, it is a practical possibility to run the *X*-11 program every month, rather than once a year, and the results in this section show that this is advantageous. Running the program every month implies that the first-announced adjusted figure is produced by the filter described in Fig. 1, rather than filters of the kind described in Fig. 5, and it is clear that the former is preferable. In the previous paragraph we have seen that the year-ahead filters *enhance* the irregular variance, moreover the break between these filters and the filters $a_i(L)$ indicates that greater first annual revisions are necessary when year-ahead adjustment is employed. These results suggest that year-ahead adjustment should be abandoned.

6. *X*-11-FORECAST PROCEDURES

As an initial step in the study of the properties of "*X*-11-forecast" procedures we ask the question, for what autoregressive model is the filter $a_{12}(L)$ applied to a series extended by 12 forecasts equivalent to the filter $a_0(L)$. For comparison with the approach of Kenny and Durbin (1982) we postulate the integrated-autoregressive model

$$(1-L)\,\phi(L)\,x_t = \varepsilon_t,$$

where $\phi(L)$ is of degree 25. For this model we construct the forecast coefficients for one-step, two-step, ..., twelve-step forecasts:

$$\hat{x}_{T+k} = \sum_{j=0}^{25} f_{kj}\,x_{T-j}, \quad k = 1, \ldots, 12.$$

We seek the ϕ-coefficients that make the one-sided filter implied by the application of $a_{12}(L)$ to the extended series, namely

$$\sum_{j=0}^{m} \left(a_{12,j} + \sum_{k=1}^{12} f_{kj}\,a_{12,-k} \right) L^j,$$

as close as possible to $a_0(L)$. A non-linear least squares solution to this problem is as follows:

i	1	2	3	4	5	6	7	8	9	10	11	12	
ϕ_i	−0·60	−0·59	−0·03	−0·15	0·13	0·08	0·16	−0·01	0·13	−0·10	0·25	−0·36	
i	13	14	15	16	17	18	19	20	21	22	23	24	25
ϕ_i	−0·24	0·42	−0·05	0·07	−0·10	0·06	−0·03	0·08	−0·06	0·09	−0·21	−0·33	0·62

In forecasting next period's change in x, ϕ_1 is the coefficient that multiplies the current month's change and ϕ_{25} the coefficient for the same month 2 years ago. A series having this autoregressive structure can be equally well adjusted either by the usual one-sided filter $a_0(L)$ or by applying $a_{12}(L)$ to a data series extended by 12 forecasts. That is, for such a series the *X*-11 filter $a_0(L)$ already minimizes the mean square of the first annual revision $r_t^{(0,12)}$.

It is common practice to compare such alternative methods by applying them to a sample of actual series, but calculations such as those presented above enable us to predict the results of these comparisons: for series with autoregressive structure close to that given above little difference in the methods will be found, while for series with considerably different autoregressive structure the "X-11-forecast" methods will perform much better than conventional X-11, provided that the model on which the forecasts are based gives a good representation of the behaviour of the series. To the extent that X-11-forecast procedures improve over the basic X-11 for certain series, the structure of those series differs from that already implicit in the X-11 weights.

7. CONCLUSION

By studying the linear filters implicit in seasonal adjustment procedures we are able to focus directly on the intrinsic properties of the statistical methods, in contrast to previous research, which has attempted to elucidate those properties by applying the procedures to a small sample of actual series and studying the results. A general characterization of the procedure, away from a particular context, allows the effect of seasonal adjustment on subsequent data analysis to be considered. This paper has shown how such a characterization can be obtained, and has indicated some of the questions to which it can be applied. A specific conclusion is that the practice of calculating seasonal factors for use up to 1 year ahead should be replaced by running the adjustment program every month.

In addition to presenting an approach to the study of seasonal adjustment procedures, and as an example of that approach, we have considered the X-11 method and some recent extensions. These were by-and-large taken as given, and the literature on alternative models, methods and criteria for seasonal adjustment has been left aside. Nevertheless, time series modelling considerations are not far away; for if a given procedure for adjusting historical data is acceptable, then the procedure for adjusting current data that minimizes revisions integrates the best forecasting rule with the historical procedure. Thus, once more, the problem becomes one of modelling.

ACKNOWLEDGEMENTS

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APPENDIX

Convolution of Two Linear Filter Arrays

The linear filter approximations to the X-11 procedure are obtained by calculating the net effect of a sequence of filtering operations. This is done by repeated application of an algorithm for the product or convolution of two sets of "time-varying" filters, which is described in this Appendix. If the output from filtering operation A is the input to filtering operation B, we calculate the coefficients of $C = B * A$. Note that if A and B each represent a single symmetric filter, then $B * A = A * B$, but this is not true of the more general case considered here.

To facilitate computer programming the filter coefficients have positive indices, and for ease of presentation we write the data as $X(t)$, $t = 1, \ldots, T$, in this Appendix. We define a filter array $A(k, j)$, $k = 1, \ldots, m_A + 1$, $j = 1, \ldots, 2m_A + 1$ where m_A is the half-length of the symmetric filter. $A(1, \cdot)$ is one-sided, with the first m_A coefficients zero, and $A(m_A + 1, \cdot)$ is symmetric. In general $A(k, j) = 0$ for $j \leq m_A - k + 1$, the first non-zero coefficient being $A(k, m_A - k + 2)$. Given data up to time T, the $m_A + 1$ most recent values of the filtered series are given as

$$X_A(T+1-k) = \sum_{j=m_A-k+2}^{2m_A+1} A(k,j)\, X(T-k+m_A+2-j), \quad k = 1, \ldots, m_A+1;$$

earlier values are obtained using the symmetric filter. In the calculation of $X_A(t)$, the coefficient of $X(t)$ is always the central coefficient $A(\cdot, m_A + 1)$. For example with these conventions, the 5-term Henderson m.a. discussed in Section 3 is set up as follows:

$$\begin{bmatrix} 0 & 0 & 0{\cdot}670 & 0{\cdot}403 & -0{\cdot}073 \\ 0 & 0{\cdot}257 & 0{\cdot}522 & 0{\cdot}294 & -0{\cdot}073 \\ -0{\cdot}073 & 0{\cdot}294 & 0{\cdot}558 & 0{\cdot}294 & -0{\cdot}073 \end{bmatrix}$$

We wish to calculate the coefficients in

$$X_C(T+1-k) = \sum_{j=m_C-k+2}^{2m_C+1} C(k,j)\, X(T-k+m_C+2-j), \quad k = 1, \ldots, m_C+1,$$

where C is the convolution of B and A, that is,

$$X_C(T+1-k) = \sum_j B(k,j)\, X_A(T-k+m_B+2-j).$$

B and C are set up according to the above conventions, and $m_C = m_A + m_B$. Since the asymmetric filters are represented by a complete row of the relevant array, with initial entries being zero, the lower limit in the above summations could always be taken as $j = 1$, but our algorithm keeps track of non-zero coefficients. The preceding equation can be written more precisely as

$$X_C(T+1-k) = \begin{cases} \displaystyle\sum_{j=m_B-k+2}^{2m_B+1} B(k,j)\; X_A(T-k+m_B+2-j), & k = 1, \ldots, m_B, \\ \displaystyle\sum_{j=1}^{2m_B+1} B(m_B+1, j)\, X_A(T-k+m_B+2-j), & k = m_B+1, \ldots, m_C+1 \end{cases}$$

and it is convenient to separate the two cases.

(i) $k \leqslant m_B$

We put $i = j - (m_B - k + 1)$, then

$$X_C(T+1-k) = \sum_{i=1}^{m_B+k} B(k, i-k+m_B+1)\, X_A(T-1-[\text{illegible}]$$

$$= \sum_{i=1}^{m_B+k} B(k, i-k+m_B+1) \sum_{h=m_A-i-2}^{2m_A+1} A(i, h) [\text{illegible}] - m_A - 2 - h)$$

$$= \sum_i \sum_l B(k, i-k+m_B+1)\, A(i, l-i+k-m_B [\text{illegible}] -k+m_C+2-l),$$

where $l = i+h-k+m_B$, hence the C-coefficients are given by

$$C(k, l) = \sum_i B(k, i-k+m_B+1)\, A(i, l-i+k-m_B), \quad l = m_C [\text{illegible}] 2m_C-1$$

the limits of summation for i being $\max(1, l+k-m_A-m_C-1)$, $\min$ [illegible] $- m_B - 1, m_B + k)$.

(ii) $k \geqslant m_B + 1$

$$X_C(T+1-k) = \sum_{j=1}^{2m_B+1} B(m_B+1, j)\, X_A(T-k+m_B+2-j)$$

$$= \sum_j B(m_B+1, j) \sum_i A(k+j-m_B-1, i)\, X(T-(k+j [\text{illegible}] -1)+m_A+2-i)$$

$$= \sum_j \sum_l B(m_B+1, j)\, A(k+j-m_B-1, l-j+1)\, X(T [\text{illegible}] m_C+2-l)$$

hence we have

$$C(k, l) = \sum_j B(m_B+1, j)\, A(k+j-m_B-1, l-j+1), \quad l = m_C [\text{illegible}] 2m_C+1.$$

where the limits of summation for j are $\max(1, l-2m_A)$, $\min(l, 2m_B$ [illegible]

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APPLICATION OF THE GENERAL LINEAR MODEL TO SEASONAL ADJUSTMENT OF ECONOMIC TIME SERIES

BY RICHARD C. HENSHAW, JR.[1]

For a catholic seasonal adjustment method for monthly economic time series, the general linear model and mutually independent random disturbances with zero mean and constant variance, in the special case with components consisting of twelve seasonal polynomials in t (time) of low degree and a nonseasonal polynomial in t of higher degree have been employed. A cogent set of test results consisting of best (minimum-variance) linear seasonal estimations and adjustments for the common logarithms of the monthly economic time series, "Shipments of Portland Cement in the United States, 1957–61," indicates that this is a theoretically and computationally promising approach now that large-capacity, high-speed electronic computers are available. The author has been attempting since 1959 to validate empirically the feasibility of this model.[2]

The history of statistical theories of seasonal adjustment is also briefly reviewed.

1. INTRODUCTION

THE SEASONAL components of economic time series are unknown functions of climatic patterns, social customs, and calendar variations. We consider here the problem of best (minimum-variance) linear seasonal estimations and adjustments of monthly economic time series. We shall not be concerned with methods in actual use that employ link relatives or ratios to moving averages, since no statistical theory has been or probably even could be developed for such models [**11**, p. 16] and [**19**, p. 684].

Since the exact nature of interactions between the seasonal and nonseasonal components in economic time series is unknown, it is not possible to claim complete unbiasedness for the estimates. As a reminder of possible biases, the seasonal and nonseasonal components being estimated will be referred to as periodicities and nonperiodicities, respectively.

Proposals that polynomials be employed in seasonal adjustment models have been closely associated with efforts to develop statistical theories of seasonal adjustment. Those of Cowden (1942) [**3**], Jones (1943) [**18**], Hald (1948) [**8**], and an

[1] The problem of seasonal adjustment has held a strong fascination for me ever since I was introduced to it by J. R. Stockton in 1948. Through the years I have also been helped in my research, writing, and computation on this subject by many others including T. Farrell, R. L. Gustafson, C. G. Hildreth, H. L. Jones, R. E. Lane (deceased), J. N. K. Rao, W. L. Ruble, W. Sparks, G. Tintner, and various editors and referees including a very articulate referee of [**15**] who surely must be a great teacher. D. W. Jorgenson provided several important references after reading [**11**] in February, 1963. I, of course, am solely responsible for errors and deficiencies.

[2] The estimates in this paper were presented in [**11**] which was read September 7, 1963 on the Econometric Society Program in Cleveland, Ohio. This paper is an up-dated revision of [**11**] and incorporates, among other things, some references to, and discussion of, subsequent works.

important paper by Hannan (1960) [9] should be mentioned. Cowden suggested treating an evolving seasonal pattern by sequentially fitting 12 polynomials in t by least squares, one to the set of January specific seasonals (i.e., ratios to 12-month moving averages), another to the set of February specific seasonals, etc. Jones and Hald considered the problem of a fixed seasonal pattern only and may be considered to have suggested simultaneously fitting by least squares 12 zero-degree polynomials to the seasonal components and a general order polynomial to the nonseasonal component of a monthly economic time series. They represented the disturbances by mutually independent normal deviates with zero mean and constant variance. Cowden and Jones did not attempt to test their proposals empirically but Hald did make an unsuccessful attempt to verify the model empirically.

Most econometricians and statisticians, however, seemed to reject the reasonableness of a general assumption of mutually independent random disturbances in monthly econometric models, e.g., Hurwicz (1950) [**17**, p. 336], Klein (1953) [**20**, p. 314], Hannan (1960) [**9**, p. 1] and (1963) [**10**, p. 31], Durbin (1961) [**4**, p. 131] and (1963) [**5**, p. 3], and Ladd (1964) [**21**, p. 410]. Under Hannan's general assumptions of a stationary autocorrelated residual, least squares and moving average estimators of fixed or evolving seasonal variations are efficient, but the estimators do not possess best linear properties.

Interest in using polynomials in seasonal adjustment models also subsided until the above papers by Hannan and Durbin, Lovell (1963) [**22**], Rosenblatt (1963) [**23**], and the author (1961) [**15**], (1963) [**11**], and (1964) [**12**].

Undoubtedly the likelihood of autocorrelated disturbances in econometric models does increase sharply when annual data are disaggregated into shorter time periods, such as quarterly or monthly series, etc. This opinion seems to be so thoroughly acceptable to researchers that no papers based upon a general assumption of mutually independent random disturbances in seasonal adjustment models had appeared since Hald's 1948 paper until the above mentioned papers by the author. A 1962 paper by Abel [**1**] was based upon a *special* assumption of mutually independent random disturbances in a monthly series on hog production. These papers were followed by another paper on this subject by Jorgenson (1964) [**19**]. The papers by the author mentioned above, like this one, were primarily devoted to an attempt to verify empirically the validity of a *general* assumption of mutually independent random disturbances when a general linear model, with polynomial components like the ones in Hannan's 1960 paper, is employed.

The surprising claims made in Jorgenson's 1964 paper must be viewed with considerable skepticism, except possibly for the special case of the polynomial model considered in this paper, in view of the prevailing negative attitude of econometricians and statisticians in regard to a general assumption of mutually independent random disturbances in monthly econometric models. Jorgenson did not present any empirical verification of his assumptions anu conclusions nor did he refer to any specific empirical work that supported them.

The point of view developed in this paper is that the general linear model with polynomial components and regression methods currently offer the most promising solutions to problems of seasonal estimations and adjustments.

2. THE MODEL

The basic model is:

$$(1) \qquad y_t = C_t + S_t + e_t, \quad t = 1, 2, \ldots, T,$$

where C_t is the nonperiodicity (better known as trend-cycle or just trend in the nomenclature of economic statistics), S_t is the (seasonal) periodicity, and the e_t are supposed to be mutually independent normal deviates with zero mean and constant variance. C_t will be describable by a polynomial in t of degree c; S_t will be describable by 12 polynomials in t, one for each month of the year, and each of degree s such that $c > s \geq 0$.

In practical applications the 12 S_t polynomials can be expected to be of lower degree than the C_t polynomial; C_t itself should be of low enough degree so that estimates with good small sample properties can be computed. There are sound economic reasons for expecting the 12 S_t functions to be of very low degree.

(a) Intra-annual climatic variations are important causes of seasonal variations but inter-annual climatic variations are neither very great nor systematic.

(b) The basic social customs that significantly affect seasonal variations such as Christmas giving, summer vacations, etc., are characterized by a high degree of inertia.

(c) Likewise, intra-annual calendar variations are important causes of seasonal variations but inter-annual calendar variations, with the possible exception of the changing date of Easter, are neither great nor very systematic. The model presented here can be modified for application to data which have been affected by the changing date of Easter, but this problem is not considered here.

Of course, the C_t function will be of relatively high degree compared to the S_t functions. For heavily aggregated, monthly economic time series, however, the degree c of the trend polynomial has not been found to be so high that good estimates of C_t are computationally impractical. Applications of the variate difference method by the late 0. Anderson [2], G. Tintner [25], and others probably constitute the largest body of empirical evidence on this point. We remark that this view of C_t differs sharply from that which has been stated by Hannan [9, p. 10].

If (1) were a behavioral equation, we might not be able to state it sufficiently well to eliminate large systematic variables in e_t because of the unavailability of accurate data for some of the explanatory variables and our ignorance of their precise relationships with y_t. We shall assume that it is possible to eliminate systematic variables in e_t of model (1) by employing polynomials in t of appropriate degrees and configurations.

In cases where a simple interaction (multiplicative) model is more realistic, the observed values should be transformed to logarithms in order to reduce the model to the additive form and remove the biases due to interactions among the components. Also, for such cases, least squares residuals should be more valid as estimates of mutually independent homoscedastic random disturbances.

We may write the nonperiodicity C_t as follows:

$$(2) \qquad C_t = \sum_{v=0}^{c} \lambda_v t^v .$$

The periodicity S_t may be written:

$$(3) \qquad S_t = \sum_{j=1}^{12} \sum_{v=0}^{s} (\lambda_{vj} - \lambda_v) \Upsilon_{jt}^v$$

where $\Upsilon_{jt}^v = t^v$ if $t-j$ divided by 12 is an integer and is zero otherwise.

A widely accepted convention in empirical work is based upon the premise that seasonal variations neither add to nor diminish the aggregate value of an economic time series. Thus, we may set:

$$(4) \qquad \sum_{t=1}^{T} S_t = 0$$

by defining λ_v as the weighted average of λ_{vj} with weights Υ_{jt}^v.

$$(5) \qquad \lambda_v = \frac{\sum_{tj} \lambda_{vj} \Upsilon_{jt}^v}{\sum_{tj} \Upsilon_{jt}^v}, \quad v=0, 1, \ldots, s .$$

Therefore, model (1) may be formulated directly as a set of polynomials in t by adding equations (2) and (3) and collecting like terms.

$$(6) \qquad y_t = \sum_{j=1}^{12} \sum_{v=0}^{s} \lambda_{vj} \Upsilon_{jt}^v + \sum_{v=s+1}^{c} \lambda_v t^v + e_t$$

The variables Υ_{jt} and t^v in (6) may be referred to as attributive and nonattributive time variables, respectively. For a given value v the differences in the coefficients λ_{vj}, $j=1, 2, \ldots, 12$, one for each of the 12 seasonal polynomials, respectively, are attributable to the different months of the year. The coefficient λ_v is, however, the same for every month of the year, and, hence, is not attributable to the different months of the year, i.e., there is only one nonseasonal polynomial. One can also view the model as consisting of 12 polynomials with unique lower degree coefficients and identical higher degree coefficients; see [**11**, p. 686].

Even if the normality assumption is relaxed, it is well known that the calculation of best linear unbiased estimates $\hat{\lambda}$, of the coefficients in (6), conditional upon prescribed values of s and c, can be accomplished by least squares. It is probably not feasible, however, to transform the matrix so that the parameters are orthogonal. This means that a high order matrix must be inverted but it will be well conditioned; nevertheless a double precision computer program may be required if the degree c of the trend polynomial C_t is larger than seven.

Consider the following modification in the specifications of model (1) *et al.*

$$y_t = C_t' + S_t' + u_t, \quad t = 1, 2, \ldots, T,$$

(7)

$$u_t = \delta u_{t-1} + e_t', \quad 0 \leq \delta < 1,$$

where u_t is a first-order autoregressive disturbance, the e_t' are mutually independent normal deviates with zero mean and constant variance, C_t' is a polynomial of arbitrary degree c', and S_t' is a set of 12 polynomials, each of arbitrary degree s' such that $c' > s' \geq 0$. By assumption, in cases where $c' \geq c$ and $s' \geq s$, δ will be zero, and the coefficients λ_{vj} and λ_v of higher degrees than v equal s and c, respectively, will be zero. Therefore, model (1) is a special case of model (7) in which $c' = c$ and $s' = s$; in cases in which $s' \neq s$ or $c' \neq c$, efficient estimates, but not estimates with optimal small sample properties, of the periodicities S_t and nonperiodicities C_t can be calculated by ordinary least squares.

We need to decide what departures from serial independence of the disturbances are likely to occur when model (7) is applied. Experience with economic time series indicates a test against positive autocorrelation and in the most usual case a test against positive autocorrelation at the 5 per cent level has been employed [7, p.161].

Responsibility for the selection of a criterion for accepting or rejecting the null hypothesis H_0: $(\delta = 0)$ must, of course, rest with the analyst, based upon his judgment and the objectives of his study. The procedural criterion for estimating s and c recommended here is as follows.

In the initial trial we advise the analyst to assume values of s and c at least large enough in his opinion to make the null hypothesis H_0: $(\delta = 0)$ the valid one as opposed to the alternative hypothesis H_a: $(\delta > 0)$. Then he should calculate the von Neumann ratio of least-squares estimated regression disturbances d[3] and compare it with the 5 per cent significance point of d. If H_0: $(\delta = 0)$ is acceptable in the test against H_a: $(\delta > 0)$ at the 5 per cent level, we suggest that he determine the highest significance level at which it is acceptable and proceed to find, by trial and error, the lowest degree $(\hat{s}, \hat{c})$ set of polynomials in t for which H_0: $(\delta = 0)$ is acceptable at approximately that significance level; then adopt (tentatively) these values of s and c. A comparison of the result of the significance point test of d with the usual analysis of variance F and t tests of the estimated regression coefficients

[3] d is defined in a footnote to Table I.

may be used to help confirm the (tentative) conclusions reached by means of the former test.

If H_0: ($\delta = 0$) is rejected at the 5 per cent level in the initial trial and it is not computationally feasible to try larger assumed values of s or c, determine the actual significance level of the calculated value of d and proceed, by trial and error, to find the lowest degree ($\hat{s}$, $\hat{c}$) set of polynomials in t for which H_0: ($\delta = 0$) is acceptable at approximately that level. Then apply the Hildreth–Lu procedure, which is completely described in [**16**], to obtain maximum likelihood estimates of δ and λ.

There are great advantages of working with a standard form such as the general linear model with independent disturbances. Employment of a maximum likelihood procedure, such as that by Hildreth and Lu, which is quite expensive computationally, should be regarded as a last resort. In all cases, when the null hypothesis H_0: ($\delta = 0$) can be accepted at the 5 per cent level it seems better to take the risk that H_a: ($\delta > 0$) is, in fact, the true hypothesis and to go ahead and incur the possible losses in reliability in estimating the periodicities and nonperiodicities by ordinary least squares, rather than take the alternative risk of using the Hildreth-Lu procedure when it is not required.

The best known significance point test of d is the Durbin-Watson "bounds test" [**6**] and [**7**, pp. 159–62]; a major drawback of this test is that often it may be inconclusive when model (7) is applied. Also, the Theil-Nagar test [**24**] will not be applicable because the first differences of the attributive time variables in model (7) will be large in absolute value compared to the corresponding range of the variable itself.

All things considered, we suggest that the approximate procedure, which Durbin and Watson [**7**] recommend when the bounds test is inconclusive, be used immediately whenever model (7) is applied. The bounds test has not been tabulated for the values that would be required in typical applications of model (7); since the approximate procedure should be sufficiently accurate for the types of problems considered here and the bounds test often may be inconclusive, the expense of calculation required to apply it seems unjustified. A complete description of the Durbin-Watson approximate procedure appears in [**7**, pp. 161–6] so it need not be discussed here. All of the above mentioned tests and procedures are described and compared in a recent paper by the author [**14**] along with a new test that is always conclusive and should be accurate even in problems in which the number of degrees of freedom is small.

The derivation of best linear estimators $\hat{y}_t$, $\hat{S}_t$, and $\hat{C}_t$, conditional upon prescribed values of s and c, is a straightforward application of the invariance property of extremum estimates. There are two ways of making seasonal adjustments:

(8) $\hat{C}_t = \hat{y}_t - \hat{S}_t$, and

(9) $\tilde{C}_t = y_t - \hat{S}_t$.

Since $\hat{C}_t$ provides a best linear seasonal adjustment, it will be preferred within the range of time periods included in the regression, i.e., $t=1, 2, \ldots, T$. $\tilde{C}_t$ may be of interest, however, for making preliminary seasonal adjustments in subsequent periods, i.e., $T+1, T+2$, etc. We remark that $\tilde{C}_t$ is similar to the method of seasonal adjustment recommended by Hannan [**10**, p. 32] although it differs in important respects.

3. AN EXAMPLE

The markets for Portland Cement are long established and well developed so the seasonal and nonseasonal components of demand undoubtedly interact. In such cases, the simplest model that is likely to be of much use in estimating seasonal variations and making adjustments is a multiplicative one. Therefore, in the following example we transform the data to logarithms in order to calculate conditional best linear seasonal estimates and adjustments for the monthly time series, "Shipments of Portland Cement in the United States, 1957–61." Initially we assume that model (7) is being applied and then we show that model (1) with $\hat{s}=0$, $\hat{c}=4$ appears to give a good approximation.

The experience of this writer in analyzing heavily aggregated monthly economic time series, such as shipments of Portland Cement in the United States, suggests that after a transformation to logarithms, the S_t periodicities are likely to be describable by 12 polynomials of degree at most 1 and the C_t nonperiodicities will be describable by a polynomial of degree roughly equal to the number of years included in the time series. The incremental cost of revising the analysis of a time

TABLE I

STATISTICS USED IN DETERMINING $(\hat{s},\hat{c})$

Assumed Values (s,c)	No. of Degrees of Freedom[1] $(60-k)$	von Neumann Ratio[2] d	Estimates[3] p	q	Durbin-Watson Approximate Test: Statistic[4] F	$F_{.50}(2q; 2p)$
(1,7)	30	2.51	19.0	14.5	0.44	1.01
(0,4)	44	2.23	26.5	23.0	0.93	1.00
(0,3)	45	1.84	26.0	23.5	1.28	1.00

1. $k = 12(s+1) - s + c$.

2. $d = \dfrac{\sum_{t=2}^{T} (\hat{u}_t - \hat{u}_{t-1})^2}{\sum_{t=1}^{T} \hat{u}_t^2}$, where $\hat{u}_t$ is a least squares residual for period t.

3. p and q are parameters of the Beta function; see [7, p. 165].

4. $F = \dfrac{p(4-d)}{qd}$ with $n_1 = 2q$, $n_2 = 2p$ degrees of freedom; see [7, p. 165].

series as new data become available can be expected to be small compared to the cost of the initial analysis since an approximate idea of the degree $(\hat{s}, \hat{c})$ of the set of polynomials will be available when up-dating the analysis.

After calculating the von Neumann ratio (**d**) for only three assumed sets of (s, c): (1, 7) (0, 4), and (0, 3), it appeared that (0, 4) was a good choice for $(\hat{s}, \hat{c})$. These results are reported in Table I.

In the initial trial we assumed $s=1$ and $c=7$ in order to allow for the possibility of linear trends in the 12 periodicital functions, and to be reasonably certain of fitting a polynomial of sufficiently high degree to the nonperiodicities. *A priori* we did not expect to find changing periodicities of higher degree than $s=1$ or nonperiodicities of higher degree than $c=7$ in the logarithms of the data.

On the basis of the statistics presented in Table I we see that the null hypothesis H_0: $(\delta=0)$ is acceptable at greater than the 50 per cent level by the Durbin-Watson approximate procedure [7, pp. 161–6] when assumed values of $s=1$ and $c=7$ were tried. Acceptance of the null hypothesis H_0: $(\delta=0)$ by a 50 per cent or greater significance point test of d may be interpreted to be unconditionally

TABLE II

ESTIMATES $\hat{\lambda}$, $\hat{Z}$, $\hat{\lambda}^{\cdot}$, AND THEIR t VALUES

Estimator $\hat{\lambda}$	Least Squares Estimates	Standard Errors	t Values[1]	Estimator[2] $\hat{Z}$	Estimates	Standard Errors	t Values[1]
$\hat{\lambda}_{0,1}$	4.129	.0277	149.3	$\hat{Z}_{0,1}$	−.265	.0153	−17.3
$\hat{\lambda}_{0,2}$	4.146	.0284	145.9	$\hat{Z}_{0,2}$	−.247	.0152	−16.3
$\hat{\lambda}_{0,3}$	4.308	.0291	148.2	$\hat{Z}_{0,3}$	−.086	.0151	−5.7
$\hat{\lambda}_{0,4}$	4.421	.0296	149.4	$\hat{Z}_{0,4}$	.027	.0151	1.8
$\hat{\lambda}_{0,5}$	4.493	.0300	149.6	$\hat{Z}_{0,5}$	.099	.0150	6.6
$\hat{\lambda}_{0,6}$	4.519	.0304	148.7	$\hat{Z}_{0,6}$	.125	.0150	8.3
$\hat{\lambda}_{0,7}$	4.505	.0306	147.0	$\hat{Z}_{0,7}$	.111	.0150	7.4
$\hat{\lambda}_{0,8}$	4.559	.0308	147.9	$\hat{Z}_{0,8}$	.165	.0150	11.0
$\hat{\lambda}_{0,9}$	4.525	.0309	146.4	$\hat{Z}_{0,9}$	.131	.0151	8.7
$\hat{\lambda}_{0,10}$	4.526	.0309	146.3	$\hat{Z}_{0,10}$	.131	.0151	8.7
$\hat{\lambda}_{0,11}$	4.369	.0309	141.4	$\hat{Z}_{0,11}$	−.025	.0152	−1.7
$\hat{\lambda}_{0,12}$	4.230	.0308	137.3	$\hat{Z}_{0,12}$	−.165	.0153	−10.8
$\hat{\lambda}_1$	−.0114	$.0^2573$	−2.0				
$\hat{\lambda}_2$	$-.0^2102$	$.0^3375$	2.7				
$\hat{\lambda}_3$	$-.0^4271$	$.0^5919$	−3.0				
$\hat{\lambda}_4$	$.0^6224$	$.0^7747$	3.0				
Estimator $\hat{\lambda}^{\cdot}$							
$\bar{\hat{\lambda}}_0$	4.394	.0260	169.3				

1. The 1 and 5 per cent one-tail significance points of t, with 44 degrees of freedom, are approximately 2.42 and 1.68, respectively.
2. $\hat{Z}_{vj} = (\hat{\lambda}_{vj} - \hat{\lambda}_v)$.

favorable evidence in support of the null hypothesis $H_0:(\delta=0)$, since such a result implies a negative serial correlation in the residuals which, under the specifications of model (7), could only occur due to chance. The lowest degree ($\hat{s}$, $\hat{c}$) set of 12 S_t polynomials and one C_t polynomial for which the null hypothesis $H_0:(\delta=0)$ is accepted by the 50 per cent significance point test of d, appears to occur for (0, 4). When (0, 3) was tried the test statistic fell within the critical region of the test.

As a further check, the usual analysis of variance F test was made of the null hypothesis H_0: $(\lambda_{1,j}-\lambda_1=\lambda_5=\lambda_6=\lambda_7=0)$ for $j=1, 2, \ldots, 12$, simultaneously. $F(14;30)=.420$ was obtained which is much less than $F_{.50}(14;30)=.970$ (the 50 per cent significance level of F).

The estimates and their standard errors, presented in Tables II and III, were calculated on the assumption that model (1) with $\hat{s}=0$, $\hat{c}=4$ provides a valid approximation. The usual t values of the coefficients in Table II, like the F test above and the Durbin-Watson test, provide evidence which supports the decision to select $\hat{s}=0$ and $\hat{c}=4$.

It is clear from the conditional standard errors of the projected estimates for 1962 in Table III that for forecasting purposes $\hat{Y}$ and $\hat{C}$ may have some value within the near range of a few months. Subsequently, not only do the standard errors rise sharply but even more serious are the obvious biases which afflict the estimates. Of course, extrapolation is not a major objective of seasonal adjustment procedures since they are primarily used to adjust current and historical data. In fact, seasonal adjustment procedures do not employ behavioral models so it is not to be expected that any of them will give an extrapolation outside of one or two periods forward in time that is good enough for any practical purpose.

The estimator $\tilde{C}=Y-\hat{S}$ is not really projectable because the actual values y_t with $(t=T+1, T+2, \ldots)$ for which the estimates are to be made must be available. Obviously in applications in which the periodicital polynomials are of degree $\hat{s}=0$, the analyst may be willing to use the estimator $\tilde{C}$ outside the range of the regression if the standard errors of the estimates are not too large for his purposes, and he is confident that the periodicities have not started changing.

In applications where the periodicital polynomials are of degree $\hat{s}\geq 1$, the standard errors and biases of the estimator $\tilde{C}=Y-\hat{S}$ will increase as the time variable t takes on values outside the range of the regression in a similar manner but not as rapidly as $\hat{C}$ and $\hat{Y}$. Generally the reliability of such estimates will be too low to be of much value. In fact, even for a case of $\hat{s}=0$, we see from Table III that the reliability of $\tilde{C}$ within the range of the regression was not very great. The standard errors of $\tilde{C}$ also can be used as an indicator of the optimal reliability of seasonal adjustments by moving average methods in which adjustments are made by dividing the seasonal estimations into the observed data or by subtracting when the log form is used.[4]

[4] This statement does not apply to methods which utilize ratios to moving averages. It applies to methods in which moving averages are utilized in order to annihilate the trend [9], [10], and [19].

Moving Seasonal Adjustment of Economic Time Series

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SUMMARY

An attempt is made to provide a theoretically sound method of seasonal adjustment, based on the traditional model of trend, seasonal and irregular components of a time series. The trend is removed by an optimal 13-term moving average, and the seasonal component estimated by smoothing the harmonic amplitudes of the deviations from trend (allowing for the loss of amplitude due to the moving average filter). The missing trend-values at the ends of the series due to taking a moving average are estimated by the adaptive prediction procedure of Box and Jenkins (1962). A computer programme has been developed, which shows the method to be flexible enough to leave no residual seasonality, while not being liable to large revisions when updating.

1. INTRODUCTION

THREE kinds of techniques have been employed in seasonally adjusting economic time series, all assuming the classical model (Trend–Seasonal–Irregular).

The first is the method of linear regression, for which the model must specify the exact form of the trend and seasonal components: assuming an additive model, the trend would be represented by a polynomial or low-frequency periodic function of time, and the seasonal component by a series of dummy variables corresponding to the months of the year. If the seasonal pattern is believed to vary over time, further dummy variables can be introduced to allow the amplitudes of the periodic components to be themselves polynomial functions of time. A great disadvantage of this method is that economic series are often found not to be expressible in terms of simple mathematical functions: for example, a trend may be quite flat over a number of years and then rapidly change to a steeper gradient, or it may contain a lot of low-frequency cyclical components whose period is unknown.

Consequently a second method has been much more popular: the use of a moving average to eliminate the trend, followed by a linear regression analysis of the deviations from trend, i.e. assuming that the seasonal factors are constant or vary linearly with time. The advantage of this method is that a suitably chosen moving average will remove polynomial trends up to say the fourth degree and *all* low frequency components. The disadvantage is that one is still assuming that variations in the seasonal pattern take a simple mathematical form, while empirical evidence suggests that they may at times change abruptly and then settle at the new level.

The third method, which is fully adaptive, is the type considered in this paper. For each month of the year deviations from trend are smoothed by moving averages over a run of years. This method avoids the disadvantages of the other two at the expense of having a very flexible model in which the distinction between trend, seasonal and irregular components has become blurred. The danger now is that there is no "right" answer and some widely used methods are too flexible, i.e. the results too sensitive to the arrival of additional data.

It is the purpose of this paper to show that the large number of empirical rules employed in adaptive methods can be replaced by a more rational procedure, guided by the results of some simple spectral analysis. This procedure is stable (in the sense that the arrival of fresh data does not much disturb previous seasonal adjustments) but flexible enough to leave no significant residual seasonality in the adjusted series.

2. Classical Method

The standard method of moving seasonal adjustment (see Shiskin and Eisenpress, 1957), known as the Bureau of the Census Method II (hereafter called Census II), is used all over the world. A similar method has been developed by the Bureau of Labor Statistics (B.L.S. method) (see Employment Statistics Committee, 1962). Both assume that seasonality is multiplicative, i.e. a percentage upward or downward bias in each month of the year. The model is

$$Y_t = TSI,$$

where Y_t is the time series, T is the trend (thought of as a slow-moving curve associated with business cycles and long-term growth), S is a periodic function associated with the months of the year and such that the average of its 12 values is unity and I is the irregular or random component. The stages in the Census II calculation are detailed below. But for those who are not familiar with this subject, the large number of empirical rules may be rather confusing: they may pass straight to Section 3 where the basic features of the method are discussed.

1. Form centred 12-month moving average of observations (i.e. 2-term average of 12-term average, so that it is centred on the middle of the months).
2. Divide moving average into the observations to obtain monthly ratios.
3. Identify and replace "extreme" ratios (see below). Six missing ratios at each end, due to use of 12-month moving average, are extrapolated by repeating first or last available ratios for these months.
4. Centre the 12 ratios for each year, i.e. divide them by their arithmetic mean.
5. Compute [3] [3] moving average of ratios for the same month over successive years; missing averages at the end of the series are extrapolated by adding two extra ratios equal to the mean of the last two; a similar procedure is used at the beginning of the series. These averages are the preliminary monthly seasonal factors (M.S.F.).
6. Divide preliminary M.S.F. into original observations to obtain preliminary adjusted series.
7. Form 15-month moving average (Spencer's formula) of adjusted series. To avoid loss of trend at end of series, extrapolate 7 extra terms by the average of last 4 terms of adjusted series; treat beginning of series similarly.
8. Divide preliminary adjusted series by its moving average to obtain "irregular ratios". Mean of absolute first differences of these (called d) gives a measure of the irregular component of the series.
9. Repeat steps 2–4.
10. Compute [3] [3] or [3] [5] moving average of ratios for each month—the choice depending on whether $d < 2$ per cent or $d \geqslant 2$ per cent; extrapolate end averages as before. These are the final M.S.F.
11. Divide the final M.S.F. into original observations to obtain final adjusted series.
12. Extrapolate the final M.S.Fs for the next year thus: factors for latest year plus half the difference between the latest factors and those for the previous year.

The method of replacing extreme values of monthly ratios is not important to our purpose; but we observe that it depends on estimating a standard error for each month, based on the deviations of the ratios from their 5-term moving average.

The following serious criticisms have been made of Census II:

(i) If the model is multiplicative, it seems illogical to take arithmetic instead of geometric means at steps 1, 5, 7 and 10.

(ii) The centred 12-month moving average suppresses the seasonality completely but is too inflexible in measuring the trend. The Spencer average is flexible enough, but cannot be used at first because it does not suppress all the seasonality. The first round in Census II will measure not only the purely seasonal component, but also that part of the trend which the 12-month moving average has failed to remove. On the second round the Spencer moving average should separate the remainder of the trend from the seasonal component. If not, a further round could be introduced using the Spencer average on the "final" adjusted series to obtain a new trend and a third set of seasonal factors—and so on.

Presumably this iterative process converges to a limit. But Hannan (1963) has shown that the convergence could be slow and that the two stages in Census II might still leave a considerable bias in the estimates. It should also be noticed that, each time an averaging process is employed, additional observations are needed or extrapolated at the ends of the series: for example, to obtain a final set of seasonal factors for a single year needs a total of 134 observations, if extrapolation is to be avoided!

(iii) The moving average in steps 5 and 10 takes the place of free-hand curve fitting in the older manual methods. It seems likely that a different moving average should be used for smoothing different months of the year (see Marris, 1961). In cases where the seasonal pattern changes slowly relative to the irregular component, the [3] [5] and, more particularly, [3] [3] averages are likely to be too much influenced by local irregularities—leading to large subsequent revisions when more data become available. On the other hand, if a series of choices of smoothing average is given [e.g. the X-10 version of Census II developed by the Bureau of the Census (1962)], the M.S.F. are no longer centred (i.e. arithmetic mean $\neq 1$). Subsequent centring will tend to upset the smoothing of the individual months.

(iv) The method of extrapolation in steps 3, 5, 10 and 12 is too rigid: it should depend on more or fewer of the most recent M.S.F. according as the irregular component is large or small relative to the movement in the seasonal ratios. An attempt has been made to take care of this in the X-10 version, by linking the choice of extrapolator with the choice of smoothing average.

(v) The decision about which monthly ratios are "extreme" is based on the evidence of a short series of years for each month. The value of 2σ limits in such small samples must be very dubious; and it is found in practice that quite small revisions to previous data have a marked effect through increasing or decreasing the number of "extreme" observations.

(vi) The method of extrapolating the trend in step 7 is extremely crude and can lead in practice to overweighting the influence of the last 4 terms in assessing the latest seasonal pattern.

(vii) A more fundamental criticism has been made by Nerlove (1962), which depends on the theory of spectral analysis. If a time series is analysed as a whole, the trend is represented as a peak in the spectrum at low frequencies. If the series consists of monthly data showing an annual seasonal pattern, there will be subsidiary peaks at the corresponding frequencies, i.e. $2\pi k/12$ $(k = 1, 2, \dots, 6)$. The irregular component may be equated to the continuous spectrum at higher frequencies (say, above $\pi/3$). Seasonal adjustment of the data should be equivalent to chopping off the seasonal peaks in the spectrum, leaving the remainder unchanged.

Nerlove used a simple variate difference filter to remove the enormous low-frequency power from the spectrum, analogous to the removal of trend by the Spencer moving average, in order to avoid distortion of his spectral estimates at the higher frequencies. After estimation, the spectrum was "recoloured", i.e. the loss due to the use of a filter was restored. The spectrum of the seasonally adjusted series, after using Census II and the analogous B.L.S. method, showed a considerable loss of power compared with the original series at frequencies other than the seasonal ones. Of course, it is impossible to remove seasonality in such a way as to leave the neighbouring frequencies entirely unaffected; and the seasonal peaks will not be sharp spectral lines owing to changes in the seasonal pattern. But Nerlove concludes that the Census II and B.L.S. methods remove much more power than necessary in attempting to measure the moving seasonality.

3. Comparison of Classical and Revised Methods

The first step in a more logical approach is a logarithmic transformation. Our model is then one of additive seasonality,

$$\log Y_t = \log T + \log S + \log I,$$

which we write as

$$y_i = m_i + u_i + \epsilon_i$$

with

$$\sum_{i=1}^{12} u_i = 0.$$

Of course, after seasonal adjustment an exponential transformation is made.

If the ratios in steps 2, 4, 6, 8 and 11 of Census II are replaced by subtractions, we can summarize the procedure under the following heads:

(i) Removal of trend by subtracting a moving average to give monthly deviations.
(ii) Centring the 12 monthly deviations in each year, i.e. deducting their mean, so that they now add to zero.
(iii) Smoothing the deviations for each month over successive years to obtain average monthly seasonal differences (A.M.S.D.).
(iv) Extrapolating the A.M.S.D. to cover the ends of the series where terms have been lost by averaging at stages (i) and (iii).
(v) Subtracting the A.M.S.D. from the original series to give a seasonally adjusted series.

Somewhere in this scheme there must be a rule for eliminating and replacing "extreme" observations, which cannot for lack of information be adjusted beforehand (e.g. a month of exceptional weather or a strike).

These topics will now be dealt with in turn.

4. Trend Removal

Hannan (1963) and Durbin (1963) have shown that the repetition of moving averages to remove trend is unnecessary. If the trend is removed by a single symmetric moving average (e.g. the Spencer average), the residual series will still have a seasonal pattern, but different from that of the original series. It is in fact a linear function of the u_i. So that, after estimating the seasonal pattern of the residual series, unbiased estimates of the u_i can be obtained by solving a set of 12 linear equations. (The matrix of these is of rank 11, but a twelfth equation is provided by $\Sigma u_i = 0$.)

This may be seen more clearly if the various linear processes are considered as spectral filters. In spectral analysis a time series is analysed into a continuous range of frequencies—the spectrum. These frequencies are measured on the angular scale, i.e. 2π is the frequency of a wave whose period is equal to the interval of data measurement. The highest frequency that can be measured is π, which corresponds to a period of two intervals, i.e. an oscillation which rises and falls in alternate intervals. The spectrum therefore ranges from 0 to π. The properties of a moving average can be described by its effect on a time series $e^{i\omega t}$ consisting of a sine wave of angular frequency ω and constant amplitude. Thus, if $[d_s]$ $(s = -m, -m+1, \ldots, m)$ is the moving average, the result of applying it to $e^{i\omega t}$ is

$$\sum_{s=-m}^{m} d_s e^{i\omega(t+s)} = \left(\sum_{s=-m}^{m} d_s e^{i\omega s}\right) e^{i\omega t}.$$

Thus the original wave emerges unaltered in frequency but multiplied by the frequency-response function, which is a complex number:

$$f(\omega) = \sum_{s=-m}^{m} d_s e^{i\omega s}.$$

The modulus of this number is the amplitude "gain" and the angle of its argument represents a phase shift. In the special case of the symmetric moving average, with which we shall be solely concerned,

$$f(\omega) = d_0 + 2\sum_{s=1}^{m} d_s \cos \omega s,$$

which is real. If $f(\omega)$ is positive, there is no phase shift; if it is negative, the shift is π (half a wavelength).

A moving average to remove trend has a frequency-response function (F.R.F.) close to 1 at low frequencies and close to 0 at higher frequencies (see columns L and S of Table 1 which relate to the centred 12-month average and the Spencer average respectively). If the column symbols in Table 1 stand for the F.R.F. of various moving averages [e.g. L stands for $L(\omega)$], the deviations from the 12-month moving average have an F.R.F. $M(\omega) = 1 - L(\omega)$.

$K(\omega)$ is the F.R.F. of the process of averaging over the years the deviations for a single month. In Census II it is [3][3] or [3][5] and in Table 1 we first consider the former. Thus MK is the F.R.F. for the preliminary monthly seasonal differences.

In the version of Census II described above the centring of the seasonal differences is achieved by subtracting from each the mean of the differences for the *calendar* year in which it lies. This is not a simple moving average process, because the mean moves forward in jerks every 12 months, and so it cannot be represented in Table 1. In a later version of Census II the centring consists of taking deviations from a *moving* centred 12-month average of the seasonal differences, i.e. a repetition of M. In this version, which is illustrated in Table 1, the combined F.R.F. of the centred seasonal differences is therefore M^2K.

Subtracting the provisional differences from the original series gives a combined F.R.F. of $N = 1 - M^2K$. The F.R.F. of the final trend is NS, where S corresponds to the Spencer average, and $Q = 1 - NS$ represents monthly deviations from trend. Finally $R_1 = QMK$ is the F.R.F. of the final centred seasonal differences.

TABLE 1

Frequency response functions of moving averages used in seasonal adjustment procedures

…gular …ency …grees)	K $=[3]\,[3]$	L $=[12]\,[2]$	M $=1-L$	N $=1-M^2K$	S = Spencer average	Q $=1-NS$	R_1 $=QMK$	R_2 (same as R_1 with $K=[5]$)	R_3 (Hannan–Durbin with $K=[5]$)
0	1	1	0	1	1	0	0	0	0
5	0·444	0·955	0·045	0·999	1·000	0·001	0·000	0·000	0·000
10	0	0·824	0·176	1	1·003	−0·003	0	0·000	0·000
15	0·111	0·633	0·367	0·985	0·984	0·031	0·001	0·003	0·001
20	0	0·409	0·591	1	0·952	0·048	0	0·002	−0·006
25	0·444	0·188	0·812	0·707	0·895	0·337	0·122	0·036	0·017
30	1	0	1	0	0·809	1	1	1	0·191†
35	0·444	−0·133	1·133	0·429	0·696	0·701	0·353	0·109	0·069
40	0	−0·198	1·198	1	0·564	0·437	0	−0·066	−0·104
45	0·111	−0·201	1·201	0·840	0·425	0·644	0·086	0·168	0·138
50	0	−0·155	1·155	1	0·293	0·707	0	−0·145	−0·163
55	0·444	−0·080	1·080	0·482	0·180	0·913	0·438	0·186	0·177
60	1	0	1	0	0·094	1	1	1	0·906†
65	0·444	0·065	0·935	0·612	0·037	0·977	0·406	0·181	0·180
70	0	0·103	0·897	1	0·006	0·994	0	−0·178	−0·178
75	0·111	0·109	0·891	0·912	−0·005	1·004	0·099	0·179	0·179
80	0	0·086	0·914	1	−0·005	1·005	0	−0·184	−0·184
85	0·444	0·045	0·955	0·595	−0·002	1·001	0·425	0·191	0·191
90	1	0	1	0	0	1	1	1	1
95	0·444	−0·038	1·038	0·521	−0·002	1·001	0·462	0·208	0·208
100	0	−0·061	1·061	1	−0·007	1·007	0	−0·214	−0·214
105	0·111	−0·064	1·064	0·874	−0·012	1·010	0·119	0·215	0·215
110	0	−0·051	1·051	1	−0·015	1·015	0	−0·214	−0·213
115	0·444	−0·027	1·027	0·532	−0·016	1·008	0·460	0·208	0·209
120	1	0	1	0	−0·013	1	1	1	1·013†
125	0·444	0·022	0·978	0·575	−0·008	1·004	0·437	0·197	0·197
130	0	0·034	0·966	1	−0·003	1·003	0	−0·194	−0·194
135	0·111	0·034	0·966	0·896	0·000	1·000	0·107	0·193	0·197
140	0	0·026	0·974	1	0·001	0·999	0	−0·195	−0·195
145	0·444	0·013	0·987	0·567	−0·000	1·000	0·439	0·197	0·198
150	1	0	1	0	−0·003	1	1	1	1·003†
155	0·444	−0·009	1·009	0·547	−0·005	1·003	0·450	0·203	0·203
160	0	−0·013	1·013	1	−0·005	1·005	0	−0·204	−0·204
165	0·111	−0·011	1·011	0·886	−0·004	1·004	0·113	0·203	0·203
170	0	−0·006	1·006	1	−0·003	1·003	0	−0·202	−0·202
175	0·444	−0·002	1·002	0·554	−0·001	1·000	0·445	0·201	0·201
180	1	0	1	0	0	1	1	1	1

† Before amplitude restoration

It is now clear that the filter for smoothing the seasonal component must be looked at in a dual role: besides this smoothing it also captures some of the power of the original spectrum at non-seasonal frequencies.

Nerlove's criticism that Census II removes too much power at these frequencies is seen to be amply justified: e.g. at higher frequencies where $S \Leftrightarrow 0, R_1 \Leftrightarrow MK$. For [3] [3], K removes four-ninths of the amplitude 5° away from a seasonal frequency. In traditional language, the estimated seasonal component has absorbed part of the irregular component.

If the seasonal pattern is absolutely stationary, it should produce sharp lines in the spectrum at the seasonal frequencies. But if the pattern changes slowly over the years, this is equivalent to saying that each seasonal amplitude consists of a time series with its own spectrum in which the low frequencies predominate. In Nerlove (1962) it is shown that the result is a broadening of the spectral lines into peaks of finite width. For instance, suppose the amplitude at $\pi/6$ is $(A\cos\omega t + B\sin\omega t)$ where ω is some low frequency: this can be represented as the sum of two waves with frequencies $(\pi/6) \pm \omega$. If we attempt to measure changes in amplitude over as short a period as three years, this would imply spectral power 10° either side of the seasonal frequencies, i.e. the peaks are 20° wide. An amplitude smoothing filter capable of estimating such rapid changes would inevitably collect much of the power at non-seasonal frequencies, and in particular most of the high-frequency "noise" or irregular component.

If instead we assume that five years is the shortest period in which measurable changes of seasonal amplitude can occur, the spectral peaks extend only 6° each side of the seasonal frequency. It is therefore proposed that the amplitude smoothing filter have a low F.R.F. beyond 5° from a seasonal frequency. This rules out [3] and [3] [3] which have F.R.Fs of 0·667 and 0·444 at 5°. The most flexible average which has acceptable spectral properties is [5], for which the F.R.F. is 0·2 at 5° but has a numerical maximum of 0·25 at 8° 45′.

R_2 in Table 1 is the F.R.F. of the combined filter for Census II using [5] as the amplitude smoothing filter. Much less is now removed from the spectrum at high frequencies. Hannan's criticism of this iterative procedure is that, if some of the trend is not removed on the first round, it causes variation of the estimated seasonal component. So it will be subtracted from the original series in obtaining the preliminary adjusted series; and all subsequent estimates of the trend will be defective. If this is not so, the first estimate of the trend was a good one and no iteration is needed.

We are now in a position to describe the Hannan–Durbin method in spectral terms. If the series is filtered to remove trend by the Spencer moving average, the amplitudes of the two lowest seasonal harmonics ($\pi/6$ and $\pi/3$) are reduced. However, the calculation of centred average monthly differences over a block of years is algebraically equivalent to a harmonic analysis of this block at the seasonal frequencies and the re-combination of the 11 amplitudes to form a 12-valued periodic function. So, if the harmonic analysis is performed and the amplitudes restored before re-combination by dividing by $1-S(k\pi/6)$, the result will be an undistorted estimate of the seasonal pattern. If a weighted average, e.g. [3] [3], is used for seasonal smoothing, this involves the averaging of three harmonic analyses for overlapping three-year periods.

If this method is applied with the Spencer average to remove trend and an amplitude smoothing filter of [5], the F.R.F. is $R_3 = MK(1-S)$, apart from the process of amplitude restoration, which cannot be represented by a single moving-average filter.

Summing up this Section, therefore, we see that the Spencer moving average has an excellent F.R.F. for trend removal—it is what the spectral analysts call a "low-pass filter". The 11 seasonal amplitudes can be estimated for any complete number of years by harmonic analysis of the residual series, followed by amplitude restoration. The monthly seasonal components factors follow immediately by re-combination of the 11 amplitudes.

5. OPTIMAL TREND-REMOVAL FILTERS

The Spencer 15-term moving average was discovered empirically to be extremely good for trend removal, and it can be seen from Table 1 that its frequency response function is that of an almost ideal low-pass filter.

It is, however, often inconvenient to use a 15-term average because seven terms are lost at each end, so that a series of exactly N years gives only $(N-2)$ complete years of deviations from trend. Also the weights of the end terms of the Spencer average are very small. So we now see whether there is any systematic procedure for finding ideal filters, which will enable us to use a 13-term average instead. We can check whether the procedure is efficient by applying it to the 15-term case and seeing if the answer resembles the Spencer average.

An ideal low-pass filter would have a frequency-response function of the form

$$f(\omega) = \begin{cases} 1 & (0 \leqslant \omega \leqslant k\pi), \\ 0 & (k\pi < \omega \leqslant \pi). \end{cases}$$

where $0<k<1$, the actual value of k being chosen to fit the particular problem. We can select a symmetric moving average filter of the form $[d_7 d_6 \ldots d_1 d_0 d_1 \ldots d_7]$ whose F.R.F., say $F(\omega)$, is fitted by least squares to the ideal filter $f(\omega)$. That is, it is the first eight terms of the Fourier expansion of $f(\omega)$, subject to the constraint:

$$\sum_{-7}^{7} d_r = 1,$$

i.e. $F(0) = 1$.

Using a Lagrange multiplier, let

$$U = \frac{1}{2}\int_0^{\pi} [f(\omega) - d_0 - 2d_1 \cos\omega \ldots - 2d_7 \cos 7\omega]^2 \, d\omega + \lambda \sum_{-7}^{7} d_i.$$

Differentiate with respect to d_r $(r = 0, 1, \ldots, 7)$ and equate to 0.

$$-\int [f(\omega) - d_0 - 2d_1 \cos\omega \ldots - 2d_7 \cos 7\omega] \cos r\omega \, d\omega + \lambda = 0.$$

$$d_0 \int \cos r\omega \, d\omega + 2d_1 \int \cos\omega \cos r\omega \, d\omega \ldots + 2d_7 \int \cos 7\omega \cos r\omega \, d\omega + \lambda$$

$$= \int f(\omega) \cos r\omega \, d\omega \quad (r \neq 0).$$

All integrals on the left vanish, except the coefficient of d_r.

$$\int_0^{\pi} \cos^2 r\omega \, d\omega = \tfrac{1}{2}\pi,$$

$$\int_0^{\pi} f(\omega) \cos r\omega \, d\omega = \int_0^{k\pi} \cos r\omega \, d\omega = \frac{\sin rk\pi}{r}.$$

So

$$d_r\pi+\lambda = \frac{\sin rk\pi}{r} \quad (r\neq 0).$$

Similarly

$$d_0\pi+\lambda = \int_0^\pi f(\omega)\,d\omega = k\pi.$$

Writing the unconstrained coefficients with primes,

$$d'_0 = k, \tag{5.1}$$

$$d'_r = \frac{\sin rk\pi}{r\pi} \quad (r\neq 0), \tag{5.2}$$

then

$$d_r = d'_r - \frac{\lambda}{\pi} \quad (\text{all } r), \tag{5.3}$$

$$1 = \sum_{-7}^{7} d'_r - 15\frac{\lambda}{\pi},$$

giving

$$\frac{\lambda}{\pi} = \frac{1}{15}\left(\sum_{-7}^{7} d'_r - 1\right). \tag{5.4}$$

Column 1 of Table 2 shows $F(\omega)$, the F.R.F. of this solution of the problem with $k = \frac{1}{4}$; the choice of this value is suggested by the fact that the Spencer F.R.F. is about 0·5 at $\pi/4$. This F.R.F. is not very close to that of Spencer, particularly for low frequencies, so we must look for another condition to impose.

The Spencer formula is noteworthy for exactly removing a cubic polynomial trend, i.e. the filter $(I-F)$, where I is the identity operator, contains Δ^4 as a factor. Thus $F(\omega)$ satisfies the condition:

$$1-F(\omega) = (e^{i\omega}-1)^4 h(\omega),$$

where $h(\omega)$ is a polynomial in $e^{i\omega}$ and $e^{-i\omega}$. Hence

$$\frac{dF}{d\omega} = \frac{d^2F}{d\omega^2} = \frac{d^3F}{d\omega^3} = 0 \quad \text{at} \quad \omega = 0.$$

Now

$$F(\omega) = \sum_{-7}^{7} d_r \cos r\omega,$$

$$\frac{dF}{d\omega} = -\Sigma r d_r \sin r\omega = 0 \quad \text{at} \quad \omega = 0,$$

$$\frac{d^2F}{d\omega^2} = -\Sigma r^2 d_r \cos r\omega = -\Sigma r^2 d_r \quad \text{at} \quad \omega = 0,$$

$$\frac{d^3F}{d\omega^3} = \Sigma r^3 d_r \sin r\omega = 0 \quad \text{at} \quad \omega = 0.$$

So the condition that the filter eliminates a cubic trend exactly is simply that $\Sigma r^2 d_r$ should vanish. Much more important—since no one believes that time series can be well represented by polynomials over an extended period—this constraint ensures that $F(\omega)$ remains close to 1 at low frequencies. Introducing the second constraint by a Lagrange multiplier μ, Equation (5.3) becomes (absorbing the factor π into the multipliers)

$$d_r = d'_r - \lambda - \mu r^2. \tag{5.5}$$

Summing the equations,

$$15\lambda + \mu \sum_{-7}^{7} r^2 = \sum_{-7}^{7} d'_r - 1. \tag{5.6}$$

Multiplying by r^2 and summing the equations,

$$\lambda \sum_{-7}^{7} r^2 + \mu \sum_{-7}^{7} r^4 = \sum_{-7}^{7} r^2 d'_r. \tag{5.7}$$

The solution to these equations, with $k = \frac{1}{4}$, has the F.R.F. shown in Column 2 of Table 2, which is rather closer to the Spencer average than Column 1. Shifting to $k = \frac{2}{9}$ (Column 3) gives a better approximation, but still not as close to 0 at high frequencies.

Let us try to lay down a little more precisely the conditions for an optimal trend-removing filter for monthly observations. First, $F(\omega)$ should be close to 1 at least up to $\pi/9$ (a period of 18 months which may be important in connection with the trade cycle). Secondly, it should be as small as possible above $\pi/3$ (period of 6 months) in order to cut out rapid fluctuations or "noise". Thirdly, it must not exceed about 0·8 at $\pi/6$, since otherwise there will be a loss of accuracy in estimating the amplitude of the annual harmonic: this implies that the amplitude restoration factor will not be more than 5.

These conditions suggest that it might be an improvement to use a modified $f(\omega)$,

$$f(\omega) = \begin{cases} 1 & (0 \leqslant \omega \leqslant k\pi), \\ \dfrac{m\pi - \omega}{(m-k)\pi} & (k\pi < \omega \leqslant m\pi), \\ 0 & (m\pi < \omega \leqslant \pi), \end{cases}$$

since the steepness of the F.R.F. around the cut-off point is not so important. We have, as is obvious geometrically,

$$\int_0^{\pi} f(\omega) = \tfrac{1}{2}(k+m)\pi,$$

also

$$\begin{aligned} \int_0^{\pi} f(\omega) \cos r\omega \, d\omega &= \int_0^{k\pi} \cos r\omega \, d\omega + \int_{k\pi}^{m\pi} \frac{m\pi - \omega}{(m-k)\pi} \cos r\omega \, d\omega \\ &= \frac{\sin rk\pi}{r} + \frac{m}{m-k}\left(\frac{\sin rm\pi}{r} - \frac{\sin rk\pi}{r}\right) \\ &\quad - \frac{1}{(m-k)\pi r^2} \int_{rk\pi}^{rm\pi} x \cos x \, dx \quad (\text{where } x = r\omega). \end{aligned}$$

The last integral becomes $rm\pi \sin rm\pi - rk\pi \sin rk\pi + \cos rm\pi - \cos rk\pi$. So

$$\int_0^{\pi} f(\omega) \cos r\omega \, d\omega = \frac{1}{r^2(m-k)\pi} (\cos rk\pi - \cos rm\pi).$$

Finally,

$$d'_0 = \tfrac{1}{2}(k+m),$$

$$d'_r = \frac{1}{r^2(m-k)\pi^2}(\cos rk\pi - \cos rm\pi) \quad (r \neq 0). \qquad (5$$

Equations (5.5) to (5.7) apply as before. For the Spencer average, $F(\omega) = 0{\cdot}5$ near $42\frac{1}{2}^\circ$, so k and m are selected equidistant from it. The choice of $m = \frac{1}{3}$ (60°) is indicated by the descent to near 0 at this point, so we have $k = \frac{5}{36}$ (25°). The resulting F.R.F. in Column 4 of Table 2 is very close to that of the Spencer average and the weights (multiplied by 320 are): [−2, −7, −6, 3, 22·5, 46, 66, 74, ...] compared with Spencer's [−3, −6, −5, 3, 21, 46, 67, 74, ...].

TABLE 2

Frequency-response functions of optimal trend-removal averages

	15-term					13-term
	(1)	(2)	(3)	(4)	(5)	(6)
Lower cut-off frequency	—	—	—	25°	Spencer	20°
Single cut-off frequency	45°	45°	40°	—	moving	—
Upper cut-off frequency	—	—	—	60°	average	60°
2nd derivative at zero frequency	≠0	0	0	0	0	0
0	1	1	1	1	1	1
5	1·017	1·000−	1·000−	1·000−	1·000−	1·000−
10	1·058	0·998	0·996	0·997	1·003	0·997
15	1·104	0·989	0·964	0·984	0·984	0·985
20	1·128	0·965	0·939	0·954	0·952	0·955
25	1·106	0·921	0·866	0·898	0·895	0·899
30	1·023	0·851	0·758	0·812	0·809	0·813
35	0·878	0·752	0·619	0·699	0·696	0·697
40	0·686	0·627	0·460	0·565	0·564	0·558
45	0·474	0·484	0·300	0·423	0·425	0·407
50	0·272	0·334	0·157	0·285	0·293	0·259
55	0·106	0·193	0·045	0·165	0·180	0·130
60	−0·006	0·072	−0·028	0·073	0·094	0·031
65	−0·061	−0·018	−0·060	0·011	0·037	−0·032
70	−0·068	−0·070	−0·059	−0·021	0·006	−0·058
75	−0·045	−0·085	−0·038	−0·029	−0·005	−0·053
80	−0·011	−0·071	−0·010	−0·022	−0·005	−0·029
85	0·018	−0·039	0·014	−0·009	−0·002	0·003
90	0·031	−0·001	0·025	0·004	0	0·030
120	−0·014	−0·019	−0·012	−0·007	−0·013	−0·034
150	−0·003	0·028	0·003	0·004	−0·003	0·034
180	0	−0·029	0·001	−0·003	0	−0·034

Steepness of function is indicated by markers at 90 per cent and 10 per cent levels.

The formulae developed for the 15-term average apply quite generally, with a suitable change in the summation limits in (5.6) and (5.7). Column 6 of Table 2 shows the results of (5.5) to (5.9) for a 13-term average with $k\pi = 20°$, $m\pi = 60°$: these values are chosen to give virtually the same F.R.F. as Spencer up to $\pi/6$ and to minimize the absolute magnitude of the function above $\pi/3$. On the latter test it is not quite so good as Spencer, i.e. for eliminating "noise" from the trend, but the loss of efficiency is not important in practice. The actual weights are: [−0·0331, −0·0208, 0·0152, 0·0755, 0·1462, 0·2039, 0·2262, ...].

6. HARMONIC ANALYSIS

Having removed the estimated trend by our optimal 13-term moving average, there are two decisions we must make in analysing the resulting monthly deviations (say x_t):

(a) Whether to use monthly deviations from the average of a calendar year (or years) or the algebraically equivalent harmonic analysis at seasonal frequencies.

(b) The length of record analysed, or in other words the shortest stretches over which the seasonal pattern should be estimated.

Algebraically we have:

(i) $$u_i = \frac{1}{p}\sum_{j=1}^{p} x_{ij} - \frac{1}{12p}\sum_{j=1}^{p}\sum_{i=1}^{12} x_{ij} \quad (i = 1, 2, \ldots, 12)$$

or

(ii) $$a_k = \frac{2}{12p}\sum_{t=1}^{12p} x_t \cos\omega_k t \quad (k = 1, 2, \ldots, 5)$$

$$b_k = \frac{2}{12p}\sum_{t=1}^{12p} x_t \sin\omega_k t \quad (k = 1, 2, \ldots, 5),$$

and

$$a_6 = \frac{1}{12p}\sum_{t=1}^{12p} x_t \cos\omega_6 t,$$

where $x_{ij} = x_t$ is the deviation from trend for the ith month in the jth year, $\omega_k = 2\pi k/12$ and p is the number of years analysed as a single section.

The great advantage of using the harmonic analysis is that it overcomes the conflict between using different averages for smoothing the seasonal factors for different months and the need for the constraint $\Sigma u_i = 0$ (see Marris, 1961). In other words the sine and cosine amplitudes a_k, b_k are orthogonal to each other and can be smoothed and extrapolated independently. When they are re-combined to form average seasonal deviations, the latter automatically sum to zero.

For the length of section, it seems best to take $p = 1$ initially in order that the maximum information may be gained about changes in the seasonal pattern. Of course more powerful smoothing will be needed afterwards in order to damp down the irregular component. Let $g(\omega) = 1 - F(\omega)$, i.e. the F.R.F. of the monthly deviations from trend. After amplitude restoration we have

$$a_{kj} = \frac{2}{12g(\omega_k)}\sum_{i=1}^{12} x_{ij} \cos i\omega_k \quad (k = 1, 2, \ldots, 5), \tag{6.1}$$

$$b_{kj} = \frac{2}{12g(\omega_k)}\sum_{i=1}^{12} x_{ij} \sin i\omega_k \quad (k = 1, 2, \ldots, 5), \tag{6.2}$$

$$a_{6j} = \frac{1}{12g(\omega_6)}\sum_{i=1}^{12} x_{ij} \cos i\omega_6. \tag{6.3}$$

We have now 11 time series of a_{kj} and b_{kj}, where the range of j is 1 less than the number of years in the series. If the trend has been completely removed, if the seasonal pattern is stationary, and if the irregular component is a series of independent random variables with constant variance, we can make certain deductions about the time series of a_{kj} and b_{kj}. Some simple algebra shows that except for $k = 1$ successive terms of each time series are uncorrelated and for $k = 2, 3, 4$ or 5 have a variance close to $\frac{1}{6}$; for $k = 6$ the variance is about $\frac{1}{12}$. These statements would be exact if there had been no trend removal—a not unexpected result for frequencies at which the trend-removal F.R.F. is close to zero. But correlation $(a_{1j}, a_{1,j+1}) = -0{\cdot}42$ for both Spencer and our optimal 13-term average and correlation $(b_{1j}, b_{1,j+1}) = -0{\cdot}27$ and $-0{\cdot}28$ respectively. $\text{Var}(a_{1j}) \doteq 1$ in both cases and $\text{var}(b_{1j}) = 0{\cdot}36$ and $0{\cdot}39$. Consequently we must expect the first cosine amplitude time series to be both large and erratic (because negatively auto-correlated) and this is found to be the case.

7. SMOOTHING THE SEASONAL FACTORS

Marris (1961) used the series of moving averages [3], [3] [3], [3] [5], [3] [9] and [3] [15]—the choice for each month depending on the relative importance of movement in the seasonal factor compared with the irregular component. For reasons given in Section 4—distortion of the spectrum at non-seasonal frequencies—it is better to omit the first two of these and substitute [5]. But since exponential weights are used for extrapolation (see next section), it seems natural to consider the two-tailed exponential as an alternative for smoothing, i.e. $[\ldots, \lambda^2, \lambda, 1, \lambda, \lambda^2, \ldots]$ multiplied by $(1-\lambda)/(1+\lambda)$.

TABLE 3

Properties of smoothing averages

Simple smoothing averages			*Exponential smoothing*		
Average	*Variance*	*Max. F.R.F.*†	*Average*	*Variance*	*Max. F.R.F.*†
			$\lambda = 0{\cdot}9$	0·026	0·01
[3] [15]	0·063	0·09	0·8	0·056	0·05
[3] [9]	0·100	0·15	0·7	0·091	0·11
[3] [5]	0·164	0·13	0·6	0·133	0·21
[5]	0·200	0·25	0·5	0·185	0·33
			0·4	0·254	0·47
[3]	0·333	0·67	0·3	0·347	0·62

† Numerical maximum 5° or more away from seasonal frequency.

Table 3 shows the error variance reductions achieved by the two sets of averages. Those with approximately the same variance reduction are shown on the same line. The F.R.Fs of the two sets of averages, applied to every twelfth monthly observation, have peaks at the origin and at the seasonal frequencies $(2\pi k/12)$. In order to obtain a rough measure of the amount of spectral distortion at non-seasonal frequencies, the third and sixth columns give the numerically greatest value of the F.R.F. in the regions 5° or more from the peaks. The F.R.F. of the exponential smoothing is $(1-\lambda)^2/(1-2\lambda\cos 12\omega+\lambda^2)$. Since this function is monotone decreasing from 0 to

= 12, the greatest values in these regions are at exactly 5° from the peak. In practice the series of amplitudes is short enough to truncate noticeably the exponential smoothing, even when there are enough terms to complete the corresponding simple average, e.g. [3] [9] or [3] [15]. For instance, if there are exactly 5 terms either side of the amplitude to be smoothed, for the exponential smoothing with $\lambda = 0{\cdot}7$ the variance of the truncated version rises to 0·178, but the maximum F.R.F. is almost unchanged (0·12).

The table suggests that for $\lambda > 0{\cdot}6$ exponential smoothing produces smaller off-seasonal spectral losses than the simple averages. We therefore use the following smoothing averages:

[5]

[3] [5]

Exponential ($\lambda = 0{\cdot}7$)

Exponential ($\lambda = 0{\cdot}8$)

Exponential ($\lambda = 0{\cdot}9$)

Average all terms.

The question of when to use which average is deferred to Section 9. But, assuming the smoothing done, the results will be denoted by $\bar{a}_{kj}, \bar{b}_{kj}$. Then u_{ij}, the seasonal component in the ith month of year j, is defined as

$$u_{ij} = \sum_{k=1}^{6} \bar{a}_{kj} \cos i\omega_k + \sum_{k=1}^{5} \bar{b}_{kj} \sin i\omega_k. \tag{7.1}$$

8. Extrapolation of Seasonal Factors

It is necessary to extrapolate each amplitude by one year at either end: first, in order to provide seasonal adjustments for the 6 data at each end that have been "lost" in taking a 13-term moving average; secondly, to provide 6 further factors at the front for manual adjustment of the next 6 months' data as they become available. By arranging for the computer programme to process either a whole number of years or a whole number plus a half-year, the seasonal factors can be up-dated by computer every six months. Where there is an odd half-year, the harmonic analysis and trend removal can be performed on the whole number of years excluding the *first* six months' data; this means that 12 extrapolated factors will be needed at the beginning of the series.

A programme was written initially to extrapolate by a 3-term autoregressive scheme. Letting a_{kj} and b_{kj} be represented by z_n,

$$\hat{z}_{n+1} = az_n + bz_{n-1} + cz_{n-2},$$

where a, b, c were estimated by least squares. With the short series of terms available, the estimating equations were not well behaved, in the sense that quite often the determinant was very small, and $(a+b+c)$ was frequently far from 1. This latter fault could have been corrected by using a constraint, but the final consideration was the instability of the parameters as additional data were introduced. It became clear that the shortness and "noisiness" of the series excluded all but single-parameter extrapolators.

The obvious choice was therefore the exponential:

$$\hat{z}_{n+1} = (1-\lambda)(z_n + \lambda z_{n-1} + \lambda^2 z_{n-2} + \ldots), \tag{8.1}$$

which has a variance of $(1-\lambda)/(1+\lambda)$. As usually employed, the residual weight is attached to the last term, i.e. the last coefficient becomes λ^{n-1} instead of $(1-\lambda)\lambda^{n-1}$. For these short series, it is more satisfactory to distribute the weight evenly over all the terms, i.e.

$$z_{n+1} = \frac{z_n + \lambda z_{n-1} + \lambda^2 z_{n-2} + \ldots + \lambda^{n-1} z_1}{1 + \lambda + \lambda^2 + \ldots + \lambda^{n-1}}. \tag{8.2}$$

The question now arises: should one extrapolate and then smooth the extended series, or should the extrapolation be treated as an estimate of the next term of the smoothed series? Where the smoothing is exponential it makes no difference. The smoothed extrapolation

$$\bar{z}_{n+1} = \frac{1-\lambda}{1-\lambda^{n+1}}(\hat{z}_{n+1} + \lambda z_n + \lambda^2 z_{n-1} + \ldots + \lambda^n z_1).$$

$$\text{Expression in brackets} = \hat{z}_{n+1} + \lambda \hat{z}_{n+1}\left(\frac{1-\lambda^n}{1-\lambda}\right)$$

$$= \hat{z}_{n+1}\left(\frac{1-\lambda^{n+1}}{1-\lambda}\right).$$

So

$$\bar{z}_{n+1} = \hat{z}_{n+1}. \tag{8.3}$$

But for [3] [5] and [5] there is a choice. If the extrapolation is not smoothed, there is the problem of "splicing"—the weights must change smoothly from the average $\bar{z}_{n-2}$ (using [5]) or $\bar{z}_{n-3}$ (using [3] [5]) to the extrapolated $\hat{z}_{n+1}$. Any choice will be somewhat arbitrary. In the early versions of the computer programme there was no smoothing of the extrapolation on the grounds that the latter is then more responsive to recent changes. But it was found that the problem of stability (when additional data are introduced) was much more serious than that of residual seasonality (failing to follow changes in the seasonal pattern closely enough). It was judged more important to reduce the variance of the extrapolated terms by smoothing (see Table 4), except where the series is already smooth and there is evidence of a linear trend.

TABLE 4

Variance of extrapolated term

1. Exponential weights:												
$\lambda =$	0·6	0·5	0·4	0·3	0·2	0·1	0	−0·1	−0·2	−0·3	−0·4	−0·5
variance	0·25	0·33	0·43	0·54	0·67	0·82	1·00	1·22	1·50	1·86	2·33	3·00
2. Smoothed exponential weights:†												
variance	0·30	0·38	0·43	0·49	0·55	0·62	0·68	0·76	0·85	0·97	1·12	1·34

† All smoothed by [5] except $\lambda = 0{\cdot}6$ for which [3] [5] is used.

The table extends only as far as $\lambda = -0{\cdot}5$ although the exponential weights are stable for $1 > \lambda > -1$. Negative values of λ will pick up part of any linear trend, though less rapidly and completely than the adaptive 2-term or 3-term predictor developed by Box and Jenkins (1962). But, for the short series of 5–15 terms dealt with here, the use of λ below $-0{\cdot}5$ is found to be unsatisfactory—see Section 9. For $\lambda = -0{\cdot}5$, provided there is other evidence of a linear trend, the extrapolation is spliced and not smoothed, in order to allow a greater response to the trend. The effect, in the ideal case where successive terms differ by a constant amount d, is an extrapolation only $\frac{1}{3}d$ (instead of d) beyond the last actual term. Thus the procedure is quite conservative about assuming continuation of a trend.

It will be noted that an indicated value of λ below $0{\cdot}5$—suggesting sharp or persistent changes in the amplitude—is allowed to influence the extrapolation, but not the method of smoothing. This compromise is dictated by the need for stability and avoidance of spectral distortion, while still allowing fairly rapid response to big changes in the apparent seasonal pattern.

9. Choice of Amplitude Smoothing and Extrapolation

A simple test of whether a series of N amplitudes z_t is random is to see whether the von Neumann ratio,

$$V = \frac{\sum_2^N (z_t - z_{t-1})^2}{\sum_1^N (z_t - \bar{z})^2} \quad \left[\bar{z} = \frac{1}{N}\sum_1^N z_t\right],$$

is significantly low at some suitable probability level—say 10 per cent. If it is not significant, the appropriate choice of smoothed amplitudes is $\bar{z}$, the average of all terms. If V is significant, we re-calculate it for the series $(z_t - \bar{z}_t)$, where $\bar{z}_t$ are the smoothed amplitudes using one of the averages in Section 7. Starting with the exponential ($\lambda = 0{\cdot}9$), we calculate V for successively more flexible averages and choose the first one for which $(z_t - \bar{z}_t)$ appears to be random.

If the smoothing chosen is an exponentially weighted average, the extrapolation at each end is carried out with the same value of λ, i.e. $\hat{z}_{N+1} = \hat{z}_N, \hat{z}_0 = \hat{z}_1$, because this is consistent with equation (8.3). If the most flexible average [5] is chosen, the value of $\lambda \leqslant 0{\cdot}5$ for extrapolating is selected by the method of least squares. If the average [3] [5] is chosen, it is natural to extrapolate using $\lambda = 0{\cdot}6$.

Suppose successive estimates $\hat{z}_t$ of z_t are made according to the modified formula (8.2):

$$\hat{z}_t = \frac{z_{t-1} + \lambda z_{t-2} + \lambda^2 z_{t-3} + \ldots + \lambda^{t-2} z_1}{1 + \lambda + \lambda^2 + \ldots + \lambda^{t-2}} \quad (2 \leqslant t \leqslant N). \tag{9.2}$$

The prediction errors $e_t = z_t - \hat{z}_t$ $(t \geqslant 2)$, e_1 is taken as zero, and we find the value of λ which minimizes Σe_t^2.

Now suppose the z_t really are random and uncorrelated, i.e. $z_t = \mu + \epsilon_t$, where $E(\epsilon_t) = 0$, $\text{var}(\epsilon_t) = \sigma^2$ and $\text{cov}(\epsilon_s, \epsilon_t) = 0$. We find, on taking expectations, that

$$E\left[\sum_1^N e_t^2\right] = \frac{2}{1+\lambda}\left[(1+\lambda) + \frac{1+\lambda+\lambda^2}{1+\lambda} + \frac{1+\lambda+\lambda^2+\lambda^3}{1+\lambda+\lambda^2} + \ldots + \frac{1+\lambda+\lambda^2+\ldots+\lambda^{N-1}}{1+\lambda+\lambda^2+\ldots+\lambda^{N-2}}\right]\sigma^2$$

$$= K(\lambda)\,\sigma^2 \quad \text{(say)}.$$

The function $K(\lambda)$ is found to be monotone increasing with decreasing λ:

$$K(1) = (N-1)+1+\tfrac{1}{2}+\tfrac{1}{3}+\ldots+\frac{1}{N-1},$$

$$K(0) = 2N-2.$$

Thus, for a truly random uncorrelated series, the modified formula is unbiased, i.e. it is most likely to select the value $\lambda = 1$, "average all terms". But the corresponding function $K(\lambda)$ for the unmodified form (8.1) is biased: for $N = 6$ it has a minimum at $\lambda = 0{\cdot}6$, for $N = 7$ to 13 at $\lambda = 0{\cdot}7$, for $N = 20$ to 40 at $\lambda = 0{\cdot}8$ and only when N reaches 50 does the minimum attain $\lambda = 0{\cdot}9$. For $\lambda < 1$, the two forms of the extrapolator are asymptotically identical.

The least squares method of selection was written into the author's computer programme, the search for a minimum ranging over values of λ down to $-0{\cdot}4$. Lower values of λ than this give weights in equation (9.2) which oscillate rather violently over the first five or six terms of the series and a minimum of Σe_t^2 in this range does not seem—on graphical inspection—to be very informative about the structure of the series.

It was therefore decided to ignore values of λ below $-0{\cdot}4$; but if $\lambda = -0{\cdot}4$ is optimal, the opportunity for a linear trend to show itself is given by considering the predictor

$$\hat{z}_t = 2z_{t-1} - z_{t-2}$$

which is appropriate for a pure linear trend. If the latter gives a lower value of Σe_t^2, the unsmoothed extrapolation with $\lambda = -0{\cdot}5$ described in Section 8 is selected.

The same procedure is repeated with the amplitudes in reverse order to select the best value of λ for extrapolating at the beginning of the series.

The selection of smoothing and extrapolation was tested for stability by comparing the choices made with those which would have been made if the original series had been truncated (i.e. the 6 latest data ignored). This is permissible because a shift of six months in the harmonic analysis corresponds to a phase shift of π. So, if seasonality were fairly slow moving and the irregular component small, the amplitudes would be the same except for a change of sign in the odd-numbered harmonics. Nine series, all at least nine years long and with a wide variation in the amount of irregularity, were used for the test. The results were unsatisfactory: of the $9 \times 11 = 99$ harmonics about 40 per cent had choices of λ for smoothing or extrapolating which differed by 0·2 or more.

It seems necessary to look deeper to determine whether a particular harmonic amplitude is changing over time. The harmonic analysis in Equations (6.1)–(6.3) relates to blocks of months labelled 1 to 12, but this particular time division is not privileged. If the harmonic analysis starts in month n of the jth year, we define analogous amplitudes $a'_{kj}(n)$ and $b'_{kj}(n)$:

$$a'_{kj}(n) = \frac{2}{12g(\omega_k)} \sum_{i=n}^{n+11} x_{ij} \cos(i-n+1)\,\omega_k \quad (k = 1,2,\ldots,5),$$

$$b'_{kj}(n) = \frac{2}{12g(\omega_k)} \sum_{i=n}^{n+11} x_{ij} \sin(i-n+1)\,\omega_k \quad (k = 1,2,\ldots,5),$$

where if $i>12$, x_{ij} is interpreted as $x_{i-12,j+1}$. The primes are to indicate that these component amplitudes are not comparable with the unprimed ones $a_{kj}(1)$, $b_{kj}(1)$ (unless $n=1$) because the time origin has been shifted by $(n-1)$ months. The components comparable with $a_{kj}(1)$, $b_{kj}(1)$ are

$$a_{kj}(n) = a'_{kj}(n)\cos(n-1)\,\omega_k - b'_{kj}(n)\sin(n-1)\,\omega_k,$$

$$b_{kj}(n) = a'_{kj}(n)\sin(n-1)\,\omega_k + b'_{kj}(n)\cos(n-1)\,\omega_k,$$

which reduce to

$$a_{kj}(n) = \frac{2}{12g(\omega_k)}\sum_{i=n}^{n+11} x_{ij}\cos i\omega_k,$$

$$b_{kj}(n) = \frac{2}{12g(\omega_k)}\sum_{i=n}^{n+11} x_{ij}\sin i\omega_k.$$

Thus the comparable amplitudes are simply moving sums of 12 terms taken from the same basic series:

$$x_{ij}\cos i\omega_k \quad \text{or} \quad x_{ij}\sin i\omega_k$$

This is indeed obvious if we think of the case where the x_{ij} are exactly periodic: when n increases by 1, we add the same term at the end as is dropped at the beginning. The same result holds for $a_{6j}(n)$ where the cosine terms are alternately ± 1.

We can now calculate the values of V for each of the 12 sequences $a_{k1}(n), a_{k2}(n), \ldots$ $(n = 1,2,\ldots,12)$ and average them together in some way. In the author's programme the method adopted is to add together the numerators and denominators of the 12 values of V:

$$\bar{V} = \frac{\sum_{n=1}^{12}\sum_{t=2}^{N(n)} [z_{kt}(n)-z_{kt-1}(n)]^2}{\sum_{n=1}^{12}\sum_{t=1}^{N(n)} [z_{kt}(n)-\bar{z}_k(n)]^2},$$

where

$$z_{kt}(n) = a_{kt}(n)-\bar{a}_{kt}(n)$$

or

$$b_{kt}(n)-\bar{b}_{kt}(n),$$

the bar sign denoting the smoothed amplitudes;

$$\bar{z}_k(n) = \frac{1}{N(n)}\sum_{t=1}^{N(n)} z_{kt}(n),$$

$$N(1) = N$$

and

$$N(n) = N-1 \quad (n\neq 1)$$

since, if the number of x_{ij} is $12N$, the total number of amplitudes is $(12N-11)$. The significance point of $\bar{V}$ is taken to be the 10 per cent level for $(N-1)$ terms, since 11 out 12 components of the pooled estimate have that number of terms.

If the behaviour of $\bar{V}$ indicates the selection of the [5] average for smoothing, determine the method of extrapolation by using the pooled sum of squares of the errors,

$$\sum_{n=1}^{12} \sum_{t=1}^{N(n)} e_t^2(\lambda, n),$$

and finding the value of λ which minimizes this.

As before, the value of λ for extrapolation at the beginning of the series is chosen separately.

10. Extrapolation of Trend

The seasonal components (u_{ij}) are now subtracted from the terms of the original series (y_t) to give a seasonally adjusted series. However, there has been some loss of information, in that the first 6 and last 6 terms have not been used directly in calculating the seasonal adjustments, since they appear only in the moving average for trend-removal. In the B.L.S. method this disadvantage is accepted on the assumption that it will make the seasonal factors more stable. In Census II a crude method of extrapolating the trend is used—step 7 of the description in Section 2 above—after the calculation of preliminary seasonal factors.

Durbin (1963) has shown that trend-removal, where a fixed seasonal pattern is assumed, has the effect of adding an adjustment to the simple "mean of each month minus grand mean", this adjustment depending only on the first $2m$ and last $2m$ terms of the series, where the trend-removal average has $(2m+1)$ terms. He pointed out that if the trend is removed by a centred 12 months' moving average, [12] [2], this will entirely eliminate a parabolic trend from the estimate of seasonal pattern. He also showed that the 6 terms at each end could be brought into the calculation of the (fixed) seasonal adjustments and that provided the end-terms adjustment for trend-removal was suitably modified, the procedure would still eliminate a parabolic trend. It is learned that Durbin has since found a method (not yet published) of generalizing this result to the case of moving averages which eliminate a cubic trend (e.g. Spencer).

It seemed that the 3-term adaptive predictor of Box and Jenkins (1962) might provide a satisfactory solution to the case of moving seasonality. So after the calculation of the preliminary adjusted series in the author's programme, a routine was inserted to estimate the Box–Jenkins parameters from the last 60 terms to obtain six extrapolated terms; to these are added the extrapolated seasonal components (u_{ij}), giving an extrapolation of the original series (60 terms is sufficient for satisfactory estimates). The same routine, *working backwards* on the first 60 terms, gives 6 extrapolated terms at the beginning; but this is only necessary when there are a whole number of years' data, as the extended series must be a whole number of years for harmonic analysis. The whole procedure of seasonal adjustment is repeated on the extended series, so that the first and last 6 terms of the original series now contribute to the estimate of the seasonal pattern. Moreover, the parameters already obtained from the preliminary adjusted series can be used to extrapolate the final adjusted series.

11. Replacement of Extreme Values

A seasonally adjusted series contains the trend and irregular components: we can attempt to separate these by means of the trend-removal filter used previously (operating on the logarithms of the adjusted series in the multiplicative case). Let ϵ_t

be the true irregular component and y'_t the adjusted series. The estimated irregular component e_t is

$$y'_t - \sum_{i=-6}^{6} d_i y'_{t+i},$$

and the variance $\sigma^2(\epsilon_t)$ is estimated from the equation

$$\sigma_1^2 = \left[(1-d_0)^2 + 2\sum_{1}^{6} d_i^2\right]\sigma^2 = \frac{1}{N-12}\sum_{7}^{N-6} e_t^2,$$

where σ_1^2 is the variance of the e_t and N the number of data in the series. So σ^2 will be an unbiased estimate of $E(\epsilon_t^2)$, if the seasonal factors have not captured any of the irregular component; in so far as they have, the estimate will be too low.

All economic time series are liable to contain a few mavericks—months in which there was a strike or other statistical freak—and the disturbance these cause in the estimation of the seasonal component must be minimized. A criterion for "extremeness" is needed—some multiple of σ_1—and a method of replacing the mavericks. The B.L.S. method uses a parabolic formula for this criterion based on the length of the series, which ranges from $2{\cdot}7\sigma_1$ for 6 years to $3\sigma_1$ for 15 years. If the ϵ_t were normally distributed—which seems unlikely—these limits give the expected number of "extremes" as half an observation. In view of the uncertainty surrounding the distribution of the ϵ_t, the author's programme contains a fixed criterion of $2{\cdot}5\sigma_1$.

It has been pointed out by the B.L.S. that the use of an ordinary weighted moving average for estimating the trend is unsatisfactory, if it is proposed to use the trend value for replacing the maverick. This is because the trend value gives a weight d_0 to the maverick, so that only $(1-d_0)$ of the "extremeness" would be removed. The B.L.S. method uses a "hole-in-the-middle" average with $d_0 = 0$. Some exploration of the optimal choice, by the method of Section 5, revealed that these filters have a peculiar F.R.F.: the trend removal properties are not much affected, but the function becomes substantially negative over most of the high-frequency range. If one tries to avoid this (e.g. by using a simple unweighted average), the filter fails to pick up the bulk of the trend. As it seems that trend is more important than the irregular component—one does not want all turning points of the trend classified as extreme! —the poor high-frequency behaviour was accepted. The "optimal" hole-in-the-middle average turns out to be virtually the same as that obtained by making the central weight of the ordinary optimal 13-term average zero and dividing the other weights by $(1-d_0)$; indeed the F.R.F. of the latter is slightly superior.

Thus, in the author's programme, the ordinary 13-term average is used on the preliminary adjusted series to calculate the residuals (e_t), and any data (y_t) designated extreme are replaced by $[y_t - e_t/(1-d_0)]$, i.e. terms containing supposedly only the trend and seasonal component (a false supposition when two extremes are close together). This replacement process is also applied to the preliminary adjusted series before the Box–Jenkins extrapolations. Mavericks can now be detected in the first 6 and last 6 terms of the original series, using the extrapolated terms, and these also are replaced. So the procedure for finding the final seasonal factors starts from the *extended* (6 terms at each end) *modified* (mavericks replaced) *original* series.

12. Constraint on Seasonal Factors

All the foregoing discussion applies equally to the additive and multiplicative models—provided that a logarithmic transformation is first applied in the multiplicative case. It is unquestioned that $\Sigma u_i = 0$ should apply to an additive model:

to 4, which is slightly below expectation. The results for the nine series are shown in Column 8 of Table 5. At the 10 per cent level of the seasonality test there are 10 non-random months—in accord with expectation. The corresponding mean differences between the M.S.F., excluding the first 12 and last 18/12 terms, are in Column 6; and the corresponding results for a fixed seasonal adjustment programme ($\lambda = 1$ for all amplitudes) which is otherwise identical with the author's moving seasonal adjustment programme, are given in Column 5. The figures in Column 6, which result from using the 25 per cent criterion, are of course larger than those obtained by using 10 per cent, but they are still only slightly above those in Column 5—indeed for series 4 and 8 the moving seasonal programme is the more stable.

If we increase the amplitude smoothing criterion still further to 30 per cent or 35 per cent, the number of non-random months in the residual seasonality test (at the 5 per cent level) is unchanged, but the mean differences between M.S.Fs begin to increase quite sharply. So the criterion of 25 per cent appears to be the best in practice.

TABLE 5

Test results

Series no.	*Mean irregular component, σ (per cent)*	*No. of years' data, N*	$\frac{2\sigma}{N}$	*Mean difference between M.S.F. of full and truncated series (per cent)*			*Residual seasonality test: number of significant months*	*No. of changes in $\lambda (>0{\cdot}1)$ from truncated to full series*	*No. of amplitudes with $\lambda < 0{\cdot}9$†*	*No. of "extreme" terms*
				Fixed M.S.F.	*Moving M.S.F.*					
					All except end-terms	*Last 12 terms*				
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
2	6·6	11½	1·15	1·09	1·12	1·68	1	1	4	1
3	4·0	12½	0·64	0·38	0·48	0·66	0	2	3	0
4	3·3	12½	0·53	0·49	0·45	0·44	2	0	0	2
5	4·0	12½	0·64	0·44	0·55	0·71	0	0	2	2
8	2·9	12½	0·46	0·22	0·19	0·22	0	0	1	1
10	2·4	12	0·40	0·22	0·35	0·96	0	0	5	6
11	14·6	9	3·25	2·02	2·23	2·21	0	1	1	4
12	1·9	15	0·26	0·42	0·42	0·42	0	0	1	4
13	0·9	10½	0·18	0·09	0·15	0·14	1	0	3	1

† Full series.

The series shown in Table 5 were selected to cover a wide range of values for the mean irregular component (σ). Common sense suggests that the mean difference between M.S.F. will rise with σ and fall with increasing N; the table shows that the mean difference is usually between σ/N and $2\sigma/N$. Column 7 shows the mean difference between M.S.F. for the last 12 terms of the truncated series (last 18 less last 6 of the full series). They are considerably larger than those in Column 6 for series 2, 3 and this ensures that the sum of the seasonally adjusted data for each year is equal to the sum of the unadjusted data. But what is the appropriate constraint for a multiplicative model? If (i) the trend is flat and (ii) the irregular component is small, the equality between the yearly totals will be approximately preserved by making $\Sigma S_i = 12$, where the S_i are the multiplicative factors. But in general these assumptions are not true.

Nevertheless Census II and B.L.S. both adopt this constraint on the S_i. Another variant of Census II uses a constraint that makes the moving 12-months' average of seasonal factors as close to unity as possible. The logarithmic transformation in the author's programme implies that the geometric mean of the S_i, rather than the arithmetic mean, equals unity. It was found with 9 series tested by the author that in all but one case the arithmetic mean exceeded unity by under 1 per cent in any year. The exception was a series with very strong seasonality for which the arithmetic mean grew over the years to more than 2 per cent above unity.

It is by no means clear what is the correct criterion to adopt in general, though where the data are "flow" rather than "stock" figures (e.g. imports and exports), the argument for trying to keep the annual totals invariant is overwhelming. It was decided to make the average of the final seasonal factors unity, by dividing the factors for each year by their arithmetic mean. The effect on the residuals was negligible, as the "bias" changed very slowly from year to year, but the seasonal factors were scaled down and the trend was scaled up slightly.

13. TESTS OF PROGRAMME

If seasonality has been effectively removed, the final residuals should show no seasonal pattern; in particular, the series of residuals for a particular month should not show any persistent bias or long runs of the same sign. The bias can be detected by an analysis of variance of the residuals by rows and columns (years and months). The randomness of each monthly series can be tested by the von Neumann ratio and this has been incorporated in the computer programme.

The other side of the problem is to see whether the adjustment procedure is conservative enough for stability, i.e. how large are the changes occurring in the seasonal factors when more recent data become available. The instability can be measured by running the complete programme with each series truncated (i.e. omitting last 6 terms) and comparing the monthly seasonal factors with those of the full series. The changes occurring in the seasonal factors tend to be larger at the ends of the series, because of the element of extrapolation; so the first 12 and last 12 factors of the truncated series (i.e. the last 18 factors of the full series) can be omitted from the comparison. To see how much stability has been lost by allowing for moving seasonality, one can compare the mean differences with those obtained by using a fixed seasonal pattern, i.e. a single harmonic analysis of the whole series of deviations from trend.

It was found that if the choice of amplitude smoothing (Section 9) was based on the 10 per cent probability level, very little moving seasonality was recognized, i.e. in nearly all cases the programme selected $\lambda = 1$. But the number of significantly non-random monthly sets of residuals (at the 5 per cent level) was 10 in nine series (= 108 monthly sets)—much higher than expected by chance. This indicates that the method is too inflexible and has left some residual seasonality in the series.

If the criterion for amplitude smoothing is raised from 10 per cent to 25 per cent, the residual seasonality test is satisfactory: the number of non-random months drops

10—for all of which moving seasonality is important (see Column 10). The amplitudes are hardly ever smoothed by the [5] average, since $\lambda < 0{\cdot}6$ for only 3 out of 99. Column 9 shows that the number of abrupt changes in the choice of λ ($> 0{\cdot}1$) between truncated and full series has been reduced to negligible proportions—4 out of 99.

The number of terms of each series classified as "extreme" (preliminary residual numerically $> 2{\cdot}5\sigma_1$) will often be different for full and truncated series (as residuals near the borderline move across it). So the author's programme has been provided with a method of specifying in advance which terms are to be regarded as extreme, an idea borrowed from the B.L.S. method. The results quoted in Table 5 used this facility, the terms specified being those thrown out as mavericks in an earlier run of either the full or the truncated series. If the extremes are not matched, the mean differences in Columns 5–7 tend to be increased.

The comparison of seasonal adjustments calculated at six-monthly intervals is a severe test, more severe than the annual revision procedure most often used, since the years into which the data are divided for harmonic analysis are alternately January/December and July/June. It is recommended that even if computing time is available for semi-annual runs, revisions at the mid-year should not disturb the first three years of the series (where end-effects are largest and revisions likely to be reversed); indeed it may be better to revise only the last two of three years semi-annually. It is also recommended that when up-dating, "extremes" should be specified (in all but the last 18 terms) in the positions found in the previous run; the programme provides for extremes in the last 18 terms to be determined by the usual criterion.

14. CONCLUSIONS

A method of moving seasonal adjustment of time series has been developed which is theoretically sound and has relatively few arbitrary empirical rules (chiefly the range of averages considered for smoothing amplitudes in Section 7, and the method of choosing between them in Section 9). It is adaptive to sharp changes in the seasonal pattern but has much less flexibility than the original Census II and the B.L.S. methods, though the X10 version of Census II is somewhat better in this respect. Finally, the method described here avoids the iteration of earlier methods—the repetition described in Sections 10 and 11 is performed on the *original* series, changed merely by the addition of 6 extrapolated terms at each end and the replacement of "extreme" data (if any).

It may be useful to summarize the final procedure emerging from this paper. For simplicity the additive model is described, and the alterations necessary to deal with the multiplicative case are given at the end.

1. Find deviations from trend measured by optimal 13-term moving average.
2. Perform harmonic analysis of the deviations in non-overlapping blocks of 12 —for all 12 possible partitions into blocks.
3. For each of the 11 harmonics:
   (a) Arrange the amplitudes for successive blocks in a time series—one series for each partition.
   (b) For a given smoothing average, test the randomness of the series of differences between amplitudes and smoothed amplitudes, by calculating the von Neumann ratio. This ratio is derived for each partition and the results pooled. Choose the least flexible smoothing average for which the ratio is not significant at the 25 per cent level, selecting [5] if the rest fail.

(c) If the smoothing average is the exponentially weighted or the [3] [5], choose the corresponding value of λ for extrapolation.

(d) If the smoothing average is [5], find by least squares the value of $\lambda \leqslant 0{\cdot}5$ which is best for extrapolation—dealing with the two ends of the series independently.

4. Apply the chosen methods of extrapolation and smoothing to the principal partiton: the one containing the block of the last 12 terms.
5. Re-combine the 11 smoothed amplitudes to give 12 monthly seasonal deviations.
6. Deduct these from the original series to give preliminary seasonally adjusted series (P.S.A.).
7. Using the optimal 13-term average again, find residuals, i.e. deviations from trend of P.S.A., and their root mean square (σ_1).
8. Treat as extreme any term of P.S.A. whose residual is numerically more than $2{\cdot}5\sigma_1$, and replace it by its trend value (estimated from neighbouring terms).
9. Extrapolate the modified P.S.A. 6 terms at the end and—if original data span a whole number of years—6 terms at the beginning. The method of extrapolation is the Box–Jenkins 3-term adaptive predictor determined from the last (first) 60 terms.
10. By adding back preliminary monthly seasonal deviations to extrapolated P.S.A. terms, obtain 6 terms at the end (beginning) of original unadjusted series. Modify any extreme terms of original series, using same criterion as in step 8.
11. Repeat steps 1–6 on extended modified original series, to give the final seasonally adjusted series (F.S.A.).
12. Repeat step 7, except that trend is estimated from seasonally adjusted *modified* series, so that residuals provide full measure of deviations of extreme terms from trend.

In the multiplicative model, steps 1, 7 and 9 are performed on the logarithms of the original series and the F.S.A. undergo the exponential transformation before being printed out. Also the twelve seasonal factors (exponentials of the final seasonal deviations) are adjusted to sum to 12.

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Seasonal Adjustment by Signal Extraction

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Summary

If an ARIMA model has been fitted to a time series, the model spectrum can be partitioned into trend, seasonal and irregular components. The corresponding linear filters are used for signal extraction to provide a theoretically based method of seasonal adjustment. The flexibility, stability and residual seasonality obtained by this method and two others are compared empirically.

Keywords: SEASONAL ADJUSTMENT; ARIMA MODEL; SPECTRUM; SIGNAL EXTRACTION; PARTIAL FRACTIONS; HARMONIC FUNCTION; AUTOCOVARIANCE GENERATING FUNCTION; LINEAR FILTER; FORECAST; BACKCAST; TREND; IRREGULAR; TRANSIENT; EXTREME VALUES; ANNUAL REVISIONS; RESIDUAL SEASONALITY

1. Introduction

Seasonal adjustment as a large-scale practical technique was introduced by the US Bureau of the Census in 1957. Their method developed over the next 8 years until the publication of the X.11 version in 1965. Various other methods appeared in the next 10 years—see Bongard (1960), Burman (1965), Mesnage (1968), Nullau *et al.* (1969), Stephenson and Farr (1972), den Haan (1974) and Durbin and Murphy (1975). Stephenson and Farr is a regression method, Durbin and Murphy partly of this type; but all the others are moving average methods, in which an attempt is made to decompose the series into trend, seasonal and irregular components. This is done by a sequence of linear filters.

Why are there so many competing methods? It is because, although the decomposition is intuitively appealing, none of the methods can be shown to have optimal properties—except, perhaps, for the trend removal filter used in Burman (1965); otherwise they are all *ad hoc* techniques. Indeed, these properties depend on the purpose for which seasonal adjustments have been performed. The most common is to provide an estimate of the current trend so that judgemental short-term forecasts can be made. Alternatively, it may be applied to a large number of series which enter an economic model, as it has been found impracticable to use unadjusted data with seasonal dummies in all but the smallest models: this is often called the historical mode of seasonal adjustment.

It seems that, in the time domain, the decomposition is incapable of precise definition. However, in the frequency domain, the trend and seasonal components of a series can be more clearly defined. The seasonal component comprises the peaks in its spectrum at the basic seasonal frequency and the multiples of this, and the trend is represented by a broad peak at low frequencies. Seasonal adjustment is the operation of removing the seasonal peaks while leaving the rest of the spectrum undisturbed (though this is not entirely feasible—see below). Non-parametric estimation of spectra goes back over 20 years—for example, Blackman and Tukey (1958), but this does not provide an easy way of designing a linear filter to suit each series.

The next development was the theory of optimal filters for estimating an unobservable component of a time series (signal extraction). This was very lucidly explained in the book on prediction by Whittle (1963), which is now unfortunately out of print. Another link in the chain was the development of methods of estimating ARIMA models for time series by Box and Jenkins (1970). This provided, *inter alia*, a way of parametrizing the spectrum of a series, so that signal extraction filters could be derived from it. Box and Jenkins' prime purpose was to fit the models for forecasting, so it is intuitively attractive that they should be used for seasonal adjustment in its forecasting mode. The recent paper by Plosser (1979) shows that X.11 is not particularly helpful for this purpose. However, Statistics Canada have developed a composite method, X.11-ARIMA (Dagum, 1975, 1978), using ARIMA models to forecast and backcast one year, before applying X.11; this seems to perform better than X.11 for all except very noisy series.

The first application of signal extraction to seasonal adjustment was in an unpublished doctoral thesis by Cleveland (1972). An ARIMA model, for which the well-known X.11 method is optimal, was given in Cleveland and Tiao (1976); and another example using the standard $(0, 1, 1)\ (0, 1, 1)_{12}$ ARIMA seasonal model appeared in Box, Hillmer and Tiao (1979). The present paper shows how a general ARIMA model (with one commensense restriction on the parameters) can be used to generate an infinite linear filter which extracts the seasonal component from a series and its forecast and backcast values.

2. PARTITIONING THE SPECTRUM OF ARIMA SERIES

Suppose an infinite time series is believed to consist of two or more independent unobservable components, whose generating processes are known. Then there exist optimal linear filters to separate them, and signal extraction is the estimation and use of these filters.

Let us take an ARIMA seasonal model:

$$z_t = f(B)\, a_t = \frac{\theta(B)}{\phi(B)\, \Phi(B^s)}\, a_t, \tag{1}$$

where B is the lag operator, s the periodicity and a_t white noise; the numerator need not be separable into seasonal and non-seasonal operators. The denominator can be rearranged into components having no common factor: $\psi_m(B)$ for the trend component and $\psi_s(B)$ for the seasonal. Obviously $\phi(B)$, which contains the differencing factors $(1-B)^d$ and the non-seasonal auto-regressive part, will belong to $\psi_m(B)$, but so will part of $\Phi(B^s)$. For a monthly series, the seasonal differencing factorizes as

$$(1-B^{12})^D = (1-B)^D (1+B+B^2, \ldots, +B^{11})^D.$$

The first factor belongs to $\psi_m(B)$, having a root at unity; and the second to $\psi_s(B)$, its roots being the 12th roots of unity. These form conjugate complex pairs which generate peaks in the spectrum of the series at $\pi/6$, $2\pi/6$, ..., $5\pi/6$, plus a real root (-1) producing a peak at π.

The other roots of $\Phi(z) = 0$ are outside the unit circle. If $\Phi(z^{-1}) = 0$ has a real positive root $\Phi_1 < 1$, the factorizing is

$$1 - \Phi_1 B^{12} = (1-\mu B)(1+\mu B + \mu^2 B^2, \ldots, + \mu^{11} B^{11}),$$

where μ is the real positive 12th root of Φ_1.

The first factor contributes to the peak in the spectrum at the origin and belongs to $\psi_m(B)$; the second contains 11 roots contributing to the seasonal peaks, as before, and belongs to $\psi_s(B)$. But, if Φ_1 is complex, say, for example, that

$$(\Phi_1)^{1/12} = \mu e^{i\alpha}$$

is the root nearest to unity. Since normally the coefficients in $\Phi(z^{-1}) = 0$ are real, there will be a conjugate $\bar{\Phi}_1$, which has a 12th root $\mu e^{-i\alpha}$. Taking the two sets of 12th roots together, they form 12 pairs displaced by an angle α each side of the 12th roots of unity. The result is a series of pairs of spectral peaks on each side of the seasonal frequencies, like the Zeeman effect in physical spectra. If Φ_1 is negative, the extreme case, the peaks occur midway between the seasonal frequencies (which correspond to the odd order harmonics of a 2-year cycle). Thus the distinction between trend and seasonal can be made only for those auto-regressive seasonal models in which $\Phi(z^{-1}) = 0$ has a real positive roots.

The spectrum of z_t is derived from the transfer function between z_t and a_t:

$$g_z(\omega) = f(e^{i\omega})f(e^{-i\omega})\,\sigma_a^2$$
$$= \frac{\theta(e^{i\omega})\,\theta(e^{-i\omega})}{\psi_m(e^{i\omega})\,\psi_m(e^{-i\omega})\,\psi_s(e^{i\omega})\,\psi_s(e^{-i\omega})}\sigma_a^2. \tag{2}$$

The conventional division of a time series into trend, seasonal and irregular components may now be made more precise. The trend and seasonal cover the permanent characteristics of the series, responsible for the spectral peaks at the origin and at seasonal frequencies respectively. The irregular component covers the transient characteristics, i.e. it should be white noise or a low order MA process.

Let

$$z_t = m_t + s_t + r_t,$$

where

$$m_t = \text{trend} = f_m(B)\,b_t;\quad s_t = \text{seasonal} = f_s(B)\,c_t;\ r_t = \text{irregular} = f_r(B)\,d_t$$

and b_t, c_t and d_t are independent white noises. So far the components have not been precisely defined. To do so, consider their spectra. The spectrum of the trend is

$$g_m(\omega) = f_m(e^{i\omega})f_m(e^{-i\omega})\,\sigma_b^2 \tag{3}$$

and $g_s(\omega)$ and $g_r(\omega)$ are similarly defined. Then the independence of b_t, c_t and d_t means that

$$g_z(\omega) = g_m(\omega) + g_s(\omega) + g_r(\omega). \tag{4}$$

In ARIMA models, σ_a^2 is usually defined so that the coefficient of B^0 in $f(B)$ is 1. For the component functions, it is more convenient to define $\sigma_b^2 = \sigma_c^2 = \sigma_d^2 = \sigma_a^2$, which determines the coefficient of B^0 in each case. Since (2) and (3) are symmetric in ω, the model and its components must be rational functions $h_z(x)$, $h_m(x)$, etc. of $x = \cos\omega$. Thus (4) becomes

$$h_z(x) = h_m(x) + h_s(x) + h_r(x). \tag{5}$$

In Burman (1976), the author suggested that one should proceed as follows:

Let

$$U(x) = \theta(e^{i\omega})\,\theta(e^{-i\omega});\quad V_m(x) = \psi_m(e^{i\omega})\,\psi_m(e^{-i\omega});\quad V_s(x) = \psi_s(e^{i\omega})\,\psi_s(e^{-i\omega}).$$

Then the polynomial quotient of $h_z(x)$ can be identified with the transient component r_t, and the remainder partitioned into partial fractions identified with m_t and s_t:

$$h_z(x) = \frac{U(x)}{V_m(x)\,V_s(x)} = Q(x) + \frac{R(x)}{V_m(x)\,V_s(x)},$$

where $Q(x)$ is the quotient, assuming $h_z(x)$ is either top heavy or balanced in degree, and $R(x)$ the remainder. Since $V_m(x)$ and $V_s(x)$ have no common factors, we can find functions $\lambda(x), \mu(x)$ by the usual method of calculating an H.C.F. such that

$$\lambda(x)\,V_m(x) + \mu(x)\,V_s(x) \equiv 1,$$

so

$$\frac{\mu(x)}{V_m(x)} + \frac{\lambda(x)}{V_s(x)} \equiv \frac{1}{V_m(x)\,V_s(x)}.$$

Multiply through by $R(x)$ and find quotients and remainders $R_m(x)$ and $R_s(x)$ of the left-hand side. The quotients must cancel, since this is an identity. Thus:

$$h_z(x) \equiv Q(x) + \frac{R_m(x)}{V_m(x)} + \frac{R_s(x)}{V_s(x)} \tag{6}$$

$$\equiv h_r(x)+h_m(x)+h_s(x) \quad \text{(say)}.$$

The first term is MA (q^*-p^*), where q^* is the degree of the numerator of the model and p^* is the degree of the denominator. The second and third terms are of the right character for the spectra of trend and seasonal components, but they may not be positive for all values of ω.

3. Minimum Signal Extraction

However, this partition is not unique because constants can be added to the second and third components without altering the character of the spectra. Box *et al.* pointed out that only the minimum amount of variance should be removed from the series in seasonal adjustment. So, if $g_s(\omega)$ has a minimum ε_s, we replace it by the non-negative $g_s^*(\omega) = g_s(\omega)-\varepsilon_s$, and add the same amount to the transient. Similarly, to obtain the smoothest trend, we replace $g_m(\omega)$ by $g_m^*(\omega) = g_m(\omega)-\varepsilon_m$; and $g_r(\omega)$ is replaced by $g_r^*(\omega) = g_r(\omega)+\varepsilon_s+\varepsilon_m$. For bottom-heavy models it turns out that ε_s can be slightly negative, but ε_m is a much larger positive, so the partition still produces valid spectra.

Both Cleveland and Box *et al.* assume that the irregular component is white noise, but for top-heavy models $(q^*>p^*)$ our partition will give a moving average irregular. In fact Cleveland discusses a top-heavy ARIMA model for which the X.11 seasonal adjustment method is optimal:

$$z_t = \frac{1+0\cdot26B+0\cdot3B^2-0\cdot32B^3}{(1-B)^2}b_t+\frac{1+0\cdot26B^{12}}{1+B\ldots+B^{11}}c_t+d_t.$$

It would be possible (though difficult) to rearrange our partition of a model with $q^*-p^* = 1$, so as to absorb the first degree term in $h_r(x)$ into the other two components. But the restriction on the irregular component to be white noise seems to be unnecessary. On the other hand, it should be only a low order moving average process: some of the models fitted to monthly series in Dagum (1978) would give an irregular component of order 10 or 11, which could have peaks or troughs in its spectrum at seasonal frequencies. However, later work at Statistics Canada (Lothian and Morry, 1979) indicates that the seasonal operator $(0, 1, 1)_s$ is adequate for all these series and so the irregular is of low order.

Whittle (1963) showed that the best (minimum mean square error) linear estimator of a component m_t given z_t is

$$\hat{m}_t = \frac{f_m(B)f_m(F)}{f(B)f(F)}z_t \quad (\text{since } \sigma_b^2 = \sigma_a^2)$$

Let $g_s^*(\omega)\equiv h_s^*(x)\equiv H_s(B,F)$ where $x = \frac{1}{2}(e^{i\omega}+e^{-i\omega})$ is replaced by $\frac{1}{2}(B+F)$. Then the minimum signal extraction filter for the seasonal component is

$$\frac{h_s^*(x)}{h_z(x)} = \frac{H_s(B,F)}{\psi_s(B)\psi_s(F)}\frac{\psi_s(B)\psi_s(F)\psi_m(B)\psi_m(F)}{\theta(B)\theta(F)}$$

$$= \frac{H_s(B,F)\psi_m(B)\psi_m(F)}{\theta(B)\theta(F)} = \frac{C_s(B,F)}{\theta(B)\theta(F)} \quad \text{(say)}, \tag{7}$$

where $C_s(B,F)$ is a symmetric polynomial in B and F of degree p^*. Similarly, the minimum trend removal filter is

$$\frac{h_m^*(x)}{h_z(x)} = \frac{C_m(B,F)}{\theta(B)\theta(F)} \quad \text{(say)} \tag{8}$$

where $C_m(B,F)$ is also symmetric and of degree p^*. Whittle's original formulation of signal extraction applies to a doubly infinite series and filter. Cleveland (1972) proved that the expected values of the signal series can be obtained by extending the orginal series with forecasts and backcasts. A referee has pointed out that this is a case of the general formula:

$E(X \mid Z) = E[E(X \mid \dot{Z}, Y) \mid Z]$ where Z is a finite series, Y the unobserved future and past values of the series and X the signal component. In practice, the parameters of seasonal models can be close to the boundary of invertibility, which causes very slow convergence of the filter, so that more than 1000 forecasts and backcasts could be needed for reasonable accuracy.

However, this difficulty can be completely avoided by a most ingenious suggestion made to the author by Dr G. Tunnicliffe Wilson. First, the two-sided filters (7) and (8) are each partitioned into two one-sided filters:

$$\frac{C(B,F)}{\theta(B)\,\theta(F)} \equiv \frac{G(B)}{\theta(B)} + \frac{G(F)}{\theta(F)}, \tag{9}$$

where $G(\ )$ is a polynomial of degree $r = \max(p^*, q^*)$. This identity gives rise to $(r+1)$ equations which determine the $(r+1)$ coefficients of $G(\ )$. The details are in the Appendix.

Secondly, we apply the forward and backward filters to z_t, which is now assumed to be extended by forecasts and backcasts.

Let

$$x_{1t} = \frac{G(F)}{\theta(F)} z_t = f_1(F)\, z_t \quad \text{(say)},$$

where f_1 is an infinite series. Forecast z_t:

$$\Phi^*(B)\, z_t = \theta(B)\, a_t \quad (t = N+1, \ldots, N+q^*+r),$$

where $\Phi^*(B) = \phi(B)\Phi(B^s)$ and N is the number of observations. Construct an intermediate series:

$$w_t = G(F)\, z_t \quad (1 \leqslant t \leqslant N+q^*)$$

Now

$$\begin{aligned}\Phi^*(B)\, x_{1t} &= \Phi^*(B) f_1(F)\, z_t = f_1(F)\, \Phi^*(B)\, z_t \\ &= 0 \text{ for } t \geqslant N+q^*+1.\end{aligned}$$

Thus we have (p^*+q^*) equations to find x_{1t} $(t = N+q^*-p^*+1, \ldots N+2q^*)$:

$$\left.\begin{aligned}\theta(F)\, x_{1t} &= w_t (t = N+q^*-p^*+1, \ldots, N+q^*), \\ \Phi^*(B)\, x_{1t} &= 0 (t = N+q^*+1, \ldots, N+2q^*).\end{aligned}\right\} \tag{10}$$

The remaining x_{1t} can be found recursively from the relation in the first part of (10), working backwards to $t = 1$. The mirror image of these steps is applied to the backcast of z_t to give x_{2t}. Finally, the filtered component is the sum of x_{1t} and x_{2t}. The whole process is applied with $G_m(\ )$ for the trend and $G_s(\ )$ for the seasonal component. Details of the method and the matrix of equations (10) are in the Appendix.

The application of the seasonal filter to the original series, together with its forecast and backcast values, constitutes the Minimum Seasonal Extraction method. It will be called hereafter MSX (as MSE is already in use).

It is important at this stage to examine the reason for the paradox cited by Grether and Nerlove (1970)—that a seasonally adjusted series always has dips in its spectrum at seasonal frequencies. Tukey has pointed out that this is analogous to the fact that the fitted residuals in a linear regression are not an independent white noise series. Let $g_m(\omega)+g_r(\omega) = g_y(\omega)$, the spectrum of the adjusted series. The symmetric filter for extracting the seasonal component has a transfer function which is just the square of the filter:

$$\left\{\frac{g_s(\omega)}{g_z(\omega)}\right\}^2.$$

So the *estimated* spectrum of this component is

$$\hat{g}_s(\omega) = \left\{\frac{g_s(\omega)}{g_z(\omega)}\right\}^2 g_z(\omega) = \frac{\{g_s(\omega)\}^2}{\{g_s(\omega)+g_y(\omega)\}^2} g_z(\omega).$$

The estimated spectrum of the adjusted series is

$$\hat{g}_y(\omega) = \frac{\{g_y(\omega)\}^2}{\{g_s(\omega)+g_y(\omega)\}^2} g_z(\omega)$$

so $\hat{g}_s(\omega)+\hat{g}_y(\omega) < g_z(\omega)$.

There is a "deficiency" in both spectra where $g_s(\omega)\, g_y(\omega) \neq 0$, that is, near the seasonal peaks. Not only are there seasonal dips in the estimated spectrum of the adjusted series, but the peaks in the estimated spectrum of the seasonal component are lower than they should theoretically be. It is suggested that this deficiency should be called "the silent spectrum", since it does not relate to either time series component, but only to the cross-spectrum.

4. THE PROGRAM

A Fortran program has been written for MSX; this is in two parts. Part 1 estimates an ARIMA model by maximum likelihood, that is, without backcasting, using the very efficient method described by Osborn (1977). (The method is only full ML for a pure IMA model.) The parameter values and a sufficient number of forecasts and backcasts are passed over to Part 2. This partitions the model spectrum, generates the trend and seasonal filters, and applies them to the series together with its forecasts and backcasts. The forward estimates of the parameters are used for the backcasts, since the backcast ML estimates have been found to be almost identical to the forecast estimates.

The program can handle any ARIMA model up to the third order for the non-seasonal parameters, second order for the MA seasonal and first order for the AR seasonal. It would be easy to remove these restrictions on Q and P, but so far no need has been found. The effective number of observations of changes in the seasonal pattern is small, so that number of seasonal parameters that can be identified must also be small.

Many of the operations to obtain the partition of the spectrum $g_z(\omega)$ have been described in terms of polynomial functions of $x = \cos\omega$. But, in writing the computer program, it was realized that a more convenient representation is in terms of harmonic functions, i.e. linear in $\cos\omega$, $\cos 2\omega$, $\cos 3\omega$, etc. There is a (1, 1) correspondence between the two representations, and multiplication and division of functions is only slightly harder in the harmonic forms.

The seasonal filter has a transfer function with a local minimum at zero and others close to $2\pi(j-\frac{1}{2})/s$ $(j = 2, \ldots, s)$. For the model (0, 1, 1) (0, 1, 1), the minimum minimorum is at either 0 or $\pi(s-1)/s$ (see Burman, 1976); but this is not true generally. The trend filter usually has a simple minimum at π, but a complex model has been constructed which has a second minimum near $\pi/2$.

All widely used methods of seasonal adjustment have a procedure for modifying extreme values. For MSX, we make preliminary estimates $\hat{m}_t$ and $\hat{s}_t$ and set $\hat{r}_t = z_t - \hat{m}_t - \hat{s}_t$. If the RMS of $\hat{r}_t$ is $\hat{\sigma}_r$, multiples α, β of the latter are chosen to determine a modified series:

$$z_t' = z_t - \text{MOD}_t.$$

A term is classed as a partial extreme if $|\hat{r}_t| > \alpha\hat{\sigma}_r$ and a full extreme if $|\hat{r}_t| > \beta\hat{\sigma}_r$.

For an *isolated* extreme the program takes

$$\text{MOD}_t = \lambda_t \hat{r}_t / w_0, \tag{11}$$

where

$$\lambda_t = (|\hat{r}_t| - \alpha\hat{\sigma}_r)/\{(\beta-\alpha)\,\hat{\sigma}_r\} \quad \text{for } \alpha\hat{\sigma}_r < |\hat{r}_t| < \beta\hat{\sigma}_r$$
$$= 1 \quad \text{for } |\hat{r}_t| > \beta\hat{\sigma}_r$$

and

$$w_0 = 1 - 2g_{m0} - 2g_{s0}$$

The scaling factor w_0 is needed because a fraction $2g_{m0}$ of z_t enters the trend estimate, and a fraction $2g_{s0}$ enters the seasonal estimate; where g_{m0} is the coefficient of B^0 in $G_m(B)/\theta(B)$ and g_{s0} similarly in $G_s(B)/\theta(B)$—see equation (9). Thus when r_t is extreme, $\hat{r}_t$ understates the extent of this, unless corrected by w_0.

Extremes quite often occur in *pairs* of opposite sign, especially in series of flows. If these represent displacement effects, it seems natural to adjust them together. Thus, if extremes occur at $(t-1)$ and t, a natural choice would be:

$$\left.\begin{aligned} \text{MOD}'_{t-1} &= -\tfrac{1}{2}(\hat{r}_t - \hat{r}_{t-1}), \\ \text{MOD}'_t &= \tfrac{1}{2}(\hat{r}_t - \hat{r}_{t-1}). \end{aligned}\right\} \tag{12}$$

Continuity between isolated and paired extremes (of opposite sign) can be obtained by only applying (12), when $\lambda_t = \lambda_{t-1} = 1$. For intermediate cases one could take an average of (11) and (12):

$$\text{MOD}^*_{t-1} = (1-\lambda_t)\,\text{MOD}_{t-1} - \lambda_{t-1}\,\lambda_t\,\tfrac{1}{2}(\hat{r}_t - \hat{r}_{t-1}),$$
$$\text{MOD}^*_t = (1-\lambda_{t-1})\,\text{MOD}_t + \lambda_{t-1}\,\lambda_t\,\tfrac{1}{2}(\hat{r}_t - \hat{r}_{t-1}).$$

Let

$$\text{CG (centre of gravity)} = \tfrac{1}{2}(\hat{r}_{t-1} + \hat{r}_t).$$

Then

$$\text{MOD}^*_{t-1} = (1-\lambda_t)\,\text{MOD}_{t-1} + \lambda_{t-1}\,\lambda_t(\hat{r}_{t-1} - \text{CG}),$$
$$\text{MOD}^*_t = (1-\lambda_{t-1})\,\text{MOD}_t + \lambda_{t-1}\,\lambda_t(\hat{r}_t - \text{CG}).$$

An alternative, which generalizes more easily to cover triplets, has been embodied in the program:

$$\left.\begin{aligned} \text{CG} &= (\lambda_{t-1}\,r_{t-1} + \lambda_t\,r_t)/(\lambda_{t-1} + \lambda_t), \\ \text{MOD}^*_{t-1} &= (1-\lambda_t)\,\text{MOD}_{t-1} + \lambda_{t-1}(\hat{r}_{t-1} - \text{CG}), \\ \text{MOD}^*_t &= (1-\lambda_{t-1})\,\text{MOD}_t + \lambda_t(\hat{r}_t - \text{CG}). \end{aligned}\right\} \tag{13}$$

There is no obvious reason for the occurrence of pairs of alternating extremes in a series of levels, but they appear nevertheless. More likely *a priori* would be a triplet of alternating extremes, in which the side terms are "shadows" caused by a genuine extreme in the middle. Since such triplets have not yet been met with in practice, no provision has been made for them in the program. Estimates for pairs of the same sign influence each other and are therefore obtained by solving two simultaneous equations, which are an extension of (11). At present α and β are taken as 2·0 and 2·5.

After modification of extremes, the same ARIMA model is refitted to z_t and revised trend and seasonal components estimated by MSX. The latter are subtracted from the *original* series to give revised residuals, following the convention that extremes should not be modified in a seasonally adjusted series.

The treatment of bias in multiplicative models also needs a mention. For an additive model, the seasonal filter (7) contains a factor $\psi_m(B)$, which renders the series stationary, so the expected values of the z_t and the seasonally adjusted series y_t are equal, apart from seasonal means. Hence the annual arithmetic means (AAM) of the two series have equal expectations. The same argument, applied to the residual filter generating r_t, shows that the AAM of y_t and the trend m_t have equal expectations. These statements, applied to the logarithms of the series in the multiplicative case, show that the annual geometric means (AGM) of the untransformed series have equal expectations. An AM is greater than a GM by an amount which increases with the variance of the terms contributing to the mean. The variance of z_t is greatest and of m_t least, so

$$\text{AAM}(z_t) > \text{AAM}(y_t) > \text{AAM}(m_t).$$

The implied "bias" in y_t relative to z_t can be handled in various ways. In the Bank of England program the AAMs of the seasonal component are scaled separately in calendar years. To avoid the possibility of jumps between calendar years, especially in the trend, it seems better to apply a single bias correction over the whole series. MSX calculates the overall mean of the seasonal factors and the factors representing the irregular component: call these $\bar{b}_1$ and $\bar{b}_2$. Then y_t is scaled up by b_1 and m_t by $\bar{b}_1 \bar{b}_2$. In practice, $\bar{b}_1$ rarely exceeds unity by more than 1 per cent and $\bar{b}_2$ is much smaller.

5. Some Examples

Mr Kenny, of the CSO, kindly supplied the author with seven monthly seasonal series:

1. Average earnings (1963–76).
2. Retail sales (1961–76).
3. Commercial vehicles production (1958–76).
4. Unemployment in GB (excluding school leavers under 18) (1958–76).
5. Domestic furniture deliveries (1963–76).
6. Passenger cars production (1958–76).
7. Engineering orders on hand (1958–76).

The lengths of the series vary from 14 to 19 years. Model identification was in two stages: first a standard model $(0, 1, 1)(0, 1, 1)_{12}$ was fitted to the full-length series and the first 24 autocorrelations of the residuals examined, using the Ljung and Box (1978) Q-test. For series with substantial growth a logarithmic transformation was first applied. Fits at the 5 per cent significance level were obtained for Series 1, 2 and 6. For Series 4 and 7 the autocorrelations suggested that an AR factor or extra non-seasonal differencing was needed, so the model was extended to $(1, 1, 2)(0, 1, 1)_{12}$. The estimation program automatically changes $(1-\phi B)$ into $(1-B)$ if ϕ exceeds 0·96 (about 1 standard error from the stationarity boundary) and re-estimates; it also removes the highest order θ or ϕ, if its coefficient is insignificant (i.e. less than its standard error). For Series 5 a top-heavy model $(0, 1, 2)(0, 1, 1)_{12}$ was fitted. At this stage all models had Q-values below the 1 per cent level except that for Series 3 which was a little above, but the first 6 and the 12th autocorrelations of the residuals of this series were all small and no further model extension was indicated.

The series were then progressively truncated and the same models fitted. The models are shown in Table 1: the suffix 12 has been omitted to save space.

Generally the more complex models were still needed for the shorter series, but, for some of 7 and 8 years' length, Θ exceeded 0·96: this causes cancellation of the seasonal factors in the model and the program instead removes a fixed deterministic component (following Pierce, 1976). For the resulting series the $(1, 0, 0)_s$ operator may be included in the model, but in the few cases tested so far Φ was either negative or insignificantly positive. The program therefore fits the non-seasonal part of the original model, as this is needed by MSX to determine the extreme values.

Attempts to fit more complex seasonal models like $(1, 0, 1)_{12}$ and $(0, 2, 2)_{12}$ were unsuccessful and led to ill-determined parameters and a worse fit than $(0, 1, 1)_{12}$. Our inability to identify $(1, 0, 1)_{12}$ models contrasts with Pierce's success in doing so. A possible reason for this is that in his method the non-seasonal and seasonal parts of the model are fitted in succession instead of simultaneously.

TABLE 1
Models fitted

Series	*Final year of estimation period* 7	8	n−3	n−2	n−1	n	*Series length = n (years)*
1†	(011) (000)	(011) (011)	(011) (011)	(011) (011)	(011) (011)	(011) (011)	14
2†	(011) (011)	(011) (011)	(011) (011)	(011) (011)	(011) (011)	(011) (011)	16
3	(012) (011)‡	(021) (011)‡	(011) (011)	(011) (011)	(011) (011)	(011) (011)	19
4	(112) (000)	(112) (000)	(112) (011)	(112) (011)	(112) (011)	(112) (011)	19
5†	(012) (011)	(012) (011)	(012) (011)	(012) (011)	(012) (011)	(012) (011)	14
5A†§	(011) (011)	(011) (011)	(011) (011)	(011) (011)	(011) (011)	(011) (011)	14
6	(011) (011)	(011) (011)	(011) (011)	(011) (011)	(011) (011)	(011) (011)	19
7	(111) (000)	(111) (000)	(111) (011)	(021) (011)	(021) (011)	(021) (011)	19

† Multiplicative model.
‡ Fixed seasonals on final round.
§ Series 5 adjusted for date of Easter (see below).

TABLE 2a
Preliminary values of Θ

Series	*Last year of estimation period* 7	8	n−3	n−2	n−1	n	*Series length n =*
1	Fixed†	0·93	0·79	0·62	0·66	0·71	14
2	0·72	0·68	0·72	0·67	0·67	0·73	16
3	0·96	0·87	0·80	0·82	0·82	0·83	19
4	Fixed†	Fixed†	0·75	0·72	0·72	0·69	19
5A	0·38	0·53	0·48	0·64	0·61	0·69	14
6	0·65	0·69	0·76	0·77	0·81	0·81	19
7	Fixed†	Fixed†	0·75	0·80	0·82	0·83	19

† Reached boundary imposed by program.

TABLE 2b
Final values of Θ

Series	*Last year of estimation period* 7	8	n−3	n−2	n−1	n	*Series length n =*
1	Fixed†	0·74	0·68	0·57	0·58	0·65	14
2	0·68	0·62	0·62	0·56	0·67	0·71	16
3	Fixed†	Fixed†	0·74	0·75	0·75	0·77	19
4	Fixed†	Fixed†	0·73	0·68	0·69	0·66	19
5A	0·27	0·50	0·44	0·63	0·57	0·64	14
6	0·62	0·65	0·62	0·65	0·70	0·71	19
7	Fixed†	Fixed†	0·66	0·74	0·78	0·79	19

† Reached boundary imposed by program.

The fitted models were used in MSX, as described above, and the extremes modified. It was noticed that for Series 5 (furniture deliveries) extreme residuals were concentrated in March and April, and that the pattern of these pairs (of opposite sign) was almost perfectly correlated with the position of Easter: normally April is a low month for deliveries, but, when Easter fell in March, that month was low. A new series (5A) was created in which a switch from April to March was made in the 4 years with an early Easter. This switch was estimated from the ratios March/(February, April)$^{\frac{1}{2}}$ over the whole 14 years (a convenient, but not optimal procedure). The parameter θ_2 then became insignificant throughout, and the fit of the model improved considerably, so no further results are given for the original Series 5.

Tables 2a and 2b show the preliminary estimates of Θ and the final ones based on the series modified for extremes.

The final values are always lower than the preliminary ones. This would be an advantage if there really is a moving seasonal pattern, which is easier to detect when some of the noise has been removed. For Series 1 (12–14 years) θ_1 became insignificant on the second round, but dropping it made the fit much worse. It seems that the very slight seasonal pattern of this series makes the model parameters ill-determined. The model with θ_1 was therefore retained. Preliminary and final values are quite close together, except for Series 2 (13–14 years), Series 6 (16–19 years), and—not surprisingly—some of the 7- and 8-year runs. Apart from length of series, the differences are linked with the number and size of extremes—Series 6 has the largest number (14). Even after the Easter adjustment, Series 5A has Θ values which are surprisingly low and variable for less than 12 years' data.

TABLE 3
Absorption of a_t into trend and seasonal components (full length)

	Series						
	1	2	3	4	5A	6	7
σ_r/σ_a	0·35	0·55	0·71	0·13	0·44	0·63	0·23
w_0	0·50	0·68	0·81	0·30	0·59	0·75	0·43

The proportion of the innovation variance absorbed by the irregular component (σ_r/σ_a) normally varies between 45 and 70 per cent for the balanced or top-heavy models—see Table 3, but is much lower for the bottom-heavy models (Series 7). This is because $g_r^*(\omega) = \varepsilon_s + \varepsilon_m$, which is usually fairly small. The proportion is also rather low for Series 1, whose small θ_1 makes it akin to a bottom-heavy model and very low for Series 4, which has a similar tendency, as θ_2 is small. In these cases, the trend filter picks up a large part of any extreme values, so that w_0 in equation (11) is also low.

6. COMPARISON WITH OTHER METHODS

To compare objectively different methods of adjustment is not easy: time series charts, relying on visual judgement, are useless, except for eliminating very inferior methods. Spectra of the adjusted and unadjusted series are a limited help, but tests in the time domain are more sensitive. The most important of these are tests for residual seasonality (has the method removed enough?) and for stability (has it removed too much, i.e. some of the noise?). If the latter is the case, revisions—normally annual—will tend to be larger, and this is something that both producers and users wish to avoid. In this section MSX is compared with the well-known X.11 method and also the Bank of England's official method (called here BE)—see Burman (1965). The same limits for extremes ($2{\cdot}0\sigma$ and $2{\cdot}5\sigma$) were employed throughout.

The stability of any method of seasonal adjustment depends on the degree of smoothing. As explained in Burman (1965), there is a trade-off between less smoothing, less stability, and greater sensitivity in following changes in the pattern, on the one hand, and more smoothing, more stability and less sensitivity on the other hand.

X.11 smooths with a [3] [5] moving average except the last 3 (and first 3) years.

BE smooths each of the harmonic components of the seasonal pattern independently, choosing from a range of filters: fixed, exponential weights and [3] [5]—the most flexible choice.

MSX smooths all components in the same way, the weights being determined by the Θ in the model.

Table 4 gives a measure of the flexibility or movement in the seasonal pattern estimated by the three methods over 8-year series and the full-length series. It was expected that, because of its range of smoothing filters, BE would be less flexible than X.11, but this was only true in 9 out of 14 cases. MSX was less flexible then X.11 in 11 cases. The relatively high flexibility of the MSX seasonals for Series 5A (8 years) reflects the low values of Θ in Table 2b.

TABLE 4
Flexibility
(*Mean absolute year on year changes in seasonal component, expressed as percentage of series mean*†)

	8 years			*Full length*		
Series	*X.11*	*BE*	*MSX*	*X.11*	*BE*	*MSX*
1	0·06	0·05	0·03	0·12	0·11	0·10
2	0·11	0·14	0·12	0·17	0·21	0·11
3	0·49	0·17	0‡	1·08	0·57	0·48
4	0·16	0·11	0‡	0·21	0·16	0·25
5A	0·51	0·61	0·72	0·47	0·56	0·42
6	0·87	0·34	0·86	1·16	0·58	0·79
7	0·08	0·14	0‡	0·12	0·10	0·06

† For multiplicative adjustment, the table shows the mean absolute changes in the seasonal factors (expressed as percentages).
‡ Fixed pattern.

When another year's data are added, the mean absolute revision (m.a.r.) to the seasonals in the last year of the series is likely to be relatively large; for the 2nd last year it should be less, the 3rd last year less still, and so on. Eventually the m.a.r. should settle down at a low level (when estimating filter has become nearly symmetric) or even drop to zero (when the filter is truncated as in X.11). Table 5 shows the m.a.r. for different parts of the series expressed as percentages (m.a.p.r.) of the mean of the shorter series. For multiplicative adjustment, it shows the mean absolute changes in the seasonal factors. The comparisons are made between MSX.1—the method as described so far—and the other two methods (see below for explanation of MSX.2). Column 1 gives the m.a.p.r. for the first 7 years of the 7-year and 8-year series; column 2 contains the same calculation for the 7th year of the two series. Columns 3–6 show the m.a.p.r. for the last 4 years of a series when 1 year is added; but, to reduce sampling fluctuations, they have been averaged over 3 pairwise comparisons (for example, for Series 4, these are 16 *vs* 17 years, 17 *vs* 18 years and 18 *vs* 19 years). Column 7 provides the m.a.p.r. over all years from the comparison $(n-1)$ *vs* n years.

As expected, the m.a.p.r. in columns 3–6 descend from right to left, X.11 usually more steeply than the others, but the revisons for the 4th last year are still larger than the average of all years (column 7). For the 7–8 year runs the m.a.p.r. are larger than those for the longer runs, and again they are larger for the 7th year than for the average of all 7 years (except when MSX produces a fixed pattern).

TABLE 5

Mean absolute percentage revisions to seasonal component on adding one year's data

		Lengths of estimation periods							
		7 vs 8		*Averages of 3 pairs: $(n-3)$ vs $(n-2)$, $(n-2)$ vs $(n-1)$, and $(n-1)$ vs n*				*$(n-1)$ vs n*	
		Years of comparison							
	Series	*All years*	*7th*	*4th last*	*3th last*	*2nd last*	*Last*	*All years*	*Full length of series n =*
1	X.11	0·14	0·16	0·07	0·14	0·21	0·30	0·06	14
	BE	0·06	0·07	0·07	0·13	0·20	0·28	0·07	
	MSX.1	0·06	0·10‡	0·07	0·13	0·22	0·32	0·08	
	MSX.2	0·09	0·10‡	0·09	0·15	0·24	0·33	0·08	
2	X.11	0·06	0·19	0·12	0·19	0·31	0·43	0·07	16
	BE	0·09	0·13	0·21	0·31	0·32	0·39	0·14	
	MSX.1	0·09	0·21	0·17	0·22	0·33	0·47	0·07	
	MSX.2	0·12	0·26	0·15	0·21	0·30	0·41	0·09	
3	X.11	1.62	3·57	0·48	0·79	1·23	1·80	0·28	19
	BE	1·30	1·38	0·76	0·87	1·05	1·27	0·44	
	MSX.1	0·69†	0·69†	0·55	0·69	0·89	1·15	0·42	
	MSX.2	1·02	1·13	0·55	0·64	0·75	0·90	0·39	
4	X.11	0·37	0·82	0·20	0·40	0·60	0·80	0·11	19
	BE	0·44	0·43	0·39	0·46	0·59	0·76	0·24	
	MSX.1	0·43†	0·43†	0·27	0·38	0·54	0·78	0·12	
	MSX.2	0·43†	0·43†	0·27	0·37	0·52	0·72	0·12	
5A	X.11	0·17	0·57	0·30	0·53	0·82	1·15	0·20	14
	BE	0·45	0·83	0·51	0·60	0·88	1·05	0·40	
	MSX.1	0·75	1·51	0·38	0·62	0·90	1·26	0·34	
	MSX.2	0·55	1·22	0·40	0·63	0·89	1·22	0·36	
6	X.11	1·30	3·45	0·88	1·60	2·32	2·97	0·36	19
	BE	1·68	2·20	0·99	1·22	1·26	1·53	0·66	
	MSX.1	1.21	2.35	1·07	1·35	1·72	2·25	0·46	
	MSX.2	1·25	2·20	1·04	1·22	1·43	1·66	0·48	
7	X.11	0.11	0·27	0·10	0·16	0·25	0·36	0·04	19
	BE	0·22	0·49	0·13	0·18	0·26	0·32	0·08	
	MSX.1	0·09†	0·09†	0·08	0·10	0·12	0·15	0·04	
	MSX.2	0·09†	0·09†	0·08	0·12	0·11	0·12	0·05	

† Fixed seasonals for 7 and 8 years.
‡ Fixed seasonals for 7 years.

For *Series 1* revisions are small for all three methods, which is not surprising, since the series displays very little seasonality. For *Series 2*, which has stronger seasonality, again all methods are very stable, with X.11 and MSX.1 close together. For *Series 3, 4 and 7*, MSX.1 is the most stable for the longer runs (columns 4–6), though twice X.11 overtakes it in the 4th last year; and MSX.1 is far more stable for the 7–8 year runs, since it selects fixed patterns. *Series 5A's* results are contradictory: X.11 does best by a small margin for the longer runs, but is by far the most stable for 7–8 years. MSX.1's poor showing in the latter case is due to its low values of Θ (see Table 2b), leading to high flexibility (Table 4). Finally, for *Series 6* (longer runs) the order of stability is: BE, MSX.1, X.11, until the 4th last year is reached, when X.11 is best.

Summing up, if stability near the end of a series is more important than the average over the whole series, MSX.1 does very well, in most cases. X.11 is more stable on average over a whole series, but the largest revisions in the table occur with this method (over 3 per cent in column 2, Series 3 and 6). However, MSX.1 has two unsatisfactory features: firstly, it can become very unstable for a short series (Series 5A); and secondly, for bottom-heavy, or nearly bottom-heavy, models, the irregular component is much smaller than for the other methods, because so much goes into the trend. Consequently, it identifies fewer extremes in Series 4 (unemployment) than X.11 or BE in the exceptionally cold winter of 1962–63, and the modifications are much smaller. The second defect does not seem very serious, because MSX.1 is no less stable than the other methods for series 4 and 7.

The instability in Series 5A is undoubtedly partly due to the remaining variability in March and April, but its behaviour does suggest two general lines for further research. Firstly, at what length of series does one start to estimate *moving* seasonality? For BE it is introduced at 7 years; and probably the same should apply for MSX. It would then be natural to introduce a lower bound for Θ in short series, which would be gradually relaxed as the length increased to (say) 10 years; and above that it would be dropped.

Secondly, a more fundamental point, the modification of extremes lowers Θ for every series, thus increasing flexibility and lowering stability. Examination of the preliminary and final seasonal components shows that modification of extremes nearly always produces a more flexible pattern in the months in which they occur. By contrast, with BE the smoothing parameters λ vary both ways between preliminary and final seasonals, and the (simple) averages of these values are close together.

7. EXTREMES, STABILITY AND RESIDUAL SEASONALITY

If there are no prior reasons for expecting an extreme in certain months (e.g. strikes, exceptional weather), is there any evidence that extremes represent departures from normality? For the seven series in our sample, the absolute values of the residuals in MSX exceed $2\hat{\sigma}_r$ for 4–6 per cent of the observations; but they exceed $2{\cdot}5\hat{\sigma}_r$ for an average of $2\frac{1}{2}$ per cent of the observations—twice the expected proportion for a normal distribution. So the modification of extremes can be partly justified on the grounds that (with full replacement above $2{\cdot}5\sigma$) they stand outside the process generating the main series. Against this must be set the size of revisions to estimates of the extremes, which are the immediate cause of the instability of the seasonal component in certain cases.

Is there any trade-off for the lower stability caused by modifying extremes? Stability is only one half of performance; the other half is the absence of residual seasonality. There is no generally accepted test of the latter. Idempotency is one possibility (Fase, Koning and Volgenant, 1973)—running an adjusted series through the same procedure again. Another (used in BE) is to calculate von Neumann ratios for the residuals of each month, and to check whether the number of significantly low ratios in a group of series exceeds the number expected. Yet another test is to see if the spectrum of an adjusted series is smooth and removes no power at inter-seasonal frequencies. Spectra were estimated for the three seasonally adjusted versions of each full-length series, and proved to be very close together—the only interesting feature being the appearance of the predicted Grether–Nerlove "dips" at some of the seasonal frequencies. (Copies of the charts may be obtained from the author.)

The method finally chosen for testing residual seasonality was as follows: fit a non-seasonal ARIMA model to each seasonally adjusted series, and calculate r_{12}, r_{24} and r_{36} for the residuals: then the analogue of the Ljung–Box test is

$$Q_s = n(n+2)\sum_{j=1}^{3} r_{12j}^2/(n-12j)$$

where n is the number of terms in the differenced series. The probability distribution of Q_s is not known exactly—see Pierce (1976); it is thought to lie between $\chi^2(2)$ and $\chi^2(3)$ for a one-parameter model, most probably close to $\chi^2(3)$. Table 6 shows the values of Q_s for each method, using Series 2–7: first, the preliminary seasonal adjustments applied to the original series; second, the final seasonal adjustments applied to the *modified* series. All but one of the r_{12} are negative, three-quarters of the r_{24} and all the r_{36}—this apparent "over-adjustment" is the time-domain equivalent of the Grether–Nerlove effect. We note first that the large majority of the series have Q_s significant at the nominal 1 per cent level, but since the true distribution is unknown, it is difficult to draw any conclusions from this. Secondly, it is remarkable that, for X.11, Q_s increases from the preliminary to the final round. For BE and MSX, Q_s goes in both directions, but the shift is small with MSX.1 for Series 4 and 7, for which the estimated extremes are small; and for Series 5A the instability noted for this method in Table 5 is accompanied by a worsening of Q_s. Thirdly, the final adjustments for BE have consistently lower residual seasonality than either MSX.1 or X.11.

TABLE 6

Residual seasonality test (Q_s)

		Series					
		2	3	4	5A	6	7
MSX.1	Preliminary	15·74‡	15·43‡	18·62‡	13·02‡	20·06‡	10·88†
	Final	12·78‡	11·13†	18·09‡	18·80‡	21·67‡	11·09†
MSX.2	Final	11.77‡	7·07	15·46‡	16·63‡	10·86†	7·85
BE:	Preliminary	12·09‡	12·73‡	1·81	7·91	11·49†	19·69‡
	Final	5·83	11·10†	6·73	14·44‡	3·31	10·34†
X.11:	Preliminary	12·61‡	20·19‡	7·80	15·96‡	24·91‡	28·43‡
	Final	15·94‡	30·85‡	12·81‡	19·42‡	27·37‡	33·37‡

† Significant at 5 per cent point for $\chi^2(3)$.
‡ Significant at 1 per cent point for $\chi^2(3)$.

Since MSX.1—modifying extremes and re-estimating the model—does not show a uniform reduction in Q_s, a better trade-off between stability and residual seasonality may be possible. MSX.2 is defined to be the same as MSX.1 up to the modification of extremes: but the preliminary parameter values are used again on the second round to obtain forecasts and backcasts. Table 6 shows, surprisingly, that MSX.2 has lower Q_s values than the final seasonals of MSX.1. Referring back to Table 5, MSX.2 has almost the same stability as MSX.1 for *Series 1*, but is slightly better for *Series 2*. For *Series 3* MSX.2 is definitely more stable on the longer runs, but less on the shorter ones, because there is not a fixed seasonal pattern on the first round of estimation. For *Series 4 and* 7 there is little difference between the two variants, as the identified extremes are small. The same is true of *Series 5A*, except for the 7- and 8-year runs, where the marked improvement stems from avoidance of low Θ values. Finally, for the longer runs of *Series 6* there is a dramatic improvement: it is now nearly as stable as BE.

We conclude that MSX.2 is to be preferred, and that its stability for recent observations is substantially better than that of X.11, except in the case of *Series 5A*. MSX.2 is, of course, quicker to run than MSX.1: for example, on an IBM 370/158, a 12-year monthly series takes about 8 seconds CPU time (provided reasonable starting values of the model parameters in the estimation can be obtained). This compares with 7 seconds for X.11 and 6 seconds for BE.

8. BOTTOM-HEAVY MODELS

When this paper was virtually complete, it was realized that there was a simple way of moving more of the high frequency spectral power from the trend to the irregular, which could be applied to bottom-heavy models (Series 4 and 7). Using the notation of Section 3, we first observe that $g_m(\omega)$ declines more slowly towards its minimum (say ε_0) at $\omega = \pi$ than it does for balanced models. We therefore assume that the irregular component may follow a first order moving average: $\varepsilon_s + \varepsilon_0 + \varepsilon_1(1 + \cos\omega)$; and determine ε_1 by minimizing $\{g_m(\omega) - \varepsilon_0\}/(1 + \cos\omega)$. In the two cases examined so far, the minimum of the latter expression is also at $\omega = \pi$, so the final trend spectrum—

$$g_m^*(\omega) = g_m(\omega) - \varepsilon_0 - \varepsilon_1(1 + \cos\omega)$$

is still monotone decreasing. The new values of w_0 (see Table 3) for Series 4 and 7 are 0·45 and 0·54, and the new values of σ_r/σ_a are 0·21 and 0·29 respectively. The average magnitudes of the extremes are now only a little less than those of X.11 and BE, though the individual extremes picked out by the three methods are not always the same.

The above procedure could be applied to remove still more from the trend, if a second order moving average were acceptable for the irregular component.

9. CONCLUSIONS

What conclusions can be drawn at this stage about the use of Signal Extraction for seasonal adjustment? Obviously they must be very tentative until a much larger number of series has been tested:

(i) Trend and seasonal signal extraction filters can be derived for all suitable seasonal ARIMA models; and, although these filters are doubly infinite, their effect on the original series can be obtained by simple finite operations. It is desirable to replace or modify extreme values of the series and re-apply the filters (Section 4).

(ii) Fitting ARIMA models to a large number of series is quite practical: first using the standard $(0, 1, 1)(0, 1, 1)_s$ model and then extending it to the $(1, 1, 2)(0, 1, 1)_s$ family, if the diagnostic checks suggest it. For the latter, the non-seasonal operator is automatically simplified to $(1, 1, 1)$ or $(0, 2, 1)$ if appropriate. Initially some skilled resources are needed for model identification, but thereafter the same model can generally be used for a number of years.

(iii) Only one seasonal operator is needed in practice, but there is some variety of non-seasonal operators, leading to balanced, top-heavy or bottom-heavy models. For bottom-heavy models, too much of the current observation is absorbed into the trend, compared with X.11 and the Bank of England method, resulting in implausibly small modifications for extreme values. This "defect" can probably be overcome by assuming a moving average irregular component (Section 8).

(iv) For short series the seasonal model may be degenerate, implying a fixed seasonal pattern, so that the adjustment consists simply of subtraction of seasonal means.

(v) Generally, both signal extraction (MSX) and the Bank of England method are less flexible than X.11 and give considerably smaller revisions to the *last 3 years* of a series, without leaving any more residual seasonality than X.11. MSX often produces larger average revisions than X.11 over the *whole* series (except in the case of the shorter series where a fixed seasonal pattern is obtained). However, this disadvantage could be nullified by a sensible publication policy: for example, in official publications no amendments (solely due to revised seasonal adjustment) would be made to data more than 4 years' old; but complete revisions could be made available to research workers on request.

(vi) MSX.2, which omits re-estimation of the model parameters, seems to be more stable and, at the same time, leaves less residual seasonality than MSX.1 (which includes re-estimation).

(vii) For shorter series—under 7 years—probably no attempt should be made to find moving seasonality. For intermediate lengths (7–10 years) some lower bound could be placed on the values of Θ, to improve stability, and this restriction would be gradually relaxed as the series lengthened.

(viii) Signal extraction is now ready for large-scale trials by other statisticians. Progress from ad hoc to more theoretically optimal methods of seasonal adjustment could be rapid in the next few years.

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APPENDIX: TUNNICLIFFE WILSON ALGORITHM

Let $G(B) = g_0 + g_1 B \ldots + g_r B^r$ and $C(B, F) = c_0 + c_1(B+F) \ldots + c_r(B^r + F^r)$. From (9):

$$\theta(F)\, G(B) + \theta(B)\, G(F) \equiv C(B, F).$$

Equating coefficients of B^r, B^{r-1}, etc.,

$$\begin{aligned} \theta_0 g_r + \theta_r g_0 &= c_r \\ \theta_1 g_r + \theta_0 g_{r-1} + \theta_r g_1 + \theta_{r-1} g_0 &= c_{r-1} \\ &\vdots \\ 2(\theta_r g_r + \theta_{r-1} g_{r-1} \dots + \theta_0 g_0) &= c_0. \end{aligned}$$

So if A is defined as

$$A = \begin{bmatrix} \theta_0 & 0 & \dots & \dots & 0 \\ \theta_1 & \theta_0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \theta_r & \theta_{r-1} & \dots & \dots & \theta_0 \end{bmatrix} + \begin{bmatrix} 0 & \dots & \dots & 0 & \theta_r \\ 0 & \dots & 0 & \theta_r & \theta_{r-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \theta_r & \theta_{r-1} & \dots & \dots & \theta_0 \end{bmatrix}.$$

The equations become

$$A(g_r\, g_{r-1}, \dots, g_0)' = (c_r\, c_{r-1}, \dots, c_0)'.$$

(The ordering of the matrix columns puts ones in most elements of the leading diagonal easier inversion.)

For the second stage, let $\Phi^*(B) = \phi_0 + \phi_1 B \dots + \phi_p B^p$ (with p written for p^*). Equati(10) are (with q written for q^*):

$$\begin{bmatrix} \theta_0 & \theta_1 & \dots & \theta_q & 0 & \dots & \dots & 0 \\ 0 & \theta_0 & \theta_1 & \dots & \theta_q & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & \theta_0 & \theta_1 & \dots & \dots & \theta_q \\ \phi_p & \dots & \dots & \phi_1 & \phi_0 & 0 & \dots & 0 \\ 0 & \phi_p & \dots & \dots & \phi_1 & \phi_0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & \phi_p & \dots & \dots & \phi_1 & \phi_0 \end{bmatrix} \begin{bmatrix} x_{N+q-p+1} \\ \vdots \\ \vdots \\ x_{N+q} \\ \vdots \\ \vdots \\ \vdots \\ x_{N+2q} \end{bmatrix} = \begin{bmatrix} w_{N+q-p+1} \\ \vdots \\ \vdots \\ w_{N+q} \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}.$$

The matrix is of order $(p^* + q^*)$ and has ones in the leading diagonal.

REGRESSION ANALYSIS OF SEASONAL DATA*

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When economists apply regression to monthly or quarterly data to estimate structural parameters, they generally either (a) use seasonally adjusted data or (b) use unadjusted data and allow for seasonal shifts in the intercept term. Starting from a more general model than either of these procedures, this paper spells out their implicit assumptions and derives expressions for the specification bias arising from incorrect assumptions. The expected values of the estimates in (a) and (b) are weighted averages of the coefficients in the more general model. The weights are regression coefficients. If parameters do not vary seasonally, procedure (b) yields unbiased estimates. Expected values of estimates from procedure (a) are random variables. If we apply covariance analysis to test for seasonal variation in coefficients, seasonally unadjusted data will yield unbiased estimates; seasonally adjusted data will not. Empirical results obtained from seasonally adjusted data differ substantially from those obtained from unadjusted data.

1. INTRODUCTION

Economists frequently apply multiple regression analysis to monthly or quarterly time series data to study various aspects of short-term economic behavior. It is rather common practice to use seasonally adjusted data in these regression analyses. In 1947 Hurwicz [2] raised questions about the desirability of using seasonally adjusted data. Apparently his main objection arose from the possibility that coefficients may vary from month to month or from quarter to quarter. Klein [3, p. 314] also questioned the advisability of using seasonally adjusted data, and he suggested the use of four seasonal variables, each variable having the value of one in its respective season and zero in all other seasons. In recent years, some economists have used this procedure of including seasonal variables in regression analyses of unadjusted data. What is the relation between the two procedures? What is their relation to some more general procedure? What are the implicit assumptions of each, and what are the effects if these implicit assumptions do not hold?

Some answers are provided in this paper. After a suitable notation is developed, these answers are obtained from a straightforward application of Theil's method of analysis [7, pp. 214, 327–9].

Suppose we are dealing with quarterly data (the argument can easily be generalized to monthly or weekly data by replacing the multiplier 4 by 12 or 52). Let T equal number of years in the observation period and m equal the number of independent variables. X_{ijt} represents the value of X_i in the j-th quarter of the t-th year; $i=1, 2, \cdots, m$; $j=1, 2, 3, 4$; $t=1, 2, \cdots, T$.

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$$X_{it} = \begin{bmatrix} X_{i1t} & 0 & 0 & 0 \\ 0 & X_{i2t} & 0 & 0 \\ 0 & 0 & X_{i3t} & 0 \\ 0 & 0 & 0 & X_{i4t} \end{bmatrix} \tag{1}$$

$$X_i = (X_{i1}X_{i2} \cdots X_{iT})', \quad \text{a } 4T\times 4 \text{ matrix} \tag{2}$$

$$X = (X_1X_2 \cdots X_m), \quad \text{a } 4T\times 4m \text{ matrix} \tag{3}$$

$$D_{it} = (D_{i1t}D_{i2t}D_{i3t}D_{i4t})', \tag{4}$$

a 4×1 column vector of the values of the i-th seasonal variable in the four quarters of year t. D_{ijt} might, for example, be unity in the j-th quarter and zero all other quarters.

$$D_t = (D_{1t}D_{2t}D_{3t}1), \quad \text{a } 4\times 4 \text{ matrix} \tag{5}$$

$$D = \begin{bmatrix} D_1 \\ D_2 \\ \vdots \\ D_T \end{bmatrix}, \quad \text{a } 4T\times 4 \text{ matrix} \tag{6}$$

$$I^* = 4m \times m \text{ matrix}, \tag{7}$$

in the i-th column the elements from $4(i-1)+1$ to $4(i-1)+4$ are ones; all other elements in that column are zero.

$$XI^* = (X_1^*X_2^* \cdots X_m^*) = X^* \tag{8}$$

X^* is a $4T\times m$ matrix. The i-th column contains the $4T$ values of the i-th independent variable.

$$y = (y_{11}y_{21}y_{31}y_{41}y_{12} \cdots y_{1T}y_{2T}y_{3T}y_{4T})' \tag{9}$$

y is a $4T\times 1$ column vector in which y_{jt} = observation on dependent variable in j-th quarter in t-th year.

2. GENERAL MODEL

The most general model which is linear in variables and parameters is

$$y = (XD)\begin{pmatrix} \beta \\ \beta_D \end{pmatrix} + \epsilon. \tag{10}$$

β is the $4m\times 1$ column vector $(\beta_{11}\beta_{12}\beta_{13}\beta_{14}\beta_{21} \cdots \beta_{m1}\beta_{m2}\beta_{m3}\beta_{m4})'$. β_{ij} represents the slope with respect to the i-th variable in the j-th quarter. β_D is the 4×1 column vector $(\beta_{D1}\beta_{D2}\beta_{D3}\beta_{D0})'$. This model allows each slope and the intercept to vary seasonally, but makes no assumption about the pattern of variation.

3. LINEAR SEASONAL SHIFT MODEL

This model, which utilizes seasonal variables of the type suggested by Klein, can be written as

$$y = (X^*D)\begin{pmatrix} \beta_L \\ \beta_{LD} \end{pmatrix} + \epsilon_L. \tag{11}$$

β_L is an $m\times 1$ column vector; β_{LD} is a 4×1 column vector. This model assumes $\beta_{ij}=\beta_{ik}$, for all j, k; for $i=1, 2, \cdots, m$. It assumes constant slopes but allows the intercept to vary seasonally.

To obtain an expression for the least squares estimates of β_L and β_{LD}, it is convenient to use a formula for the inverse of a partitioned matrix:

$$\begin{pmatrix} e & f \\ g & h \end{pmatrix}^{-1} = \begin{pmatrix} E & G \\ F & H \end{pmatrix} \tag{12}$$

where

$$\begin{aligned} H &= (h - ge^{-1}f)^{-1} \\ G &= - e^{-1}fH \\ F &= - Hge^{-1} \\ E &= e^{-1} - e^{-1}fF. \end{aligned} \tag{13}$$

Equations (12) and (13) can be applied to the present problem by setting

$$\begin{aligned} e &= X^{*\prime}X^{*} \\ f &= X^{*\prime}D \\ g &= D'X^{*} \\ h &= D'D. \end{aligned} \tag{14}$$

Simplification is achieved if we define

$$C = (X^{*\prime}X^{*})^{-1}X^{*\prime}D. \tag{15}$$

C is the least squares estimate of the regression of D on X^* and μ is the matrix of residuals: $D=X^*C+\mu$.

$$\mu'\mu = D'D - D'X^{*}(X^{*\prime}X^{*})^{-1}X^{*\prime}D. \tag{16}$$

Letting b_L and b_{LD} represent the least squares estimates of β_L and β_{LD} and applying equations (12) through (16), we have

$$\begin{pmatrix} b_L \\ b_{LD} \end{pmatrix} = \begin{bmatrix} (X^{*\prime}X^{*})^{-1} + C(\mu'\mu)^{-1}C' & -C(\mu'\mu)^{-1} \\ -(\mu'\mu)^{-1}C' & (\mu'\mu)^{-1} \end{bmatrix} \begin{bmatrix} X^{*\prime}y \\ D'y \end{bmatrix}. \tag{17}$$

The estimate b_L can be written as

$$b_L = (X^{*\prime}X^{*})^{-1}X^{*\prime}y - C(\mu'\mu)^{-1}\mu'y. \tag{18}$$

Substitute (10) into (18) and take expectations, assuming the fixed X or regression model. X^* and D are then fixed numbers; C is a function of fixed numbers and is, therefore, fixed; hence, μ is also fixed.

$$\begin{aligned} E(b_L) = E[&(X^{*\prime}X^{*})^{-1}X^{*\prime}X\beta - C(\mu'\mu)^{-1}\mu'X\beta \\ &+ C\beta_D - C(\mu'\mu)^{-1}\mu'D\beta_D \\ &+ (X^{*\prime}X^{*})^{-1}X^{*\prime}\epsilon - C(\mu'\mu)^{-1}\mu'\epsilon]. \end{aligned} \tag{19}$$

The third and fourth terms are easily disposed of; they sum to zero since

$(\mu'\mu)^{-1}\mu'D=I$, the identity matrix. Assuming that $E(\epsilon)=0$ in equation (10) is sufficient to assure that the fifth and sixth terms each equal zero since we are dealing with a fixed X model.

The second term is not so easily disposed of. By the definition of μ, $E(\mu'\mu)^{-1}\mu'X^*=0$. The element in the i-th row and j-th column of X^* is the sum of the four elements in the i-th row of columns $4(j-1)+1$ to $4(j-1)+4$ of X. If $E(\mu'\mu)^{-1}\mu'X^*=0$ and $E(\mu'\mu)^{-1}\mu'X\neq 0$, this implies a multivariate distribution in which: (a) $\sum_{j=1}^{4}E\mu_{jt}X_{ijt}=0$ and (b) $E\mu_{jt}X_{ijt}\neq 0$ for two or more values of j. In the absence of any strong reasons for believing that these covariances will, in fact, undergo such seasonal variation, it is reasonable to assume $E(\mu'\mu)^{-1}\mu'X=0$. This will be assumed. Then

$$E(b_L) = P_{X.X^*}\beta \tag{20}$$

where

$$P_{X.X^*} = (X^{*\prime}X^*)^{-1}X^{*\prime}X. \tag{21}$$

$P_{X.X^*}$ is an $m\times 4m$ matrix of regression coefficients. The elements in column $4(i-1)+1$ are the coefficients in the regression of the first column of X_i on all variables in X^*. The elements in column $4(i-1)+4$ are the coefficients in the regression of the fourth column of X_i on all variables in X^*.

$$P_{X.X^*} = \begin{bmatrix} P_{11}(1) & P_{12}(1) & P_{13}(1) & P_{14}(1) & \cdots & P_{m4}(1) \\ P_{11}(2) & P_{12}(2) & P_{13}(2) & P_{14}(2) & \cdots & P_{m4}(2) \\ \vdots & & & & & \\ P_{11}(m) & P_{12}(m) & P_{13}(m) & P_{14}(m) & \cdots & P_{m4}(m) \end{bmatrix}. \tag{22}$$

These $P_{ij}(k)$ are regression coefficients in

$$(X_{ijt}) = P_{ij}(1)X_{1jt} + P_{ij}(2)X_{2jt} + \cdots + P_{ij}(m)X_{mjt} \tag{23}$$

where,

$$\begin{aligned}(X_{ijt}) &= X_{ijt},\ j\text{-th quarter of each year} \\ &= 0,\ \text{all other quarters.}\end{aligned} \tag{24}$$

The variable (X_{ijt}) is the j-th column of X_i.

Each element of $E(b_L)$ is a weighted average of all elements in β. Write the k-th element of β_L as β_{Lk}. Then

$$E(b_{Lk}) = \sum_{i=1}^{m}\sum_{j=1}^{4}\beta_{ij}P_{ij}(k). \tag{25}$$

$\sum_j P_{ij}(i)=1$ and $\sum_j P_{ij}(r)=0$ for $r\neq i$. This is easily seen if we note that the sum of the four columns of X_i is equal to the i-th column of X^*. Thus, if all slopes are constant seasonally,

$$E(b_{Lk}) = \beta_{k1} = \beta_{k2} = \beta_{k3} = \beta_{k4} = \beta_k. \tag{26}$$

If the β_{kj} vary seasonally, but all other coefficients are constant seasonally,

$$E(b_{Lk}) = \sum_{j=1}^{4} \beta_{kj} P_{kj}(k). \tag{27}$$

$E(b_{Lk})$ is a weighted average of the four quarterly coefficients with the sum of the weights equal to one. If the β_{kj} are constant seasonally, but all other coefficients vary seasonally,

$$E(b_{Lk}) = \beta_k + \sum_i{}' \sum_j \beta_{ij} P_{ij}(k) \tag{28}$$

where $\sum_i'$ denotes summation over all values of i excluding $i=k$.

The estimates of β_{LD} are

$$b_{LD} = (\mu'\mu)^{-1}\mu' y. \tag{29}$$

From an argument similar to the preceding one, it follows that

$$E(b_{LD}) = \beta_D. \tag{30}$$

4. SEASONALLY ADJUSTED DATA MODEL

The use of seasonally adjusted data carries the implict assumption that slopes and intercepts vary in a certain way. A version of model (10) which permits intercepts and slopes to vary is

$$y_{jt} = \beta_{0j} + \sum_i \beta_{ij} X_{ijt} + \epsilon_{jt}. \tag{31}$$

Use s_{jt} to denote the seasonal index for y_{jt} and r_{ijt} to denote the seasonal index for X_{ijt}.

The equation customarily estimated with seasonally adjusted data is of the form

$$y_{jt}/s_{jt} = \beta_{0s} + \sum \beta_{is} X_{ijt}/r_{ijt} + \epsilon_{sjt}. \tag{32}$$

Multiplying (32) by s_{jt} and equating the result to (31) we see that the use of seasonally adjusted data implies the assumptions

$$\beta_{0s} = \beta_{0j}/s_{jt} \tag{33.a}$$

$$\beta_{is} = \beta_{ij} r_{ijt}/s_{jt} \tag{33.b}$$

$$\epsilon_{sjt} = \epsilon_{jt}/s_{jt} \tag{33.c}$$

where β_{0s} and β_{is} are the parameters estimated from seasonally adjusted data. According to these equations, the slopes and intercepts vary seasonally, and in a certain way. Although slopes and intercept may vary seasonally, there is no reason they should vary in this certain way. If s_{jt} or r_{ijt} vary secularly or cyclically (i.e., changing seasonals), the use of seasonally adjusted data then implies a certain pattern of secular or cyclical change in the slopes and intercept.

Since these assumptions are quite special, it seems worthwhile to investigate what happens when they are not met, assuming (10) is a valid model. Let

s_{jt} = seasonal index for j-th quarter of t-th year value of the dependent variable. (34)

$$S_t = \begin{bmatrix} 1/s_{1t} & 0 & 0 & 0 \\ 0 & 1/s_{2t} & 0 & 0 \\ 0 & 0 & 1/s_{3t} & 0 \\ 0 & 0 & 0 & 1/s_{4t} \end{bmatrix} \tag{35}$$

$$S = \begin{bmatrix} S_1 & 0 & \cdots & 0 \\ 0 & S_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & S_T \end{bmatrix} \tag{36}$$

S is a $4T \times 4T$ matrix.

r_{ijt} = seasonal index for j-th quarter of t-th year value of i-th independent variable in X^*. (37)

$$R_{it} = \begin{bmatrix} 1/r_{i1t} & 0 & 0 & 0 \\ 0 & 1/r_{i2t} & 0 & 0 \\ 0 & 0 & 1/r_{i3t} & 0 \\ 0 & 0 & 0 & 1/r_{i4t} \end{bmatrix} \tag{38}$$

$$R_i = \begin{bmatrix} R_{i1} & 0 & & 0 \\ 0 & R_{i2} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & R_{iT} \end{bmatrix} \tag{39}$$

R_i is a $4T \times 4T$ matrix.

$$X_A^* = (X_1^* X_2^* \cdots X_m^* e) \tag{40}$$

where e is a $4T \times 1$ column vector of ones.

$$RX_A^* = (R_1 X_1^* R_2 X_2^* \cdots R_m X_m^* e) = X_R \tag{41}$$

X_R is a $4T \times (m+1)$ matrix of seasonally adjusted values of the independent variables and a constant term.

The model using seasonally adjusted variables can now be written as

$$Sy = X_R \beta_s + \epsilon_s. \tag{42}$$

The least squares estimate of β_s is

$$b_s = (X'_R X_R)^{-1} X'_R Sy. \tag{43}$$

Multiplying both sides of equation (10) by S, substituting into (43) and taking mathematical expectations

$$E(b_s) = E[(X'_R X_R)^{-1} X'_R SX\beta + (X'_R X_R)^{-1} X'_R SD\beta_D + (X'_R X_R)^{-1} X'_R S\epsilon]. \tag{44}$$

If S were constant, the last term would be zero. It is not constant, and there are reasons to expect the last term to be nonzero. (1) S is a function of y, which in turn is a function of X and ϵ. S is not independent of X nor of ϵ. (2) R_i is a function of X_i. Hence R_i and X_i are not independent. (3) Because of their common dependence on X_i, S and R_i may be correlated.

$$E(b_s) = E(P_{SX.X_R})\beta + E(P_{SD.X_R})\beta_D + E(P_{S.X_R})\epsilon \tag{45}$$

$E(b_s)$ is a weighted sum of the β_{ij}, the β_{Dj} and the ϵ. That b_s is a function of ϵ is to be expected. That $E(b_s)$ is a function of ϵ is indeed surprising. Even though the last expectation in (45) is zero, if the assumptions (33.a) and (33.b) do not hold, each coefficient in $E(b_s)$ is a function of all elements of β and of β_D. $P_{SX.X_R}$ is the matrix of regression coefficients of (a) SX, the variables obtained by adjusting each independent variable by the seasonal index for y, on (b) X_R, the seasonally adjusted values of the independent variables. $P_{SD.X_R}$ is the matrix of regression coefficients of (a) SD, the variables obtained from adjusting the seasonal variables by the seasonal index for y, on (b) X_R. $P_{S.X_R}$ is the matrix of regression coefficients of (a) S, the seasonal index for y, on (b) the seasonally adjusted values of the independent variables. Even though we assume the fixed X or regression model, expectations enter into (45) since S is a function of y and y is a random variable.

In the case of equation (25), we were able to conclude that $\sum_j P_{ij}(i) = 1$ and $\sum_j P_{ij}(r) = 0$. No similar conclusion holds here for the elements of $P_{SX.X_R}$ or $P_{SD.X_R}$.

5. ESTIMATING EACH SEASON SEPARATELY

One might estimate the coefficients separately each quarter to test for seasonal variation in the parameters. The use of seasonally adjusted data may indicate seasonal variation where none exists or may indicate no seasonal variation where it does exist. Letting q denote the individual quarter and writing (10) and (42) for one quarter,

$$y_q = X_q\beta_q + \epsilon_q \tag{46}$$

$$S_q y_q = X_{Rq}\beta_{sq} + \epsilon_{sq} \tag{47}$$

y_q is a $T\times 1$ vector of q-th quarter observations on the dependent variable. X_q is a $T\times(m+1)$ matrix of q-th quarter observations on the independent variables, augmented by a column of ones. X_{Rq} is a similarly augmented matrix of seasonally adjusted values of the independent variables in the q-th quarter. S_q is a $T\times T$ matrix of q-th quarter seasonal indices for y. β_{sq} and β_q are column vectors of $m+1$ coefficients. The least squares estimate of β_{sq} is

$$b_{sq} = (X'_{Rq}X_{Rq})^{-1}X'_{Rq}S_q y_q \tag{48}$$

and its expected value is

$$E(b_{sq}) = E(P_q)\beta_q + E(P_{S_q.X_{Rq}})\epsilon_q \tag{49}$$

where P_q is the matrix of regression coefficients of S_qX_q on X_{Rq} and $P_{S_q.X_{Rq}}$ is the matrix of regression coefficients of S_q on X_{Rq}. Even though β_q is invariant, b_{sq} and $E(b_{sq})$ may vary seasonally. b_{sq} and $E(b_{sq})$ may not vary seasonally even though β_q does.

Suppose the linear shift model is used for quarterly estimation.

$$y_q = X_q\beta_{qL} + \epsilon_{qL}. \tag{50}$$

The estimates of β_{qL} are

$$b_{qL} = (X_q' X_q)^{-1} X_q' y_q \tag{51}$$

$$E(b_{qL}) = \beta_q + (X_q' X_q)^{-1} X_q' E(\epsilon_q). \tag{52}$$

If we assume $E(\epsilon_q) = 0$ we obtain

$$E(b_{qL}) = \beta_q. \tag{53}$$

If there are no seasonal shifts in the parameters, $E(b_{qL})$ is the same each quarter. If there is seasonal variation, $E(b_{qL})$ reflects it.

If one wants to use covariance analysis to test the hypothesis of no seasonal variation in intercepts and slopes, then, he should not use seasonally adjusted data. If the null hypothesis of no seasonal changes is accepted, the model obtained from (11) by deleting the first three columns of D and the first three elements of β_D is (conditionally) appropriate. If the covariance analysis accepts the null hypothesis of no seasonal variation in slopes, but rejects the null hypothesis of no seasonal variation in slopes and intercepts, (11) would be (conditionally) appropriate. Even if the covariance analysis accepts the hypothesis of seasonal variation in slopes and intercepts, the model (42) is appropriate only if the coefficients vary as (33.a) and (33.b).

Commonly, the parameters in equations such as (10), (11), or (42) have behavioral interpretation; marginal propensity to consume, stock-sales ratio, flexibility of price expectations, etc. It is evident that the coefficients obtained from the use of seasonally adjusted data or from the linear seasonal shift model need have no such behavioral interpretation since each estimated coefficient is a weighed sum of behavioral parameters.

The use of the seasonal shift model has this one advantage; it permits testing hypotheses about shifts in intercepts given constant slopes. The use of the seasonally adjusted model, on the other hand, has a maintained hypothesis that slopes and intercepts vary and that they vary in a specified fashion. Thus, there appear to be dangers inherent in the use of either procedure.

One possibility was mentioned earlier; that of using covariance analysis to test for differences among intercepts and among slopes. A make-shift but useful alternative to covariance analysis is the following. Use the linear shift model, which permits seasonal variations in the intercept. Then to check on the possibility that seasonal rotations occur, plot residuals from the equation against various independent variables. When seasonal differences appear to exist in slopes, add variables to the regression.

6. AUTOCORRELATED ERRORS

So long as the assumption of the fixed X or regression model is maintained (among other things this means no lag values of y appearing in X) the analysis is quite straightforward. Sometimes models appropriate for analysis of monthly or quarterly data contain lagged values of the dependent variable among the independent variables. If lagged values of y do appear among the independent variables, the preceding conclusions may be interpreted as probability limits for large sample estimates.

The existence of lagged values of y among the independent variables introduces the possibility of bias arising from autocorrelation in the disturbances. Klein [3, p. 314], Hurwicz [2] and others have pointed out the increased likelihood of autocorrelation in the errors as the unit observation period is shortened from a year, to a quarter, to a month. If lagged values of y do not appear in the X matrix, the presence of autocorrelated disturbances will not bias the estimates of the coefficients, but will bias the t and F tests. If lagged values of y do appear in the X matrix, however, biased estimates of the coefficients will result. It does not seem possible to make any general statements about the autocorrelation properties of ϵ_L or e_s. Even though the ϵ in (10) and their estimates e possess no autocorrelation, it is possible that the ϵ_L or ϵ_s will possess autocorrelation. It is a well-known fact that a series which is temporally independent can be converted to an autocorrelated series by converting it to a moving average [8, pp. 204–5]. It is also true that a series which is temporally independent can be converted to an autocorrelated series by dividing it by a moving average.

7. STATISTICAL PROCEDURE

At least three questions have been raised:

(1) Do slopes vary seasonally?
(2) Do intercepts vary seasonally?
(3) Is autocorrelation a serious problem with seasonal data?

No general answers are possible. Each case must be analyzed separately. As a beginning, some results are presented below for quarterly department store stocks, quarterly investment component of gross national product and monthly manufacturers' durable goods inventories.

Suppose we have the V equations

$$y_{jt} = \beta_{0j} + \sum_{i=1}^{m} \beta_{ij} X_{ijt} + \epsilon_{jt}; \qquad j = 1, 2, \cdots, V \tag{54}$$

where V = number of seasons per year as 4 for quarterly data or 12 for monthly data; y_{jt} = value of dependent variable in j-th season of year t; X_{ijt} = value of i-th independent variable in j-th season of year t; ϵ_{jt} = disturbance in j-th season of year t; $t = 1, 2, \cdots, T$.

The null hypothesis for questions (1) and (2) is: $H_0(1, 2)$

$$\beta_{01} = \beta_{02} = \cdots = \beta_{0V} \tag{55.0}$$

$$\beta_{11} = \beta_{12} = \cdots = \beta_{1V} \tag{55.1}$$

$$\vdots$$

$$\beta_{m1} = \beta_{m2} = \cdots = \beta_{mV}. \tag{55.m}$$

This hypothesis can be tested by an F ratio. The null hypothesis for question (2) is (55.1) $\cdots$ (55.m): $H_0(2)$. This hypothesis can also be tested by an F ratio.

No attempt is made here to answer the third question. To answer this question one might use the Durbin-Watson d statistic or the Hart-von Neumann ratio. There is experimental evidence that the latter is biased toward acceptance of the null hypothesis [1], [6]. The d statistic is not appropriate for equations that contain the just lagged dependent variable among the independent variables. The equations presented here are of this type.

8. QUARTERLY DEPARTMENT STORE STOCKS

End-of-quarter department store stocks were analyzed in three fashions: using seasonally adjusted (SA) data, using unadjusted (SU) data in the linear seasonal shift model, and covariance analyses. The following variables were used.

$I_t = (P_{Hst} + 2P_{At}) \div 3.$

P_{Hst} = Bureau of Labor Statistics Consumer Price Index for House Furnishings, 1947–49: 1.00, last month of quarter t.

P_{At} = Bureau of Labor Statistics Consumer Price Index for Apparel, 1947–49: 1.00, last month of quarter t.

i_t = indexes of department store stocks, end-of-quarter t, deflated by I_t.

s_t = sum of index of department store sales for the three months of quarter t divided by $I_t + I_{t-1/3} + I_{t-2/3}$.

$\Delta s_t = s_t - s_{t-1}$.

The other variables were mostly derivatives of the preceding variables.

$s_{2t}^2 = s_t^2$ in the second quarter, equals zero in other quarters.

$\Delta s_{3t} = \Delta st$ in the third quarter, equals zero in other quarters.

$\Delta s_{4t} = \Delta s_t$ in the fourth quarter, equals zero in other quarters.

E_t = zero in the second, third and fourth quarters, equals zero in the first quarter if Easter falls in March; equals the date in April if Easter comes in April.

t = quarterly time trend, equals one in 1948-I, two in 1948-II, etc.

In addition, in the seasonal shift model three seasonal variables defined in Table 1 were used.

The estimated equation was derived from the following argument. Desired end-of-quarter level of inventories, $\bar{i}_t$, depends on the level of sales expected for next quarter, s_{t+1}^*

$$\bar{i}_t = a s_{t+1}^* \tag{56}$$

TABLE 1. QUARTERLY VARIABLES

Quarter	Variables		
	D_1	D_2	D_3
1	−1	0	1
2	−1	0	−1
3	1	−1	0
4	1	1	0

TABLE 2. REGRESSION RESULTS USED IN COVARIANCE ANALYSES OF QUARTERLY DEPARTMENT STORE STOCKS, SEASONALLY UNADJUSTED DATA

Equation number	Coefficients and standard errors				b_0	R^2
	s_t	Δs_t	i_{t-1}	t		
1.C	1.132 (0.308)***	−0.496 (0.186)**	−0.798 (0.265)***	0.685 (0.247)***	61.703	0.897[a]
1.S	1.581 (0.225)***	−0.201 (0.125)	−0.757 (0.215)***	0.158 (0.076)**	—[b]	0.990[c]
QI	0.875 (0.666)	0.137 (0.422)	0.182 (0.707)	−0.007 (0.228)	4.847	0.992
QII	1.162 (0.098)***	−0.178 (0.090)*	−0.228 (0.090)**	0.098 (0.019)***	1.355	0.999
QIII	0.740 (0.680)	0.149 (0.565)	0.102 (0.692)	0.264 (0.162)	20.427	0.996
QIV	−0.933 (0.854)	1.779 (7.536)	1.926 (9.733)	−0.438 (2.367)	−17.874	0.920

[a] $FH_0(1, 2) = 9.32$***.

[b] Quarterly values of b_0 are: Quarter I $b_0 = 17.825$; II 15.055; III 20.696; IV 0.788

[c] $FH_0(2) = 0.88$.

The expected level of sales is taken as equal to actual current sales

$$\overset{*}{s}_{t+1} = s_t. \tag{57}$$

Actual change in inventories from the end of last quarter to the end of the current quarter, Δi_t, may differ (by a proportional factor) from desired change, $\bar{i}_t - i_{t-1}$, because of technological or logistical rigidities. Actual change in inventories is also affected by unexpected variations in sales, $s_t - s_t^* (= s_t - s_{t-1})$.

$$\Delta i_t = \gamma(\bar{i}_t - i_{t-1}) + \alpha \Delta s_t. \tag{58}$$

From these three equations we obtain the equation estimated

$$i_t = a\gamma s_t + \alpha \Delta s_t + (1 - \gamma) i_{t-1}. \tag{59}$$

The equations were estimated with i_t as the dependent variable. The equations were estimated using data for 1948-I to 1950-II and 1952-I to 1960-II. Results are presented in Tables 2 to 4. For each equation the top line presents coefficients. The second line presents standard errors. Single, double, or triple asterisks indicate, respectively, significance at the 10, 5 and 1 per cent levels.

TABLE 3. REGRESSION RESULTS IN COVARIANCE ANALYSES OF QUARTERLY DEPARTMENT STORE STOCKS, SEASONALLY ADJUSTED DATA

Equation number	Coefficients and standard errors				b_0	R^2
	s_t	Δs_t	i_{t-1}	t		
1.C	0.564 (0.125)***	0.037 (0.145)	0.528 (0.112)***	0.064 (0.087)	−7.457	0.989[a]
1.S	0.579 (0.122)***	−0.026 (0.156)	0.520 (0.110)***	0.060 (0.086)	—[b]	0.989[c]
QI	0.710 (0.284)**	−0.027 (0.403)	0.637 (0.328)*	−0.221 (0.272)	−30.580	0.986
QII	0.672 (0.157)***	−0.768 (0.326)*	0.415 (0.138)**	0.172 (0.102)	−6.481	0.998
QIII	0.098 (0.185)	−0.144 (0.251)	0.678 (0.135)***	0.365 (0.096)**	20.598	0.998
QIV	0.506 (0.210)*	0.636 (0.322)	0.846 (0.228)**	−0.208 (0.223)	−33.333	0.996

[a] $FH_0(1, 2) = 1.68$.

[b] Quarterly intercepts are: Quarter I $b_0 = -8.297$; II -7.416; III -8.588; IV -8.568

[c] $FH_0(2) = 1.89$.

Covariance analyses

In Tables 2 and 3, equations QI, QII, QIII, and QIV were estimated using data from the first, second, third, and fourth quarters, respectively. Equations 1.C, used in the F ratio for testing $H_0(1, 2)$, were estimated using pooled data measuring deviations for each variable from its over-all mean. Equations 1.S, used in the F ratio for testing $H_0(2)$, were estimated using pooled data with each observation measured from its respective quarterly mean.

The first F ratio is highly significant, and the second F ratio is nonsignificant for the unadjusted data. Hence, the linear seasonal shift model appears to be adequate for department store stocks analyses. Neither F ratio is significant for the seasonally adjusted data. Seasonal adjustment hides the seasonal shifts that do take place.

Linear seasonal shift model

The linear seasonal shift model (containing D_1, D_2, and D_3) was estimated. Plotting quarterly residuals from the SU equation suggested the addition of s_{2t}^2, Δs_{3t}, Δs_{4t}, and E_t (see Table 4). The addition of these variables resulted in a significant increase in the value of $\bar{R}^2$ by the conventional F test. Their addition also reduced the coefficients of t, D_{1t}, and D_{3t} to nonsignificance. The addi-

TABLE 4. SELECTED STATISTICS FROM LEAST SQUARES ESTIMATION OF QUARTERLY DEPARTMENT STORE STOCKS EQUATIONS, SEASONALLY ADJUSTED DATA (SA) AND UNADJUSTED DATA (SU)

Equation number	s_t	Δs_t	$\frac{s_{2t}^2}{100}$	Δs_{3t}	Δs_{4t}	E_t	i_{t-1}	t	D_{1t}	D_{2t}	D_{3t}	b_0	R^2
SU.1	1.682 (0.190)***	−0.292 (0.120)***	0.032 (0.014)**	0.691 (0.302)**	−0.149 (0.235)	0.112 (0.065)*	−0.843 (0.186)***	0.099 (0.069)	−0.591 (0.760)	−8.979 (1.283)***	2.268 (1.806)	12.083	0.996
SU.2	1.994 (0.087)***	−0.516 (0.050)***	0.041 (0.004)***	0.876 (0.240)***		0.126 (0.060)**	−1.091 (0.090)***			−8.436 (0.415)***		5.307	0.995
SA.1	0.610 (0.122)***	−0.133 (0.222)	0.010 (0.006)*	−0.092 (0.279)	0.428 (0.281)	0.004 (0.052)	0.558 (0.093)***					−15.128	0.990
SA.2	0.564 (0.125)***	0.037 (0.145)					0.528 (0.112)***	0.064 (0.087)				−7.457	0.988

tion of these same four variables—s_{2t}^2, Δs_{3t}, Δs_{4t}, and E_t—to SA equations had a nonsignificant effect on the value of R^2 (see Table 4).

There are several differences between the coefficients in Tables 2 and 4 and those in Tables 3 and 4. The coefficients of s_t obtained from SU data are more than twice as large as the coefficients obtained from SA data. The coefficients of i_{t-1} obtained from SU data are significantly negative; the coefficients obtained from SA data are significantly positive. The SU coefficient of Δs_t is significant in Table 4 and the SA coefficient is nonsignificant.

Summary

According to the results of the covariance analyses, the seasonal linear shift model is appropriate for the analysis of quarterly department store stocks. The F tests of SU data indicate seasonal variation in the intercept but not in the slopes. This seasonal variation in the intercept is submerged in the SA analyses. Results in Table 4 indicate that the coefficient of Δs_t is different in the third quarter from the other quarters and that the relation between inventory and current sales is curvilinear in the second quarter and linear in the other quarters.

Neither model (11) nor (42) is completely valid for the analysis of quarterly department store stocks, but the model (11) appears to be more nearly valid.

9. INVENTORY INVESTMENT COMPONENT OF GROSS NATIONAL PRODUCT

The variables used in this analysis are:

s_t = deflated final sales of goods, current quarter.

$\Delta i_t = i_t - i_{t-1}$.

i_t = deflated end-of-current-quarter nonfarm business inventories.

$\Delta 0_{ut-1} = 0_{ut-1} - 0_{ut-2}$.

0_{ut-1} = deflated durable goods industries manufacturers' unfilled orders, end of previous quarter.

All variables were measured in billions of dollars, except 0_{ut-1} which is in hundreds of millions of dollars.

The economic argument underlying the selection of variables is similar to the economic argument underlying the selection of variables in the previous section. Equations (56) and (57) are assumed to hold. In contrast with department stores, manufacturers may have substantial volumes of unfilled orders on hand. As these rise, manufacturers may desire to increase inventories of raw materials to guard against shortages or price increases. As steps are taken to work off unfilled orders, inventories of goods in process may rise. Equation (58) then becomes

$$\Delta i_t = \gamma(\bar{\imath}_t - i_{t-1}) + \alpha \Delta s_t + \beta \Delta 0_{ut-1} \tag{58.a}$$

and (59) becomes

$$i_t = a\gamma s_t + \alpha \Delta s_t + \beta \Delta 0_{ut-1} + (1 - \gamma) i_{t-1}. \tag{59.a}$$

Because of inertia Δi_t may be affected by Δi_{t-1}. Some equations containing this variable were also estimated.

These results are presented in Tables 5 to 7. The sample period was 1949-I, to 1960-IV. Here Δi_t was the dependent variable. This probably is the main reason why the values of R^2 are lower here than in Tables 2 to 4. In equations containing y_{t-1} as an independent variable, it makes little difference whether y_t or Δy_t is used as the dependent variable. Specifically: (1) the coefficient of y_{t-1} obtained using Δy_t will equal the coefficient obtained using y_t, minus one, (2) all other coefficients will be the same, (3) all standard errors will be the same, and (4) the value of R^2 obtained by the use of Δy_t will usually be smaller, but the values of F obtained in tests of significance of regression will be equal. (These statements are easily proved by using the inverse of a partitioned matrix $(X^*y_{t-1})'(X^*y_{t-1})$ [4]. This report presents additional results on analyses of monthly durable and nondurable goods inventories and quarterly inventories of beef, pork, cheese, butter and department store stocks, using SU data only.)

As with the previous analyses, these covariance analyses indicate seasonal variation in the intercept but none in the slopes. Again the seasonal adjustment process masks the seasonal variation in the intercept.

The coefficient of $\Delta 0_{ut-1}$ obtained with SU data is about one-third the size of the coefficient obtained with SA data although both are significant. In Table 7 the coefficient of Δi_{t-1} obtained from SU data is nonsignificant, whereas the

TABLE 5. REGRESSION RESULTS USED IN COVARIANCE ANALYSES OF QUARTERLY INVENTORY INVESTMENT COMPONENT OF GNP, SEASONALLY UNADJUSTED DATA

Equation number	Coefficients and standard errors				b_0	R^2
	s_t	Δs_t	i_{t-1}	$\Delta 0_{ut-1}$		
1.C	0.197 (0.049)***	−0.264 (0.029)***	−0.102 (0.031)***	0.032 (0.004)***	−9.093	0.783[a]
1.S	0.346 (0.072)***	−0.185 (0.086)**	−0.190 (0.043)***	0.030 (0.004)***	—[b]	0.689[c]
QI	0.346 (0.163)*	0.125 (0.186)	−0.160 (0.104)	0.017 (0.008)*	−12.146	0.793
QII	0.440 (0.252)	−0.047 (0.441)	−0.208 (0.144)	0.032 (0.008)***	−22.514	0.821
QIII	0.168 (0.134)	−0.135 (0.184)	−0.119 (0.077)	0.029 (0.007)***	−7.063	0.790
QIV	0.354 (0.115)**	−0.607 (0.189)**	−0.174 (0.077)*	0.043 (0.008)***	−15.970	0.868

[a] FH_0 (1, 2) =2.47**.

[b] Intercepts for the four quarters are: Quarter I b_0 = −14.75
II −16.45
III −16.33
IV −18.06

[c] FH_0 (2) =1.78.

TABLE 6. REGRESSION RESULTS USED IN COVARIANCE ANALYSES OF QUARTERLY INVENTORY INVESTMENT COMPONENT OF GNP, SEASONALLY ADJUSTED DATA

Equation number	Coefficients and standard errors				b_0	R^2
	s_t	Δs_t	i_{t-1}	$\Delta 0_{ut-1}$		
1.C	0.323 (0.066)***	−0.219 (0.106)*	−0.194 (0.043)***	0.110 (0.013)***	−59.874	0.699[a]
1.S	0.324 (0.065)***	−0.219 (0.104)*	−0.194 (0.042)***	0.111 (0.013)***	—[b]	0.712[c]
QI	0.176 (0.158)	0.159 (0.222)	−0.091 (0.108)	0.070 (0.034)*	−32.172	0.742
QII	0.470 (0.185)**	−0.166 (0.385)	−0.269 (0.125)*	0.111 (0.030)***	−89.903	0.821
QIII	0.151 (0.128)	−0.076 (0.234)	−0.098 (0.086)	0.107 (0.023)***	−26.277	0.791
QIV	0.375 (0.136)**	−0.501 (0.216)*	−0.231 (0.086)**	0.157 (0.034)***	−69.630	0.821

[a] FH_0 (1, 2) =0.98.
[b] Quarterly intercepts are: Quarter I $b_0 = -59.10$; II −60.44; III −60.30; IV −60.49
[c] FH_0 (2) =1.05.

coefficient obtained from SA data is significant. The partial correlations from these equations were not graphed to look for seasonal shifts in slopes.

In summary, according to the covariance analyses the linear shift model, (11), is valid for the analysis of the inventory investment component of GNP.

10. MANUFACTURERS' DURABLE GOODS INVENTORIES

The variables used in this analysis are:

P_{TDt} = wholesale price index, total durable goods, month t, 1947–49: 1.00.

P_{DMt} = wholesale price index, durable manufactured goods, month t, 1947–49: 1.00.

i_t = total durable goods industries manufacturers' inventories, millions of dollars, end of month t, divided by P_{TDt}.

s_t = manufacturers' durable goods sales, month t, millions of dollars, divided by P_{DMt}.

0_{ut} = durable goods industries manufacturers' unfilled orders, end of month t, millions of dollars, divided by P_{DMt}.

P_t = wholesale price index, durable raw or slightly processed goods, month t, 1947–49: 1.00.

$0_{u^*t-1} = -0^2_{ut-1} \times 10^{-6}$ in September,
$= 0^2_{ut-1} \times 10^{-6}$ in December,
$= 0$ all other months.

TABLE 7. LEAST SQUARES ESTIMATES OF QUARTERLY INVENTORY INVESTMENT COMPONENT OF GNP, SEASONALLY ADJUSTED DATA (SA) AND UNADJUSTED DATA (SU)

Equation number	Coefficients and standard errors									R^2
	s_t	Δs_t	i_{t-1}	$\Delta 0_{u\,t-1}$	Δi_{t-1}	D_{1t}	D_{2t}	D_{3t}	b_0	
SU.1	0.340 (0.075)***	−0.196 (0.090)**	−0.189 (0.045)***	0.027 (0.005)***	0.116 (0.123)	−0.693 (0.274)**	−0.851 (0.345)**	0.891 (0.669)	−16.101	0.839
SA. 1	0.313 (0.064)***	−0.265 (0.104)**	−0.195 (0.041)***	0.086 (0.017)***	0.250 (0.113)**				−57.866	0.730

TABLE 8. LEAST SQUARES ESTIMATES OF MONTHLY DURABLE GOODS INVENTORIES, SEASONALLY ADJUSTED DATA (SA) AND UNADJUSTED DATA (SU)

Equation number	Coefficients and standard errors												cos $30i°$	sin $90i°$	cos $120i°$	R^2
	b_3	s_{t-1}	Δs_t	$0_{u\,t-1}$	$\Delta 0_{u\,t-1}$	$\Delta 0_{u\,t-2}$	ΔP_t	ΔP_{t-1}	i_{t-1}	Δi_{t-1}	Δi_{t-2}	$0_u{}^*{}_{t-1}$				
SU. 1	−94	0.069 (0.018)***	0.061 (0.020)***	0.0020 (0.0012)*	−0.018 (0.017)	0.056 (0.018)***	−1.57 (0.30)***	0.82 (0.32)**	−0.031 (0.008)***	0.35 (0.07)***	0.16 (0.07)**	0.051 (0.017)***	86 (17)***	59 (16)***	27 (17)	0.751
SA. 1	−102	0.073 (0.021)***	0.034 (0.040)	0.0020 (0.0013)	−0.017 (0.017)	0.055 (0.017)***	−13.52 (3.11)***	6.21 (3.36)*	−0.033 (0.009)***	0.34 (0.08)***	0.16 (0.07)**					0.645

The economic argument used in the selection of variables here was similar to, but more complex than, the economic argument used in preceding sections. Equation (56) was replaced by

$$\bar{i}_t = a s^*_{t+1} + b \Delta P^*_{t+1}, \tag{56.b}$$

$\Delta P^*_{t+1} = P^*_{t+1} - P_t =$ the expected change from the current period to next period in P_t, to allow for the possibility of price speculation. Equation (57) was replaced by

$$s^*_{t+1} = s_{t-1} + \beta \Delta s_t. \tag{57.b}$$

Expected prices were assumed to be generated by

$$\Delta P^*_{t+1} = \alpha_0 \Delta P_t + \alpha_1 \Delta P_{t-1}. \tag{60}$$

Equation (58) was replaced by

$$\Delta i_t = \gamma_0(\bar{i}_t - i_{t-1}) + \gamma_1 \Delta i_{t-1} + \gamma_2 \Delta i_{t-2} + c_0 0_{ut-1} + c_1 \Delta 0_{ut-1} + c_2 \Delta 0_{ut-2}. \tag{58b}$$

The estimated equation was

$$\begin{aligned} \Delta i_t = {} & a\gamma_0 s_{t-1} + a\beta\gamma_0 \Delta s_t + c_0 0_{ut-1} \\ & + c_1 \Delta 0_{ut-1} + c_2 \Delta 0_{ut-2} \\ & + b\alpha_0\gamma_0 \Delta P_t + b\alpha_1\gamma_0 \Delta P_{t-1} \\ & - \gamma_0 i_{t-1} + \gamma_1 \Delta i_{t-1} + \gamma_2 \Delta i_{t-2} \\ & + c_3 \Delta 0_{u^*t-1}. \end{aligned} \tag{59.b}$$

$\Delta 0_{ut^*-1}$ was added after a graphic analysis of residuals.

In the analysis of unadjusted data the following monthly seasonal variables were included as independent variables: sine βi, cosine βi, sine $2\beta i$, cosine $2\beta i$, sine $3\beta i$, cosine $3\beta i$, sine $4\beta i$, cosine $4\beta i$, sine $5\beta i$, cosine $5\beta i$, and cosine $6\beta i$ where β equals 30 degrees and i each year ranges from 1 in January to 12 in December. Many of these variables were nonsignificant.

Table 8 presents some results obtained from SU and SA data. The sample period was April, 1948 to December, 1960.

The important differences between these equations seem to be in the coefficients of Δs_t, ΔP_t, ΔP_{t-1}, and $\Delta 0_{u^*t-1}$. The SU data indicate Δs_t to be highly significant, whereas the SA data indicate it to be nonsignificant. The coefficients of the current price change variable are significant. In the SA analysis the coefficient is eight times as large in absolute magnitude as in the SU analyses. The coefficient of ΔP_{t-1}, however, is significant at the 5 or the 1 per cent level in the SU analyses and is significant at only the 10 per cent level in the analyses using SA data. The variable 0_{u^*t-1} was added to the SU analyses after a graphic study of the partial regressions. Its coefficient is highly significant by the t test. This nonlinearity did not show up in the graphs of the partial regressions of the SA data. For the other variables presented here, the two types of data lead to the same conclusions concerning the significance of the variables and concerning the magnitude of their coefficients.

Some other differences between SU and SA results were found in equations which are not presented here. For example, in a few equations, $\Delta 0_{ut}$ was used in place of $\Delta 0_{ut-2}$. Its coefficient was never significant using SA data, but was negative and significant at the 5 per cent level when using SU data. Some equations were estimated using new orders rather than unfilled orders, 0_{Nt-1} (new orders of the previous month). The analysis of SU data indicated the coefficient of this variable to be nonsignificant and the analyses of SA data indicated it to be highly significant.

Covariance analyses were not carried out. The significance of the coefficients of cos $30i°$ and sin $90i°$ rejects the hypothesis of no seasonal variation in the intercept. The fact that $\Delta 0_{u^*t-1}$ is the only seasonal rotation variable that was significant in the analyses of SU data suggests that the SA model, equation (42), is less appropriate for the analysis of monthly manufacturers' durable good inventories than is the linear seasonal shift model, equation (11).

11. RETAIL MEAT PRICES

In a recent study Logan and Boles [5] used covariance analysis to test for quarterly fluctuations in demand for beef, pork, broilers, and lamb. In every case the hypothesis of no seasonal variation in intercept was rejected. Only in the case of lamb, however, was the hypothesis of no seasonal differences in slopes rejected.

12. SUMMARY

The same set of seasonal data cannot satisfy both the assumptions implicit in the linear seasonal shift model and in the seasonally adjusted data model. The data may not satisfy the assumptions of either one. In either case, analysis of SU data may be expected to lead to different results from the use of SA data. Covariance analyses of SU data (not of SA data) can be used to test which, if either, model is appropriate for the estimation of behavioral parameters. If covariance analysis rejects the hypothesis of constant slopes and constant intercepts, this does not establish that the seasonally adjusted data model is appropriate. This model assumes a certain pattern of variation in slopes and intercept. Out of six covariance analyses, two here and four in Logan and Boles [5], five lead to the conclusion that the linear shift model is appropriate. In only one analysis—lamb demand—were significant seasonal differences in slopes obtained.

The use of the linear seasonal shift model has one advantage over the use of seasonally adjusted data model. The former permits testing of one hypothesis about seasonal variation in parameters. The latter does not permit testing any hypotheses about seasonal variation in parameters. The former has one disadvantage, provided SA data are already available: greater computational expense.

The use of either procedure may lead to specification bias in the estimated coefficients. It does not seem that convenience by itself is an adequate basis for selecting one or the other. Additional investigation is needed. If additional covariance analyses turn out as the ones described above, five to one in favor of the linear seasonal shift model, the linear seasonal shift model should become more popular.

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ESTIMATION OF QUASI-LINEAR TREND AND SEASONAL VARIATION*

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Given a series of quarterly data, estimates may be obtained for both trend and seasonal variation by minimising the sum, or more generally a linear combination, of two sums of squares, one of them based on the second differences between trend values, the other on the deviations of the observations from seasonally corrected trend values. Exact solutions are obtained for 8 and 12 observations, and an approximate solution applicable to longer time series is given. A numerical example is supplied, and the procedure outlined here is compared with the moving average method.

1. INTRODUCTION

The extension of the least squares principle to time series in the form of minimising a linear combination of two sums of squares has first been developed by Whittaker [3, 4] and Henderson [1]. Leser [2] applied the principle to the case in which one sum of squares refers to the second differences between trend values, described as permanent disturbances, and the other one to the deviations of the observations from the trend, described as temporary disturbances. The resulting trend estimates, denoted as quasi-linear trends, were explicitly obtained with up to seven observations. For the central quasi-linear trend, obtained when the two sums of squares are combined in the proportion 1:1, an exact solution was given with up to 15, and an approximate solution with any number of observations.

In the present study, it is proposed to extend the method previously developed for annual data or data not subject to seasonal variation, to time series containing an additive seasonal component. It will be assumed here that the data are quarterly ones, though generalisation to monthly or other periodic data would not present any difficulty in principle. Unless otherwise stated, it will also be assumed that the number of observations is an exact multiple of 4.

2. THE GENERAL CASE

Given $n=4m$ data for successive quarters, where m is an integer ≥ 2; these are denoted by Y_i $(i=1, 2, \cdots, n)$. Let θ_i $(i=1, 2, \cdots, n)$ represent the unknown trend values, and Σ_i $(i=1, 2, 3, 4)$ the unknown seasonal components, with sum 0. Then the model is

$$\begin{aligned} Y_{4t+u} &= \theta_{4t+u} + \Sigma_u + \epsilon_{4t+u} \qquad \begin{pmatrix} t = 0, 1, \cdots, m-1 \\ u = 1, 2, 3, 4 \end{pmatrix} \\ \theta_i &= 2\theta_{i-1} - \theta_{i-2} + \gamma_{i-1} \qquad (i = 3, 4, \cdots, n). \end{aligned} \tag{1}$$

It is envisaged that in the absence of the permanent disturbances γ_i, the trend would be a straight line, but these disturbances change the direction of the

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trend. Furthermore, in the absence of temporary disturbances, the observations would be represented by the sum of trend and seasonal variation.

Estimators T_i $(i=1, 2, \cdots, n)$ for the trend and S_i $(i=1, 2, 3, 4)$ for the seasonal variation may be obtained, starting with the basic idea that both kinds of disturbances may be expected to be small. This consideration naturally leads to the condition that a linear combination of the two sums of squares becomes a minimum, or with any non-negative number h

$$\sum_{i=2}^{n-1} (T_{i-1} - 2T_i + T_{i+1})^2 + h \sum_{t=0}^{m-1} \sum_{u=1}^{4} (Y_{4t+u} - T_{4t+u} - S_u)^2 = \text{Min.} \qquad (2)$$

subject to the condition

$$\sum_{u=1}^{4} S_u = 0. \qquad (3)$$

In differentiating (2), the use of Lagrange multipliers is not required, since the set of $n+4$ linear equations obtained is linearly dependent. Condition (3) in effect serves to identify the solution. The linear equations obtained are as follows:

$$\begin{aligned}
(1+h)T_1 - 2T_2 + T_3 &= h(Y_1 - S_1) \\
-2T_1 + (5+h)T_2 - 4T_3 + T_4 &= h(Y_2 - S_2) \\
T_1 - 4T_2 + (6+h)T_3 - 4T_4 + T_5 &= h(Y_3 - S_3) \\
\cdots\cdots\cdots\cdots\cdots\cdots & \cdots\cdots\cdots \\
T_{n-4} - 4T_{n-3} + (6+h)T_{n-2} - 4T_{n-1} + T_n &= h(Y_{n-2} - S_2) \\
T_{n-3} - 4T_{n-2} + (5+h)T_{n-1} - 2T_n &= h(Y_{n-1} - S_3) \\
T_{n-2} - 2T_{n-1} + (1+h)T_n &= h(Y_n - S_4) \\
\sum_{t=0}^{m-1} T_{4t+1} &= \sum_{t=0}^{m-1} Y_{4t+1} - mS_1 \qquad (4) \\
\sum_{t=0}^{m-1} T_{4t+2} &= \sum_{t=0}^{m-1} Y_{4t+2} - mS_2 \\
\sum_{t=0}^{m-1} T_{4t+3} &= \sum_{t=0}^{m-1} Y_{4t+3} - mS_3 \\
\sum_{t=0}^{m-1} T_{4t+4} &= \sum_{t=0}^{m-1} Y_{4t+4} - mS_4.
\end{aligned}$$

Since the sum of the first n equations in (4) is the same as the sum of the last four equations multiplied by h, one equation is redundant. Together with (3), the system is determined.

The first n equations in (4) are identical with the set of equations (4) discussed in the earlier paper [2, p. 93], provided that Y_{4i+j} is replaced by $Y_{4i+j} - S_j$. Their solution may thus be written in the form

$$T_i = \sum_{t=0}^{m-1} \sum_{u=1}^{4} K_{i,4t+u}(Y_{4t+u} - S_u)$$

or alternatively

$$T_i = \sum_{j=1}^{n} K_{i,j} Y_j - \sum_{u=1}^{4} L_{i,u} S_u \qquad (i = 1, 2, \cdots, n)$$

$$L_{i,u} = \sum_{t=0}^{m-1} K_{i,4t+u}. \tag{5}$$

If the coefficients $K_{i,j}$ are known, the expressions (5) for the trend values T_i may be substituted into any three of the last four equations in (4). Together with (3), four linear equations in S_1, S_2, S_3, and S_4 in terms of $Y_1, Y_2, \cdots, Y_n$ are then obtained. When these have been solved, the seasonal variation terms can be eliminated from (5). Thus, both trend and seasonal variation may be expressed as linear functions of the data.

In the particular cases $h \to 0$ and $h \to \infty$, a general solution, valid for any value of m, may easily be derived. This cannot, of course, be done by simply putting $h=0$ or $h=\infty$ in equations (4). The correct procedure consists in solving the equations but neglecting all terms except those of the two lowest or two highest orders in h.

For $h \to 0$, the solution is equivalent to the well-known method of constructing a linear trend based on annual data, and using the arithmetic means of the deviations from the regression line to obtain the seasonal variation estimates.

For $h \to \infty$, the method uses in effect the second differences between observations, to estimate the second differences between seasonal variation terms. This works out most easily if the number of second differences, not the number of data, is a multiple of 4; in the present case this implies adding on two observations Y_0 and Y_{n+1} if this is possible, or else ignoring Y_1 and Y_n. The formulae obtained for S_1 and S_2 would be in the former case

$$S_1 = \frac{1}{16m}\{-5Y_0 + 11Y_1 - 4(Y_2 + Y_3 + Y_4) + 12Y_5 - 4(Y_6 + Y_7 + Y_8)$$
$$+ - \cdots + 12Y_{n-3} - 4(Y_{n-2} + Y_{n-1}) + Y_n + Y_{n+1}\}$$
$$S_2 = \frac{1}{16m}\{Y_0 - 7Y_1 + 12Y_2 - 4(Y_3 + Y_4 + Y_5) + 12Y_6$$
$$- + \cdots + 12Y_{n-2} - 4Y_{n-1} - 5Y_n + 3Y_{n+1}\} \tag{6}$$

and in the latter case

$$S_1 = \frac{1}{16(m-1)}\{3Y_2 - 5Y_3 - 4Y_4 + 12Y_5 - 4(Y_6 + Y_7 + Y_8) + 12Y_9$$
$$- + \cdots + 12Y_{n-3} - 7Y_{n-2} + Y_{n-1}\}$$
$$S_2 = \frac{1}{16(m-1)}\{Y_2 + Y_3 - 4(Y_4 + Y_5) + 12Y_6 - 4(Y_7 + Y_8 + Y_9)$$
$$+ 12Y_{10} - + \cdots - 4(Y_{n-5} + Y_{n-4} + Y_{n-3}) + 11Y_{n-2} - 5Y_{n-1}\}. \tag{7}$$

H‹ as in the following, the expressions for S_3 and S_4 are obtained by writing do‹n the coefficients for S_2 and S_1 in reverse order.

Generally speaking, it seems more realistic to take an intermediate value for h, which represents the ratio between the variances of the permanent and the temporary disturbances. The problem is somewhat analogous to that of linear regression with errors in both variables, in which case, under the usual assumptions, the ratio of error variances cannot be estimated and must be assigned an a priori value. The error variances are frequently assumed to be proportionate to the observed variances of the variables; or alternatively, when the variables are measured in the same units, the error variances are taken as equal.

By a similar reasoning it seems appropriate here to take $h=1$, implying equal variances for the permanent and temporary disturbances. With this assumption, some previous results obtained [2] may be utilised, and the resulting formulae are comparatively tractable.

3. SOLUTION FOR 8 OR 12 OBSERVATIONS

For the central quasi-linear trend ($h=1$) with $n=8$ and $n=12$, the coefficients $K_{i,j}$ in (5)—or, strictly speaking, the coefficients multiplied by 11,713 for $n=8$ and by 103,000 for $n=12$—have been tabulated [2, p. 102]. With their help, the following solutions for S_1 and S_2 are derived:

$n = 8, \quad m = 2:$

$$\begin{aligned} S_1 &= \frac{1}{1{,}248}(215Y_1 - 199Y_2 - 191Y_3 - 323Y_4 \\ &\quad + 691Y_5 - 113Y_6 - 121Y_7 + 11Y_8) \\ S_2 &= \frac{1}{1{,}248}(-193Y_1 + 459Y_2 - 173Y_3 - 249Y_4 \\ &\quad - 119Y_5 + 477Y_6 - 139Y_7 - 63Y_8) \end{aligned} \tag{8}$$

$n = 12, \quad m = 3:$

$$\begin{aligned} S_1 &= \frac{1}{44{,}304}(5{,}236Y_1 - 4{,}711Y_2 - 3{,}723Y_3 - 5{,}167Y_4 \\ &\quad + 14{,}557Y_5 - 3{,}048Y_6 - 4{,}752Y_7 - 6{,}601Y_8 \\ &\quad + 13{,}435Y_9 - 3{,}297Y_{10} - 2{,}601Y_{11} + 692Y_{12}) \\ S_2 &= \frac{1}{44{,}304}(-4{,}668Y_1 + 10{,}695Y_2 - 3{,}945Y_3 - 4{,}773Y_4 \\ &\quad - 2{,}913Y_5 + 11{,}568Y_6 - 3{,}768Y_7 - 5{,}043Y_8 \\ &\quad - 3{,}495Y_9 + 10{,}965Y_{10} - 3{,}363Y_{11} - 1{,}260Y_{12}). \end{aligned} \tag{9}$$

The formulae for S_3 are obtained from those for S_2, and the formulae for S_4 from those for S_1, by reversing the order of the coefficients. Writing the expressions for the trend values as

$$T_i = \sum_{j=1}^{n} a_{i,j} Y_j$$

the coefficients $a_{i,j}$—or rather their ratio to 1,248 and 44,304 respectively are shown in Table 1.

The remaining coefficients follow from symmetry considerations, since for

$$n = 8: \quad a_{i,j} = a_{9-i,9-j}$$

$$n = 12: \quad a_{i,j} = a_{13-i,13-j}$$

It should, however, be noted that the coefficient matrix is not fully symmetrical, i.e. $a_{i,j} \neq a_{j,i}$. This is in contrast to the result obtained for the quasi-linear trend in the absence of seasonal variation, where $K_{i,j} = K_{j,i}$.

The estimators S_i given by equations (8) or (9) can be shown to possess optimal properties if considered as estimators to the seasonal variations Σ_i in model (1). Write for any $n = 4m$, m integer ≥ 2

$$S_i = \sum_{j=1}^{n} b_{i,j} Y_j \qquad (i = 1, 2, 3, 4).$$

Furthermore, write $B_{i,1}$ for the sum of the coefficients of $Y_1, Y_5, \cdots, Y_{n-3}$,

TABLE 1. COEFFICIENTS OF Y_j IN CENTRAL QUASI-LINEAR TREND, 8 OR 12 QUARTERLY DATA

$n = 8$: $D = 1,248$

j	$Da_{1,j}$	$Da_{2,j}$	$Da_{3,j}$	$Da_{4,j}$
1	835	377	87	−51
2	383	429	291	145
3	215	325	411	297
4	283	273	303	389
5	−523	−65	225	363
6	−71	−117	21	167
7	97	−13	−99	15
8	29	39	9	−77

$n = 12$: $D = 44,304$

j	$Da_{1,j}$	$Da_{2,j}$	$Da_{3,j}$	$Da_{4,j}$	$Da_{5,j}$	$Da_{6,j}$
1	31,564	13,314	2,568	−1,816	−2,288	− 174
2	13,377	16,245	10,467	4,761	741	−2,139
3	5,031	11,049	15,759	10,749	5,187	789
4	4,043	6,933	10,947	15,049	10,751	5,385
5	−11,609	−1,077	6,507	12,185	15,535	10,551
6	− 1,794	−3,012	612	5,364	10,686	15,408
7	3,744	− 30	−2,796	252	5,148	10,722
8	6,851	3,183	− 735	−3,293	767	5,439
9	− 8,879	−1,161	2,001	707	−2,171	699
10	− 507	−2,157	− 3	951	− 351	−2,193
11	2,301	57	−1.887	75	741	− 435
12	182	960	864	− 680	− 442	252

and similarly $B_{i,2}$, $B_{i,3}$ and $B_{i,4}$ for corresponding sums of coefficients; thus

$$B_{i,k} = \sum_{j=0}^{m-1} b_{i,4j+k} \qquad (k = 1, 2, 3, 4)$$

and $C_{i,k}$ for the expressions obtained by double cumulation of the terms $b_{i,j}$

$$C_{i,k} = \sum_{j=1}^{k} (k+1-j)b_{i,j}.$$

Then with the help of (1) we obtain the relations

$$\begin{aligned} S_i = {} & \left\{C_{i,n} - (n-1)\sum_{k=1}^{4} B_{i,k}\right\}\theta_1 + \left\{n\sum_{k=1}^{4} B_{i,k} - C_{i,n}\right\}\theta_2 \\ & + B_{i,1}\textstyle\sum_1 + B_{i,2}\sum_2 + B_{i,3}\sum_3 + B_{i,4}\sum_4 \\ & + \left\{C_{i,1} + (n-1)\sum_{k=1}^{4} B_k - C_{i,n}\right\}\gamma_2 \\ & + \left\{C_{i,2} + (n-2)\sum_{k=1}^{4} B_k - C_{i,n}\right\}\gamma_3 \\ & + \cdots\cdots\cdots\cdots\cdots \\ & + \left\{C_{i,n-2} + 2\sum_{k=1}^{4} B_k - C_{i,n}\right\}\gamma_{n-1} \\ & + b_{i,1}\epsilon_1 + \cdots + b_{i,n}\epsilon_n. \end{aligned} \tag{10}$$

It will be seen that with $E(\gamma_i) = E(\epsilon_i) = 0$, $E(S_i) = \Sigma_i$ if and only if

$$\begin{aligned} B_{i,i} &= \tfrac{3}{4} \\ B_{i,j} &= -\tfrac{1}{4} \qquad (j \neq i) \\ L_{i,n} &= 0. \end{aligned} \tag{11}$$

The coefficients in (8) and (9) satisfy these conditions. The estimators are therefore unbiased.

Furthermore, if conditions (11) are satisfied, equations (10) may be written more simply as

$$S_i = \textstyle\sum_i + \displaystyle\sum_{j=1}^{n-2} C_{i,j}\gamma_{j+1} + \sum_{j=1}^{n} b_{i,j}\epsilon_j \tag{12}$$

and with

$$\begin{aligned} E(\gamma_i\gamma_j) &= \delta_{ij}\sigma_\gamma^2 \\ E(\epsilon_i\epsilon_j) &= \delta_{ij}\sigma_\epsilon^2 \\ E(\gamma_i\epsilon_j) &= 0 \end{aligned}$$

the sampling variance is obtained as

$$\mathrm{Var}(S_i) = \sigma_Y^2 \sum_{j=1}^{n-2} C_{i,j}^2 + \sigma_\epsilon^2 \sum_{j=1}^{n} b_{i,j}^2. \tag{13}$$

It can be shown that when $\sigma_Y^2 = \sigma_\epsilon^2$, the coefficients in (8) and (9) minimise the sampling variance, subject to restraints (11). Under these assumptions, (8) and (9) thus give best unbiased linear estimators for $n=8$ and $n=12$ respectively.

This property does not, of course, hold for a different ratio between the error variances. If $\sigma_Y^2=0$, the linear trend method is optimal; and if $\sigma\epsilon^2=0$, the second difference method is optimal. However, the estimators S_i given in (8) for 8 observations can be shown to be at least 64.5 per cent efficient, and the estimators given in (9) for 12 observations to be at least 77.2 per cent efficient even in these extreme cases.

4. APPROXIMATE SOLUTION FOR MORE THAN 12 OBSERVATIONS

In practice, time series analysed generally cover at least four full years and are often much longer. An exact solution of (4), even with $h=1$, would be laborious to derive and cumbersome to use. An approximate solution has therefore been worked out, based on the approximation formulae given for the central quasi-linear trend in the absence of seasonal variation [2, p. 103]. By neglecting all but the two highest powers of m in the result and using numerical approximations, the following formulae are obtained:

$$S_1 = \frac{1}{m(m-1.08)}$$
$$\cdot\{(.31m-.36)Y_1 - (.33m-.35)Y_2 - (.19m-.09)Y_3 - (.19m+.08)Y_4$$
$$+(.77m-.36)Y_5 - (.25m-.35)Y_6 - (.25m-.09)Y_7 - (.25m+.08)Y_8$$
$$+(.75m-.36)Y_9 - (.25m-.35)Y_{10} - + \cdots$$
$$\cdots$$
$$\cdots - (.25m-.09)Y_{n-5} - (.27m+.08)Y_{n-4}$$
$$+(.72m-.36)Y_{n-3} - (.25m-.35)Y_{n-2} - (.13m-.09)Y_{n-1}$$
$$+(.06m-.08)Y_n\} \tag{14}$$

$$S_2 = \frac{1}{m(m-1.08)}$$
$$\cdot\{-(.31m-.37)Y_1 + (.70m-.72)Y_2 - (.27m-.28)Y_3 - (.24m-.07)Y_4$$
$$-(.25m-.37)Y_5 + (.75m-.72)Y_6 - (.25m-.28)Y_7 - (.25m-.07)Y_8$$
$$-(.25m-.37)Y_9 + - \cdots$$
$$\cdots$$
$$\cdots - (.25m-.07)Y_{n-4}$$
$$-(.29m-.37)Y_{n-3} + (.71m-.72)Y_{n-2} - (.24m-.28)Y_{n-1}$$
$$-(.06m-.07)Y_n\}.$$

They are applicable provided that $m \geq 4$ and may be applied, alternatively to the exact solution (9), to the case $m=3$, if the coefficients given for Y_{n-4} and Y_{n-3} are used as coefficients of Y_8 and Y_9.

The coefficients in (14) satisfy the first two parts of condition (11), but the last part is only approximately satisfied. The estimators are therefore slightly biased, owing to the approximations made in deriving them, but this is of no practical importance.

The trend values are obtained, once the seasonal variation has been computed, from the following set of formulae:

$$
\begin{aligned}
T_1 &= .77Y_1 + .29Y_2 + .04Y_3 - .04(Y_4 + Y_5) - .02Y_6 \\
&\quad - .48S_1 - .02S_2 + .21S_3 + .29S_4 \\
T_2 &= .29Y_1 + .41Y_2 + .24Y_3 + .08Y_4 + .01Y_5 - .01Y_6 - .02Y_7 \\
&\quad - .05S_1 - .15S_2 + .03S_3 + .17S_4 \\
T_3 &= .04Y_1 + .24Y_2 + .40Y_3 + .25Y_4 + .09Y_5 + .01Y_6 - .01Y_7 - .02Y_8 \\
&\quad + .12S_1 \qquad\qquad - .14S_3 + .02S_4 \qquad\qquad (15) \\
T_4 &= - .04Y_1 + .08Y_2 + .25Y_3 + .40Y_4 + .24Y_5 + .09Y_6 + .01Y_7 - .01Y_8 \\
&\quad - .02Y_9 + .07S_1 + .08S_2 - .01S_3 - .14S_4 \\
T_5 &= - .04Y_1 + .01(Y_2 + Y_8) + .09(Y_3 + Y_7) + .24(Y_4 + Y_6) + .39Y_5 \\
&\quad - .01Y_9 - .02Y_{10} - .09S_1 + .02S_2 + .07S_3 \\
T_6 &= - .02(Y_1 + Y_{11}) - .01(Y_2 + Y_{10}) + .01(Y_3 + Y_9) + .09(Y_4 + Y_8) \\
&\quad + .24(Y_5 + Y_7) + .38Y_6 + .02S_1 - .11S_2 + .02S_3 + .07S_4.
\end{aligned}
$$

The formulae for T_7, $T_8 \cdots$, T_{n-5} are derived from the formula for T_6 by shifting the suffix. The formulae for T_{n-4}, $T_{n-3} \cdots$, T_n are derived by writing the coefficients of the Y_i in T_5, T_4, $\cdots$, T_1 respectively in reverse order, ending with a term containing Y_n, and also the coefficients of the S_i in reverse order.

Since the solution is an approximate one only, the last four equations in (4) are only approximately satisfied. The relationship obtained by adding these equations, which is

$$\sum_{i=1}^{n} T_i = \sum_{i=1}^{n} Y_i$$

does, however, hold exactly. This implies that when trend and seasonal variation have been eliminated from the data, the sum of the residuals is zero; the sums of residuals for each quarter are, however, only approximately zero.

The method developed here thus permits the estimation of trend and seasonal variation for any series of quarterly data, the length of which is a multiple of 4. Compared with the linear trend method, it has the advantage of far greater flexibility, and it avoids the complications which arise when curvilinear trends are employed. Compared with the moving average method, it has the advantage of being more clearly based on a theoretical foundation, as well as the practical advantage that it immediately offers trend values for all observa-

TABLE 2. COEFFICIENTS OF Y_j IN APPROXIMATION FORMULAE FOR S_i

n m	12 3	16 4	20 5	24 6	28 7	32 8	36 9	40 10
D	5.76	11.68	19.60	29.52	41.44	55.36	71:28	89.20
$Db_{1,1}$	.57	.88	1.19	1.50	1.81	2.12	2.43	2.74
$Db_{1,2}$	−.64	−.97	−1.30	−1.63	−1.96	−2.29	−2.62	−2.95
$Db_{1,3}$	−.48	−.67	−.86	−1.05	−1.24	−1.43	−1.62	−1.81
$Db_{1,4}$	−.65	−.84	−1.03	−1.22	−1.41	−1.60	−1.79	−1.98
$Db_{1,5}$	1.95	2.72	3.49	4.26	5.03	5.80	6.57	7.34
$Db_{1,6} = \cdots = Db_{1,n-6}$	−.40	−.65	−.90	−1.15	−1.40	−1.65	−1.90	−2.15
$Db_{1,7} = \cdots Db_{1,n-5}$	−.66	−.91	−1.16	−1.41	−1.66	−1.91	−2.16	−2.41
$Db_{1,8} = \cdots = Db_{1,n-8}$	/	−1.08	−1.33	−1.58	−1.83	−2.08	−2.33	−2.58
$Db_{1,9} = \cdots = Db_{1,n-7}$	/	2.64	3.39	4.14	4.89	5.64	6.39	7.14
$Db_{1,n-4}$	−.89	−1.16	−1.43	−1.70	−1.97	−2.24	−2.51	−2.78
$Db_{1,n-3}$	1.80	2.52	3.24	3.96	4.68	5.40	6.12	6.84
$Db_{1,n-2}$	−.40	−.65	−.90	−1.15	−1.40	−1.65	−1.90	−2.15
$Db_{1,n-1}$	−.30	−.43	−.56	−.69	−.82	−.95	−1.08	−1.21
$Db_{1,n}$	.10	.16	.22	.28	.34	.40	.46	.52
$Db_{2,1}$	−.56	−.87	−1.18	−1.49	−1.80	−2.11	−2.42	−2.73
$Db_{2,2}$	1.38	2.08	2.78	3.48	4.18	4.88	5.58	6.28
$Db_{2,3}$	−.53	−.80	−1.07	−1.34	−1.61	−1.88	−2.15	−2.42
$Db_{2,4}$	−.65	−.89	−1.13	−1.37	−1.61	−1.85	−2.09	−2.33
$Db_{2,5} = \cdots = Db_{2,n-7}$	−.38	−.63	−.88	−1.13	−1.38	−1.63	−1.88	−2.13
$Db_{2,6} = \cdots = Db_{2,n-6}$	1.53	2.28	3.03	3.78	4.53	5.28	6.03	6.78
$Db_{2,7} = \cdots = Db_{2,n-5}$	−.47	−.72	−.97	−1.22	−1.47	−1.72	−1.97	−2.22
$Db_{2,8} = \cdots = Db_{2,n-4}$	−.68	−.93	−1.18	−1.43	−1.68	−1.93	−2.18	−2.43
$Db_{2,n-3}$	−.50	−.79	−1.08	−1.37	−1.66	−1.95	−2.24	−2.53
$Db_{2,n-2}$	1.41	2.12	2.83	3.54	4.25	4.96	5.67	6.38
$Db_{2,n-1}$	−.44	−.68	−.92	−1.16	−1.40	−1.64	−1.88	−2.12
$Db_{2,n}$	−.11	−.17	−.23	−.29	−.35	−.41	−.47	−.53

tions, without losing two quarters each at the beginning and end of the period. These advantages would seem to more than outweigh the slight drawback of a somewhat greater computational effort.

For $3 \leq m \leq 10$, the numerical values of the denominator D and the coefficients $Db_{i,j}$ in the numerator of (14) are shown in Table 2.

5. A NUMERICAL EXAMPLE

The application of formulae (14) and (15) to estimate seasonal variation and trend may be illustrated by a numerical example. The basic data used are given in Table 3.

With $n=28$, $m=7$, the appropriate column of coefficients in Table 2 is utilised to obtain numerical values for S_i. A moving average has also been worked out for comparison by the usual method, and the resulting estimates are denoted by S_i'. The data are not suitable for fitting a linear trend. The results are:

$$S_1 = 2.70, \quad S_2 = .59, \quad S_3 = -4.37, \quad S_4 = 1.08$$

as against

$$S_1' = 2.45, \quad S_2' = .88, \quad S_3' = -4.57, \quad S_4' = 1.24.$$

The trend values have been computed with the aid of (15) and are shown in Table 4, with the corresponding moving average given in brackets where applicable. The residuals obtained by deducting trend and seasonal variation from the data are also shown.

TABLE 3. DATA FOR EXAMPLE: VALUE OF IMPORTS, IRELAND 1955–61

Year	£ Million: Quarter			
	I	II	III	IV
1955	55.4	50.5	47.7	54.1
1956	52.9	46.7	39.5	43.8
1957	47.8	45.4	42.6	48.3
1958	48.7	51.5	44.5	54.3
1959	54.5	53.6	50.7	53.8
1960	57.0	55.9	52.2	61.2
1961	65.8	67.4	62.3	65.7

If we denote the second differences between trend values, or estimated permanent disturbances by g_i, the residuals or estimated temporary disturbances by e_i, and the corresponding figures based on the moving average by g'_i, e'_i, then the expressions $\sum g_i^2$ and $\sum g'^2_i$ serve as measures for the smoothness of the trend, whilst $\sum e_i^2$ and $\sum e'^2_i$ indicate the closeness of fit. Here it is found that

$$\sum_{i=2}^{27} g_i^2 = 14.77$$

$$\sum_{i=1}^{28} e_i^2 = 28.71$$

TABLE 4. TREND AND RESIDUALS, VALUE OF IMPORTS, IRELAND 1955–61*

Year	£ Million: Quarter			
	I	II	III	IV
Trend:				
1955	52.08	51.65	51.86 (51.61)	51.42 (50.82)
1956	49.47 (49.32)	46.75 (47.01)	44.61 (45.09)	43.78 (44.29)
1957	44.21 (44.51)	45.17 (45.46)	46.16 (46.14)	46.94 (47.01)
1958	47.75 (48.01)	49.16 (49.00)	50.39 (50.48)	51.72 (51.46)
1959	52.52 (52.50)	53.25 (53.21)	53.66 (53.46)	53.74 (54.06)
1960	54.19 (54.54)	55.40 (55.65)	57.39 (57.68)	60.15 (60.21)
1961	63.17 (62.91)	65.42 (64.74)	66.09	65.70
Residuals:				
1955	+ .62	−1.74	+ .21 (+ .66)	+1.60 (+2.04)
1956	+ .73 (+1.13)	− .64 (−1.19)	− .74 (−1.02)	−1.06 (−1.73)
1957	+ .89 (+ .84)	+ .36 (− .94)	+ .81 (+1.03)	+ .28 (+ .05)
1958	−1.75 (−1.76)	+1.75 (+1.62)	−1.52 (−1.41)	+1.50 (+1.60)
1959	− .72 (− .45)	− .24 (− .49)	+1.41 (+1.81)	−1.02 (−1.50)
1960	+ .11 (+ .01)	− .09 (− .63)	− .82 (− .91)	− .03 (− .25)
1961	− .07 (+ .44)	+1.39 (+1.78)	+ .58	−1.08

and if the first two and last two terms are omitted for the sake of comparability with the moving average

$$\sum_{i=4}^{25} g_i^2 = 10.32$$

$$\sum_{i=3}^{26} e_i^2 = 23.79$$

as against

$$\sum_{i=4}^{25} g_i'^2 = 7.50$$

$$\sum_{i=3}^{26} e_i'^2 = 34.86.$$

The central quasi-linear trend is thus a little less smooth than the moving average. This will generally be the case, and if a smooth trend at the expense of closeness of fit should be required, the specification $h=1$ should be replaced by a lower value for h.

On the other hand, the central quasi-linear trend, together with the seasonal element, gives a considerably closer fit to the data than the moving average. This is, of course, not surprising since the method adopted here minimises the sum of the two sums of squares, and if the first sum is relatively high, the second sum must a fortiori be relatively low.

All in all, the example shows the method outlined here to be a practicable one yielding plausible results. It is suggested that the method presents an improvement on established elementary procedures, whilst itself following a simple idea and requiring only simple calculations.

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